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# ANALYTIC DISCS, GLOBAL EXTREMAL FUNCTIONS AND PROJECTIVE HULLS IN PROJECTIVE SPACE

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ABSTRACT. Using a recent result of Lárusson and Poletsky regarding plurisubharmonic subextensions, we prove a disc formula for the quasiplurisubharmonic global extremal function for domains in $\mathbb{P}^n$. As a corollary, we get a characterization of the projective hull for connected compact sets in $\mathbb{P}^n$ by the existence of analytic discs.

## 1. Introduction

The global extremal function, also called the Siciak–Zahariuta extremal function, has proven very useful in pluripotential theory in $\mathbb{C}^n$, see [6, Section 13] and [7, Section 5] for an overview of the applications. We are however most interested in its counterpart in the theory of quasiplurisubharmonic functions on $\mathbb{P}^n$. The quasiplurisubharmonic global extremal function was defined by Guedj and Zeriahi [3] and has already proven useful, most notably in connection with projective hulls [4]. The projective hull of a compact set in $\mathbb{P}^n$ is the natural generalization of the polynomial hull of a set in $\mathbb{C}^n$.

We start by making a small generalization of a recent result of Lárusson and Poletsky [8] regarding plurisubharmonic subextensions for domains in $\mathbb{C}^n$ (Theorem 2.3). We also define a disc structure (Lemma 2.6) for sets in $\mathbb{C}^{n+1} \setminus \{0\}$ with the properties required by Theorem 2.3.

In Section 3, we turn our attention to quasiplurisubharmonic functions, or $\omega$-plurisubharmonic functions, on $\mathbb{P}^n$ which we denote by $\mathcal{PSH}(\mathbb{P}^n, \omega)$. Here, the current $\omega$ is the Fubini–Study Kähler form. Using the results from Section 2, we prove a disc formula for the global extremal function for a

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domain $W \subset \mathbb{P}^n$ (Theorem 3.2),

$$\sup\{u(x); u \in \mathcal{PSH}(\mathbb{P}^n, \omega), u|_W \leq \varphi\}$$
$$= \inf\left\{-\int_{\mathbb{D}} \log|\cdot| f^*\omega + \int_{\mathbb{T}} \varphi \circ f\, d\sigma; f \in \mathcal{A}^W_{\mathