ON SOLVABLE SUBGROUPS OF THE CREMONA GROUP

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ABSTRACT. The Cremona group $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ is the group of birational self-maps of $\mathbb{P}^2_{\mathbb{C}}$. Using the action of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ on the Picard-Manin space of $\mathbb{P}^2_{\mathbb{C}}$, we characterize its solvable subgroups. If $\mathrm{G} \subset \mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ is solvable, nonvirtually Abelian, and infinite, then up to finite index: either any element of G is of finite order or conjugate to an automorphism of $\mathbb{P}^2_{\mathbb{C}}$, or G preserves a unique fibration that is rational or elliptic, or G is, up to conjugacy, a subgroup of the group generated by one hyperbolic monomial map and the diagonal automorphisms.

We also give some corollaries.

1. Introduction

We know properties on finite subgroups ([16]), finitely generated subgroups ([6]), uncountable maximal Abelian subgroups ([13]), nilpotent subgroups ([14]) of the Cremona group. In this article, we focus on solvable subgroups of the Cremona group.

Let G be a group. Recall that $[g,h] = ghg^{-1}h^{-1}$ denotes the commutator of g and h. If Γ_1 and Γ_2 are two subgroups of G, then $[\Gamma_1, \Gamma_2]$ is the subgroup of G generated by the elements of the form $[g,h]$ with $g \in \Gamma_1$ and $h \in \Gamma_2$. We define the *derived series* of G by setting

$$\mathrm{G}^{(0)} = \mathrm{G}, \qquad \mathrm{G}^{(n+1)} = \left[\mathrm{G}^{(n)}, \mathrm{G}^{(n)}\right] \quad \forall n \geq 0.$$

The *soluble length* $\ell(\mathrm{G})$ of G is defined by

$$\ell(\mathrm{G}) = \min\left\{k \in \mathbb{N} \cup \{0\} | \mathrm{G}^{(k)} = \{\mathrm{id}\}\right\}$$

with the convention: $\min \emptyset = \infty$. We say that G is *solvable* if $\ell(\mathrm{G}) < \infty$. The study of solvable groups started a long time ago, and any linear solvable

subgroup is up to finite index triangularizable (Lie–Kolchin theorem, [23, Theorem 21.1.5]). The assumption "up to finite index" is essential: for instance, the subgroup of $\mathrm{PGL}(2,\mathbb{C})$ generated by $\left[\begin{smallmatrix} 1 & 0 \\ 1 & -1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ is isomorphic to $\mathfrak{S}_3$ so is solvable but is not triangularizable.

THEOREM A. *Let* G *be an infinite, solvable, non virtually Abelian subgroup of* $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$. *Then, up to finite index, one of the following holds*:

(1) *any element of* G *is either of finite order, or conjugate to an automorphism of* $\mathbb{P}^2_{\mathbb{C}}$;

(2) G *preserves a unique fibration that is rational, in particular* G *is, up to conjugacy, a subgroup of* $\mathrm{PGL}(2,\mathbb{C}(y)) \rtimes \mathrm{PGL}(2,\mathbb{C})$;

(3) G *preserves a unique fibration that is elliptic*;

(4) G *is, up to birational conjugacy, contained in the group generated by*

$$\left\{ \left(x^p y^q, x^r y^s \right), (\alpha x, \beta y) | \alpha, \beta \in \mathbb{C}^* \right\},$$

where $M = \left[\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right]$ *denotes an element of* $\mathrm{GL}(2,\mathbb{Z})$ *with spectral radius* > 1. *The group* G *preserves the two holomorphic foliations defined by the 1-forms* $\alpha_1 x \mathrm{d}y + \beta_1 y \mathrm{d}x$ *and* $\alpha_2 x \mathrm{d}y + \beta_2 y \mathrm{d}x$ *where* (α_1, β_1) *and* (α_2, β_2) *denote the eigenvectors of* ${}^t M$.

Furthermore if G *is uncountable, case* 3. *does not hold.*

EXAMPLES. • Denote by S_3 the group generated by $\left[\begin{smallmatrix} 1 & 0 \\ 1 & -1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. As we recall before $\mathrm{S}_3 \simeq \mathfrak{S}_3$. Consider now the subgroup G of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ whose elements are the monomial maps $(x^p y^q, x^r y^s)$ with $\left[\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right] \in \mathrm{S}_3$. Then any element of G has finite order, and G is solvable; it gives an example of case 1.

• Other examples that illustrate case 1. are the following groups

$$\left\{ (\alpha x + \beta y + \gamma, \delta y + \varepsilon) | \alpha, \delta \in \mathbb{C}^*, \beta, \gamma, \varepsilon \in \mathbb{C} \right\} \subset \mathrm{Aut}\left(\mathbb{P}^2_{\mathbb{C}}\right),$$

and

$$\mathrm{E} = \left\{ \left(\alpha x + P(y), \beta y + \gamma \right) | \alpha, \beta \in \mathbb{C}^*, \gamma \in \mathbb{C}, P \in \mathbb{C}[y] \right\} \subset \mathrm{Aut}\left(\mathbb{C}^2\right).$$

• The centralizer of a birational map of $\mathbb{P}^2_{\mathbb{C}}$ that preserves a unique fibration that is rational is virtually solvable ([9, Corollary C]); this example falls in case 2 (see Section 3.2).

• In [10, Proposition 2.2] Cornulier proved that the group

$$\left\langle (x+1, y), (x, y+1), (x, xy) \right\rangle$$

is solvable of length 3, and is not linear over any field; this example falls in case 2. The invariant fibration is given by $x = \mathrm{cst}$.

REMARK. In case 1 if there exists an integer d such that $\deg \phi \leq d$ for any ϕ in G, then there exist a smooth projective variety M and a birational map $\psi \colon M \dashrightarrow \mathbb{P}^2_{\mathbb{C}}$ such that $\psi^{-1} \mathrm{G} \psi$ is a solvable subgroup of $\mathrm{Aut}(M)$ (see Section 3.3). But there is some solvable subgroups G with only elliptic ele-

ments that do not satisfy this property: the group E introduced in Examples. Let us mention an other example: Wright constructs Abelian subgroups H of $\mathrm{Aut}(\mathbb{C}^2)$ such that any element of H is of finite order, H is unbounded and does not preserve any fibration ([28]).

In Section 3, we prove Theorem A: we first assume that our solvable, infinite and non virtually Abelian subgroup G contains a hyperbolic map, then that it contains a twist and no hyperbolic map, and finally that all elements of G are elliptic. In the last section (Section 4), we also

- recover the following fact: if G is an infinite nilpotent subgroup of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$, then G does not contain a hyperbolic map;
- remark that we can bound the soluble length of a nilpotent subgroup of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ by the dimension of $\mathbb{P}^2_{\mathbb{C}}$ as Epstein and Thurston did in the context of Lie algebras of rational vector fields on a connected complex manifold;
- give a negative answer to the following question of Favre: does any solvable and finitely generated subgroup G of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ contain a subgroup of finite index whose commutator subgroup is nilpotent? if we assume that $[\mathrm{G}, \mathrm{G}]$ is not a torsion group;
- give a description of the embeddings of the solvable Baumslag–Solitar groups into the Cremona group.

2. Some properties of the birational maps

First definitions. Let $\mathcal{S}$ be a projective surface. We will denote by $\mathrm{Bir}(\mathcal{S})$ the group of birational self-maps of $\mathcal{S}$; in the particular case of the complex projective plane the group $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ is called *Cremona group*. Take ϕ in $\mathrm{Bir}(\mathcal{S})$, we will denote by $\mathrm{Ind}\,\phi$ the set of points of indeterminacy of ϕ; the codimension of $\mathrm{Ind}\,\phi$ is ≥ 2.

A birational map from $\mathbb{P}^2_{\mathbb{C}}$ into itself can be written

$$(x : y : z) \dashrightarrow \big(\phi_0(x, y, z) : \phi_1(x, y, z) : \phi_2(x, y, z)\big),$$

where the ϕ_i's denote some homogeneous polynomials of the same degree and without common factors of positive degree. The *degree* of ϕ is equal to the degree of the ϕ_i's. Let ϕ be a birational map of $\mathbb{P}^2_{\mathbb{C}}$. One can define the *dynamical degree* of ϕ as

$$\lambda(\phi) = \lim_{n \to +\infty} \big(\deg \phi^n\big)^{1/n}.$$

More generally, let $\mathcal{S}$ be a projective surface, and $\phi \colon \mathcal{S} \dashrightarrow \mathcal{S}$ be a birational map. Take any norm $\|\cdot\|$ on the Néron–Severi real vector space $\mathrm{N}^1(\mathcal{S})$. If ϕ^* is the induced action by ϕ on $\mathrm{N}^1(\mathcal{S})$, we can define

$$\lambda(\phi) = \lim_{n \to +\infty} \big\|\big(\phi^n\big)^*\big\|^{1/n}.$$

Remark that this quantity is a birational invariant: if $\psi \colon \mathcal{S} \dashrightarrow \mathcal{S}'$ is a birational map, then $\lambda(\psi\phi\psi^{-1}) = \lambda(\phi)$.

Classification of birational maps. The algebraic degree is not a birational invariant, but the first dyna mical degree is; more precisely one has a classification of birational maps based on the degree growth. Before stating it, let us first introduce the following definitions. Let ϕ be an element of $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$. If

- $(\deg \phi^n)_{n \in \mathbb{N}}$ is bounded, we say that ϕ is an *elliptic map*,
- $(\deg \phi^n)_{n \in \mathbb{N}}$ grows linearly, we say that ϕ is a *Jonquières twist*,
- $(\deg \phi^n)_{n \in \mathbb{N}}$ grows quadratically, we say that ϕ is a *Halphen twist*,
- $(\deg \phi^n)_{n \in \mathbb{N}}$ grows exponentially, we say that ϕ is a *hyperbolic map*.

THEOREM 2.1 ([15], [20], [3]). *Let ϕ be an element of $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$. Then one and only one of the following cases holds*

- ϕ *is elliptic, furthermore if ϕ is of infinite order, then ϕ is up to birational conjugacy an automorphism of $\mathbb{P}^2_\mathbb{C}$,*
- ϕ *is a Jonquières twist, ϕ preserves a unique fibration that is rational and every conjugate of ϕ is not an automorphism of a projective surface,*
- ϕ *is a Halphen twist, ϕ preserves a unique fibration that is elliptic and ϕ is conjugate to an automorphism of a projective surface,*
- ϕ *is a hyperbolic map.*

In the three first cases, $\lambda(\phi) = 1$, in the last one $\lambda(\phi) > 1$.

The Picard–Manin and bubble spaces. Let $\mathcal{S}$, and $\mathcal{S}_i$ be complex projective surfaces. If $\pi \colon \mathcal{S}_1 \to \mathcal{S}$ is a birational morphism, one gets $\pi^* \colon \mathrm{N}^1(\mathcal{S}) \to \mathrm{N}^1(\mathcal{S}_1)$ an embedding of Néron–Severi groups. Take two birational morphisms $\pi_1 \colon \mathcal{S}_1 \to \mathcal{S}$ and $\pi_2 \colon \mathcal{S}_2 \to \mathcal{S}$; the morphism π_2 is *above* π_1 if $\pi_1^{-1}\pi_2$ is regular. Starting with two birational morphisms one can always find a third one that covers the two first. Therefore, the inductive limit of all groups $\mathrm{N}^1(\mathcal{S}_i)$ for all surfaces $\mathcal{S}_i$ above $\mathcal{S}$ is well-defined; it is the *Picard-Manin space $\mathcal{Z}_\mathcal{S}$* of $\mathcal{S}$. For any birational map π, π^* preserves the intersection form and maps nef classes to nef classes hence the limit space $\mathcal{Z}_\mathcal{S}$ is endowed with an intersection form of signature $(1, \infty)$ and a nef cone.

Let $\mathcal{S}$ be a complex projective surface. Consider all complex and projective surfaces $\mathcal{S}_i$ above $\mathcal{S}$, that is all birational morphisms $\pi_i \colon \mathcal{S}_i \to \mathcal{S}$. If p (resp. q) is a point of a complex projective surface $\mathcal{S}_1$ (resp. $\mathcal{S}_2$), and if $\pi_1 \colon \mathcal{S}_1 \to \mathcal{S}$ (resp. $\pi_2 \colon \mathcal{S}_2 \to \mathcal{S}$) is a birational morphism, then p is identified with q if $\pi_1^{-1}\pi_2$ is a local isomorphism in a neighborhood of q that maps q onto p. The *bubble space $\mathcal{B}(\mathcal{S})$* is the union of all points of all surfaces above $\mathcal{S}$ modulo the equivalence relation induced by this identification. If p belongs to $\mathcal{B}(\mathcal{S})$ represented by a point p on a surface $\mathcal{S}_i \to \mathcal{S}$, denote by E_p the exceptional divisor of the blow-up of p and by e_p its divisor class viewed as a point in $\mathcal{Z}_\mathcal{S}$. The following properties are satisfied

$$\begin{cases} e_p \cdot e_q = 0 & \text{if } p \neq q, \\ e_p \cdot e_p = -1. \end{cases}$$

Hyperbolic space $\mathbb{H}_{\mathcal{S}}$. Embed $\mathrm{N}^1(\mathcal{S})$ as a subgroup of $\mathcal{Z}_{\mathcal{S}}$; this finite dimensional lattice is orthogonal to e_p for any $p \in \mathcal{B}(\mathcal{S})$, and

$$\mathcal{Z}_{\mathcal{S}} = \left\{ D + \sum_{p \in \mathcal{B}(\mathcal{S})} a_p e_p \Big| D \in \mathrm{N}^1(\mathcal{S}), a_p \in \mathbb{R} \right\}.$$

The *completed Picard-Manin space* $\overline{\mathcal{Z}}_{\mathcal{S}}$ of $\mathcal{S}$ is the L^2-completion of $\mathcal{Z}_{\mathcal{S}}$; in other words

$$\overline{\mathcal{Z}}_{\mathcal{S}} = \left\{ D + \sum_{p \in \mathcal{B}(\mathcal{S})} a_p e_p \Big| D \in \mathrm{N}^1(\mathcal{S}), a_p \in \mathbb{R}, \sum a_p^2 < +\infty \right\}.$$

The intersection form extends as an intersection form with signature $(1, \infty)$ on $\overline{\mathcal{Z}}_{\mathcal{S}}$. Let

$$\overline{\mathcal{Z}}_{\mathcal{S}}^{+} = \{ d \in \overline{\mathcal{Z}}_{\mathcal{S}} | d \cdot c \geq 0 \; \forall c \in \overline{\mathcal{Z}}_{\mathcal{S}} \}$$

be the nef cone of $\overline{\mathcal{Z}}_{\mathcal{S}}$ and

$$\mathcal{L}\overline{\mathcal{Z}}_{\mathcal{S}} = \{ d \in \overline{\mathcal{Z}}_{\mathcal{S}} | d \cdot d = 0 \}$$

be the light cone of $\overline{\mathcal{Z}}_{\mathcal{S}}$.

The *hyperbolic space* $\mathbb{H}_{\mathcal{S}}$ of $\mathcal{S}$ is then defined by

$$\mathbb{H}_{\mathcal{S}} = \left\{ d \in \overline{\mathcal{Z}}_{\mathcal{S}}^{+} | d \cdot d = 1 \right\}.$$

Let us remark that $\mathbb{H}_{\mathcal{S}}$ is an infinite dimensional analogue of the classical hyperbolic space $\mathbb{H}^n$. The *distance* on $\mathbb{H}_{\mathcal{S}}$ is defined by

$$\cosh\left(\mathrm{dist}\left(d, d'\right)\right) = d \cdot d' \quad \forall d, d' \in \mathbb{H}_{\mathcal{S}}.$$

The *geodesics* are intersections of $\mathbb{H}_{\mathcal{S}}$ with planes. The projection of $\mathbb{H}_{\mathcal{S}}$ onto $\mathbb{P}(\overline{\mathcal{Z}}_{\mathcal{S}})$ is one-to-one, and the boundary of its image is the projection of the cone of isotropic vectors of $\overline{\mathcal{Z}}_{\mathcal{S}}$. Hence

$$\partial \mathbb{H}_{\mathcal{S}} = \left\{ \mathbb{R}_{+} d | d \in \overline{\mathcal{Z}}_{\mathcal{S}}^{+}, d \cdot d = 0 \right\}.$$

Isometries of $\mathbb{H}_{\mathcal{S}}$. If $\pi \colon \mathcal{S}' \to \mathcal{S}$ is a birational morphism, we get a canonical isometry π^* (and not only an embedding) between $\mathbb{H}_{\mathcal{S}}$ and $\mathbb{H}_{\mathcal{S}'}$. This allows to define an action of $\mathrm{Bir}(\mathcal{S})$ on $\mathbb{H}_{\mathcal{S}}$. Consider a birational map ϕ on a complex projective surface $\mathcal{S}$. There exists a surface $\mathcal{S}'$, and $\pi_1 \colon \mathcal{S}' \to \mathcal{S}$, $\pi_2 \colon \mathcal{S}' \to \mathcal{S}$ two morphisms such that $\phi = \pi_2 \pi_1^{-1}$. One can define $\phi_{\bullet}$ by

$$\phi_{\bullet} = \left(\pi_2^*\right)^{-1} \pi_1^*;$$

in fact, one gets a faithful representation of $\mathrm{Bir}(\mathcal{S})$ into the group of isometries of $\mathbb{H}_{\mathcal{S}}$ (see [6]).

The isometries of $\mathbb{H}_{\mathcal{S}}$ are classified in three types ([4], [19]). The *translation length* of an isometry $\phi_{\bullet}$ of $\mathbb{H}_{S}$ is defined by

$$L(\phi_{\bullet}) = \inf\left\{ \mathrm{dist}\left(p, \phi_{\bullet}(p)\right) | p \in \mathbb{H}_S \right\}.$$

If the infimum is a minimum, then

— either it is equal to 0 and $\phi_\bullet$ has a fixed point in $\mathbb{H}_S$, $\phi_\bullet$ is thus *elliptic*,

— or it is positive and $\phi_\bullet$ is *hyperbolic*. Hence, the set of points $p \in \mathbb{H}_S$ such that $\text{dist}(p, \phi_\bullet(p))$ is equal to $L(\phi_\bullet)$ is a geodesic line $\text{Ax}(\phi_\bullet) \subset \mathbb{H}_S$. Its boundary points are represented by isotropic vectors $\omega(\phi_\bullet)$ and $\alpha(\phi_\bullet)$ in $\overline{\mathcal{Z}}_S$ such that

$$\phi_\bullet\big(\omega(\phi_\bullet)\big) = \lambda(\phi)\omega(\phi_\bullet), \qquad \phi_\bullet\big(\alpha(\phi_\bullet)\big) = \frac{1}{\lambda(\phi)}\alpha(\phi_\bullet).$$

The axis $\text{Ax}(\phi_\bullet)$ of $\phi_\bullet$ is the intersection of $\mathbb{H}_S$ with the plane containing $\omega(\phi_\bullet)$ and $\alpha(\phi_\bullet)$; furthermore, $\phi_\bullet$ acts as a translation of length $L(\phi_\bullet) = \log \lambda(\phi)$ along $\text{Ax}(\phi_\bullet)$ (see [8, Remark 4.5]). For all p in $\mathbb{H}_S$ one has

$$\lim_{k \to +\infty} \frac{\phi_\bullet^{-k}(p)}{\lambda(\phi)} = \alpha(\phi_\bullet), \qquad \lim_{k \to +\infty} \frac{\phi_\bullet^{k}(p)}{\lambda(\phi)} = \omega(\phi_\bullet).$$

When the infimum is not realized, $L(\phi_\bullet) = 0$ and $\phi_\bullet$ is *parabolic*: $\phi_\bullet$ fixes a unique line in $\mathcal{L}\overline{\mathcal{Z}}_S$; this line is fixed pointwise, and all orbits $\phi_\bullet^n(p)$ in $\mathbb{H}_S$ accumulate to the corresponding boundary point when n goes to $\pm\infty$.

There is a strong relationship between this classification and the classification of birational maps of the complex projective plane ([6, Theorem 3.6]): if ϕ is an element of $\text{Bir}(\mathbb{P}^2_\mathbb{C})$, then

- $\phi_\bullet$ is an elliptic isometry if and only if ϕ is an elliptic map;
- $\phi_\bullet$ is a parabolic isometry if and only if ϕ is a twist;
- $\phi_\bullet$ is a hyperbolic isometry if and only if ϕ is a hyperbolic map.

Tits alternative. Cantat proved the Tits alternative for the Cremona group ([6, Theorem C]): let G be a finitely generated subgroup of $\text{Bir}(\mathbb{P}^2_\mathbb{C})$, then

- either G contains a free non-Abelian subgroup,
- or G contains a subgroup of finite index that is solvable.

As a consequence, he studied finitely generated and solvable subgroups of $\text{Bir}(\mathbb{P}^2_\mathbb{C})$ without torsion ([6, Theorem 7.3]): let G be such a group, there exists a subgroup G_0 of G of finite index such that

- either G_0 is Abelian,
- or G_0 preserves a foliation.

3. Proof of Theorem A

3.1. Solvable groups of birational maps containing a hyperbolic map. Let us recall the following criterion (for its proof see, for example, [11]) used on many occasions by Klein, and also by Tits ([26]) known as Ping-Pong Lemma: *let* H *be a group acting on a set* X, *let* Γ_1, Γ_2 *be two subgroups of* H, *and let* Γ *be the subgroup generated by* Γ_1 *and* Γ_2. *Assume that* Γ_1 *contains at least three elements, and* Γ_2 *at least two elements. Suppose that there exist two non-empty subsets* X_1, X_2 *of* X *such that*

$$X_2 \not\subset X_1, \qquad \gamma(X_2) \subset X_1 \quad \forall \gamma \in \Gamma_1 \setminus \{\text{id}\}, \qquad \gamma'(X_1) \subset X_2 \quad \forall \gamma' \in \Gamma_2 \setminus \{\text{id}\}.$$

Then Γ *is isomorphic to the free product* $\Gamma_1 * \Gamma_2$. The Ping-Pong argument allows us to prove the following.

LEMMA 3.1. *A solvable, non-Abelian subgroup of* $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ *cannot contain two hyperbolic maps* ϕ *and* ψ *such that* $\{\omega(\phi_\bullet), \alpha(\phi_\bullet)\} \neq \{\omega(\psi_\bullet), \alpha(\psi_\bullet)\}$.

Proof. Assume by contradiction that $\{\omega(\phi_\bullet), \alpha(\phi_\bullet)\} \neq \{\omega(\psi_\bullet), \alpha(\psi_\bullet)\}$. Then the Ping-Pong argument implies that there exist two integers n and m such that ψ^n and ϕ^m generate a subgroup of G isomorphic to the free group F_2 (see [6]). But $\langle \phi, \psi \rangle$ is a solvable group: contradiction. $\square$

Let G be an infinite solvable, non-virtually Abelian, subgroup of $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$. Assume that G contains a hyperbolic map ϕ. Let $\alpha(\phi_\bullet)$ and $\omega(\phi_\bullet)$ be the two fixed points of $\phi_\bullet$ on $\partial\mathbb{H}_{\mathbb{P}^2_\mathbb{C}}$, and $\mathrm{Ax}(\phi_\bullet)$ be the geodesic passing through these two points. As G is solvable there exists a subgroup of G of index ≤ 2 that preserves $\alpha(\phi_\bullet)$, $\omega(\phi_\bullet)$, and $\mathrm{Ax}(\phi_\bullet)$ (see [6, Theorem 6.4]); let us still denote by G this subgroup. Note that there is no twist in G since a parabolic isometry has a unique fixed point on $\partial\mathbb{H}_{\mathbb{P}^2_\mathbb{C}}$. One has a morphism $\kappa\colon \mathrm{G} \to \mathbb{R}_{>0}$ such that

$$\psi_\bullet(\ell) = \kappa(\psi)\ell$$

for any ℓ in $\overline{\mathcal{Z}}_{\mathbb{P}^2_\mathbb{C}}$ lying on $\mathrm{Ax}(\phi_\bullet)$.

The kernel of κ is an infinite subgroup that contains only elliptic maps. Indeed the set of elliptic elements of G coincides with $\ker\kappa$; and $[\mathrm{G}, \mathrm{G}] \subset \ker\kappa$ so if $\ker\kappa$ is finite, G is Abelian up to finite index which is by assumption impossible.

Gap property. If ψ is an hyperbolic birational map of G, then $\kappa(\psi) = L(\psi_\bullet) = \log\lambda(\psi)$. Recall that $\lambda(\phi)$ is an algebraic integer with all Galois conjugates in the unit disk, that is a Salem number, or a Pisot number. The smallest known number is the Lehmer number $\lambda_L \simeq 1.176$ which is a root of $X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1$. Blanc and Cantat prove in [2, Corollary 2.7] that there is a gap in the dynamical spectrum $\Lambda = \{\lambda(\phi) | \phi \in \mathrm{Bir}(\mathbb{P}^2_\mathbb{C})\}$: there is no dynamical degree in $]1, \lambda_L[$.

The gap property implies that in fact κ is a morphism from G to a subgroup of $\mathbb{R}_{>0}$ isomorphic to $\mathbb{Z}$.

Elliptic subgroups of the Cremona group with a large normalizer. Consider in $\mathbb{P}^2_\mathbb{C}$ the complement of the union of the three lines $\{x = 0\}$, $\{y = 0\}$ and $\{z = 0\}$. Denote by $\mathcal{U}$ this open set isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$. One has an action of $\mathbb{C}^* \times \mathbb{C}^*$ on $\mathcal{U}$ by translation. Furthermore $\mathrm{GL}(2, \mathbb{Z})$ acts on $\mathcal{U}$ by monomial maps

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} \mapsto \left((x, y) \mapsto \left(x^p y^q, x^r y^s\right)\right).$$

One thus has an injective morphism from $(\mathbb{C}^* \times \mathbb{C}^*) \rtimes \mathrm{GL}(2, \mathbb{Z})$ into $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$. Let $\mathrm{G}_{\mathrm{toric}}$ be its image.

One can now apply [12, Theorem 4] that says that if there exists a short exact sequence

$$1 \longrightarrow A \longrightarrow N \longrightarrow B \longrightarrow 1,$$

where $N \subset \mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ contains at least one hyperbolic element, and $A \subset \mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ is an infinite and elliptic[1] group, then N is up to conjugacy a subgroup of G_{toric}. Hence, up to birational conjugacy $G \subset G_{\mathrm{toric}}$. Recall now that if ψ is a hyperbolic map of the form $(x^a y^b, x^c y^d)$, then to preserve $\alpha(\psi_\bullet)$ and $\omega(\psi_\bullet)$ is equivalent to preserve the eigenvectors of the matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. We can now thus state:

PROPOSITION 3.2. *Let* G *be an infinite solvable, non-virtually Abelian, subgroup of* $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$. *If* G *contains a hyperbolic birational map, then* G *is, up to conjugacy and finite index, a subgroup of the group generated by*

$$\left\{ \left(x^p y^q, x^r y^s\right), (\alpha x, \beta y) | \alpha, \beta \in \mathbb{C}^* \right\},$$

where $\left[\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right]$ *denotes an element of* $\mathrm{GL}(2, \mathbb{Z})$ *with spectral radius* > 1.

3.2. Solvable groups with a twist. Consider a solvable, non-Abelian subgroup G of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$. Let us assume that G contains a twist ϕ; the map ϕ preserves a unique fibration $\mathcal{F}$ that is rational or elliptic. Let us prove that any element of G preserves $\mathcal{F}$. Denote by $\alpha(\phi_\bullet) \in \partial \mathbb{H}_{\mathbb{P}^2}$ the fixed point of $\phi_\bullet$. Take one element in $\mathcal{L}\overline{\mathcal{Z}}_{\mathbb{P}^2_{\mathbb{C}}}$ still denoted $\alpha(\phi_\bullet)$ that represents $\alpha(\phi_\bullet)$. Take $\varphi \in G$ such that $\varphi(\alpha(\phi_\bullet)) \neq \alpha(\phi_\bullet)$. Then $\psi = \varphi \phi \varphi^{-1}$ is parabolic and fixes the unique element $\alpha(\psi_\bullet)$ of $\mathcal{L}\overline{\mathcal{Z}}_{\mathbb{P}^2_{\mathbb{C}}}$ proportional to $\varphi(\alpha(\phi_\bullet))$. Take $\varepsilon > 0$ such that $\mathcal{U}(\alpha(\phi_\bullet), \varepsilon) \cap \mathcal{U}(\alpha(\psi_\bullet), \varepsilon) = \emptyset$ where

$$\mathcal{U}(\alpha, \varepsilon) = \{\ell \in \mathcal{L}\overline{\mathcal{Z}}_{\mathbb{P}^2_{\mathbb{C}}} | \alpha \cdot \ell < \varepsilon\}.$$

Since $\psi_\bullet$ is parabolic, then for n large enough $\psi_\bullet^n(\mathcal{U}(\alpha(\phi_\bullet), \varepsilon))$ is included in a $\mathcal{U}(\alpha(\psi_\bullet), \varepsilon)$. For m sufficiently large $\phi_\bullet^m \psi_\bullet^n(\mathcal{U}(\alpha(\phi_\bullet), \varepsilon)) \subset (\mathcal{U}(\alpha(\phi_\bullet), \varepsilon/2)) \subsetneq (\mathcal{U}(\alpha(\phi_\bullet), \varepsilon))$; hence $\phi_\bullet^m \psi_\bullet^n$ is hyperbolic. You can by this way build two hyperbolic maps whose sets of fixed points are distinct: this gives a contradiction with Lemma 3.1. So for any $\varphi \in G$ one has: $\alpha(\phi_\bullet) = \alpha(\varphi_\bullet)$; one can thus state the following result.

PROPOSITION 3.3. *Let* G *be a solvable, non-Abelian subgroup of* $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ *that contains a twist* ϕ. *Then*

- *if* ϕ *is a Jonquières twist, then* G *preserves a rational fibration, that is up to birational conjugacy* G *is a subgroup of* $\mathrm{PGL}(2, \mathbb{C}(y)) \rtimes \mathrm{PGL}(2, \mathbb{C})$,
- *if* ϕ *is a Halphen twist, then* G *preserves an elliptic fibration.*

REMARK 3.4. Both cases are mutually exclusive.

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[1] A subgroup of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ is elliptic if it fixes a point in $\mathbb{H}_{\mathbb{P}^2_{\mathbb{C}}}$.

Note that if G is a subgroup of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ that preserves an elliptic fibration, then G is countable ([5]). Let us explain briefly why. A smooth rational projective surface $\mathcal{S}$ is a *Halphen surface* if there exists an integer $m > 0$ such that the linear system $|-mK_{\mathcal{S}}|$ is of dimension 1, has no fixed component, and has no base point. The smallest positive integer for which $\mathcal{S}$ satisfies such a property is the *index* of $\mathcal{S}$. If $\mathcal{S}$ is a Halphen surface of index m, then $K_{\mathcal{S}}^2 = 0$ and, by the genus formula, the linear system $|-mK_{\mathcal{S}}|$ defines a genus 1 fibration $\mathcal{S} \to \mathbb{P}^1_{\mathbb{C}}$. This fibration is *relatively minimal* in the sense that there is no (-1)-curve contained in a fiber. The following properties are equivalent:

- $\mathcal{S}$ is a Halphen surface of index m,
- there exists an irreducible pencil of curves of degree $3m$ with 9 base points of multiplicity m in $\mathbb{P}^2_{\mathbb{C}}$ such that $\mathcal{S}$ is the blow-up of the 9 base points and $|-mK_{\mathcal{S}}|$ is the proper transform of this pencil (the base points set may contain infinitely near points).

As a corollary of the classification of relatively minimal elliptic surfaces the relative minimal model of a rational elliptic surface is a Halphen surface of index m ([22, Chapter 2, Section 10]). Up to conjugacy G is a subgroup of $\mathrm{Aut}(\mathcal{S})$ where $\mathcal{S}$ denotes a Halphen surface of index m. The action of G on $\mathrm{NS}(\mathcal{S})$ is almost faithful, and G is a discrete (it preserves the integral structure of $\mathrm{NS}(\mathcal{S})$) and virtually Abelian (it preserves the intersection form and the class of the elliptic fibration) subgroup of $\mathrm{Aut}(\mathcal{S})$. So one has the following.

COROLLARY 3.5. *If* G *is an uncountable, solvable, non-Abelian subgroup of* $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$, *then* G *doesn't contain a Halphen twist.*

EXAMPLE 3.6. Let us come back to the example given in Section 1. If $\phi \in \mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ preserves a unique fibration that is rational then one can assume that up to birational conjugacy this fibration is given, in the affine chart $z = 1$, by $y = \mathrm{cst}$. If ϕ preserves $y = \mathrm{cst}$ fiberwise, then

- ϕ is contained in a maximal Abelian subgroup denoted $\mathrm{Ab}(\phi)$ that preserves $y = \mathrm{cst}$ fiberwise ([13]),
- the centralizer of ϕ is a finite extension of $\mathrm{Ab}(\phi)$ (see [9, Theorem B]).

This allows us to establish that if ϕ preserves a fibration not fiberwise, then the centralizer of ϕ is virtually solvable. For instance, if $\phi = (x + a(y), y + 1)$ (resp. $(b(y)x, \beta y)$ or $(x + a(y), \beta y)$ with $\beta \in \mathbb{C}^*$ of infinite order) preserves a unique fibration, then the centralizer of ϕ is solvable and metabelian ([9, Propositions 5.1 and 5.2]).

3.3. Solvable groups with no hyperbolic map, and no twist. Let M be a smooth, irreducible, complex, projective variety of dimension n. Fix a Kähler form κ on M. If ℓ is a positive integer, denote by $\pi_i \colon M^\ell \to M$ the projection onto the ith factor. The manifold M^ℓ is then endowed with the Kähler form $\sum_{i=1}^\ell \pi_i^* \kappa$ which induces a Kähler metric. To any $\phi \in \mathrm{Bir}(M)$

one can associate its graph $\Gamma_\phi \subset M \times M$ defined as the Zariski closure of

$$\big\{\big(z, \phi(z)\big) \in M \times M \,|\, z \in M \setminus \mathrm{Ind}\,\phi\big\}.$$

By construction Γ_ϕ is an irreducible subvariety of $M \times M$ of dimension n. Both projections $\pi_1, \pi_2 \colon M \times M \to M$ restrict to a birational morphism $\pi_1, \pi_2 \colon \Gamma_\phi \to M$.

The *total degree* $\mathrm{tdeg}\,\phi$ of $\phi \in \mathrm{Bir}(M)$ is defined as the volume of Γ_ϕ with respect to the fixed metric on $M \times M$:

$$\mathrm{tdeg}\,\phi = \int_{\Gamma_\phi} \big(\pi_1^*\kappa + \pi_2^*\kappa\big)^n = \int_{M \setminus \mathrm{Ind}\,\phi} \big(\kappa + \phi^*\kappa\big)^n.$$

Let $d \geq 1$ be a natural integer, and set

$$\mathrm{Bir}_d(M) = \big\{\phi \in \mathrm{Bir}(M) \,|\, \mathrm{tdeg}\,\phi \leq d\big\}.$$

A subgroup G of $\mathrm{Bir}(M)$ has *bounded degree* if it is contained in $\mathrm{Bir}_d(M)$ for some $d \in \mathbb{N}^*$.

Any subgroup G of $\mathrm{Bir}(M)$ that has bounded degree can be regularized, that is up to birational conjugacy all indeterminacy points of all elements of G disappear simultaneously.

THEOREM 3.7 ([27]). *Let M be a complex projective variety, and let G be a subgroup of* $\mathrm{Bir}(M)$. *If G has bounded degree, there exists a smooth, complex, projective variety M', and a birational map $\psi \colon M' \dashrightarrow M$ such that $\psi^{-1}\mathrm{G}\psi$ is a subgroup of* $\mathrm{Aut}(M')$.

The proof of this result can be found in [21], [29]; an heuristic idea appears in [7].

4. Applications

4.1. Nilpotent subgroups of $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$. Let us recall that if G is a group, the *descending central series* of G is defined by

$$C^0\mathrm{G} = \mathrm{G}, \qquad C^{n+1}\mathrm{G} = \big[\mathrm{G}, C^n\mathrm{G}\big] \quad \forall n \geq 0.$$

We say that G is *nilpotent* if there exists $j \geq 0$ such that $C^j\mathrm{G} = \{\mathrm{id}\}$. If j is the minimum non-negative number with such a property, we say that G is of *nilpotent class j*. Nilpotent subgroups of the Cremona group have been described:

THEOREM 4.1 ([14]). *Let G be a nilpotent subgroup of* $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$. *Then*
- *either G is up to finite index metabelian,*
- *or G is a torsion group.*

We find an alternative proof of [14, Lemma 4.2] for G infinite:

LEMMA 4.2. *Let G be an infinite, nilpotent, non-virtually Abelian subgroup of* $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$. *Then G does not contain a hyperbolic map.*

Proof. The group G is also solvable. Assume by contradiction that G contains a hyperbolic map; then according to Theorem A up to birational conjugacy and finite index there exists $\Upsilon \subset \mathbb{C}^* \times \mathbb{C}^*$ infinite such that G is generated by $\phi = (x^p y^q, x^r y^s)$ and

$$\big\{(\alpha x, \beta y)|(\alpha, \beta) \in \Upsilon\big\}.$$

The group $C^1 G$ contains

$$\big\{[\phi, (\alpha x, \beta y)]|(\alpha, \beta) \in \Upsilon\big\} = \big\{(\alpha^{p-1}\beta^q x, \alpha^r \beta^{s-1} y)|(\alpha, \beta) \in \Upsilon\big\}$$

that is infinite since Υ is infinite. Suppose that $C^i G$ contains the infinite set

$$\big\{(\alpha^{\ell_i}\beta^{n_i} x, \alpha^{k_i}\beta^{m_i} y)|(\alpha, \beta) \in \Upsilon\big\}$$

(ℓ_i, n_i, k_i and m_i are some functions in p, q, r and s); then $C^{i+1} G$ contains

$$\big\{(\alpha^{(p-1)\ell_i + qm_i}\beta^{(p-1)k_i + qn_i} x, \alpha^{r\ell_i + qm_i}\beta^{rk_i + (s-1)n_i} y)|(\alpha, \beta) \in \Upsilon\big\}$$

that is still infinite. $\square$

So any nilpotent and infinite subgroup of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ falls in case (1), (2), (3) of Theorem A. If it falls in case (2) or (3) then G is virtually metabelian ([14, Proof of Theorem 1.1]). Finally if G falls in case (1), we can prove as in [14] that either G is a torsion group, or G is virtually metabelian.

4.2. Soluble length of a nilpotent subgroup of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$. Let us recall the following statement due to Epstein and Thurston ([17]): let M be a connected complex manifold. Let $\mathfrak{h}$ be a nilpotent Lie subalgebra of the complex vector space of rational vector fields on M. Then $\mathfrak{h}^{(n)} = \{0\}$ if $n \geq \dim M$; hence, the solvable length of $\mathfrak{h}$ is bounded by the dimension of M. We have a similar statement in the context of birational maps; indeed a direct consequence of Theorem 4.1 is the following property: let $G \subset \mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ be a nilpotent subgroup of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ that is not a torsion group, then the soluble length of G is bounded by the dimension of $\mathbb{P}^2_{\mathbb{C}}$.

4.3. Favre's question. In [18], Favre asked few questions; among them there is the following: does any solvable, finitely generated subgroup G of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ contain a subgroup H of finite index such that $[H, H]$ is nilpotent? We will prove that the answer is no if $[G, G]$ is not a torsion group.

Take G a solvable and finitely generated subgroup of the Cremona group; besides suppose that $[G, G]$ is not a torsion group. Assume that the answer of Favre's question is yes. Up to finite index one can assume that $[G, G]$ is nilpotent. According to Theorem 4.1 the group $G^{(1)} = [G, G]$ is up to finite index metabelian; in other words up to finite index $G^{(2)} = [G^{(1)}, G^{(1)}]$ is Abelian and so $G^{(3)} = [G^{(2)}, G^{(2)}] = \{\mathrm{id}\}$, *that is*, the soluble length of G is bounded by 3 up to finite index. Consider the subgroup

$$\left\langle \big(x + y^2, y\big), \big(x(1 + y), y\big), \left(x, \frac{y}{1+y}\right), (x, 2y) \right\rangle$$

of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$. It is solvable of length 4 (see [24]): contradiction.

4.4. Baumslag–Solitar groups. For any integers m, n such that $mn \neq 0$, the Baumslag–Solitar group $\mathrm{BS}(m;n)$ is defined by the following presentation

$$\mathrm{BS}(m;n) = \langle r,s | rs^m r^{-1} = s^n \rangle.$$

In [3], we prove that there is no embedding of $\mathrm{BS}(m;n)$ into $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ as soon as $|n|$, $|m|$, and 1 are distinct; it corresponds exactly to the case $\mathrm{BS}(m;n)$ is not solvable. Indeed $\mathrm{BS}(m;n)$ is solvable if and only if $|m| = 1$ or $|n| = 1$ (see [25, Proposition A.6]).

PROPOSITION 4.3. *Let ρ be an embedding of $\mathrm{BS}(1;n) = \langle r,s | rsr^{-1} = s^n \rangle$, with $n \neq 1$, into the Cremona group. Then*

- *the image of ρ doesn't contain a hyperbolic map,*
- *and*

$$\rho(s) = (x, y+1), \qquad \rho(r) = \big(\nu(x), n\big(y + a(x)\big)\big)$$

with $\nu \in \mathrm{PGL}(2,\mathbb{C})$ and $a \in \mathbb{C}(x)$.

Proof. According to [3, Proposition 6.2, Lemma 6.3] one gets that $\rho(s) = (x, y+1)$ and $\rho(r) = (\nu(x), n(y + a(x)))$ for some $\nu \in \mathrm{PGL}(2,\mathbb{C})$ and $a \in \mathbb{C}(x)$.

Furthermore, $\rho(s)$ can neither be conjugate to an automorphism of the form $(\alpha x, \beta y)$ (see [1]), nor to a hyperbolic birational map of the form $(\gamma x^p y^q, \delta x^r y^s)$ with $\left[\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right] \in \mathrm{GL}(2,\mathbb{Z})$ of spectral radius > 1. As a consequence, Proposition 3.2 implies that $\rho(\mathrm{BS}(1;n))$ does not contain a hyperbolic birational map. $\square$

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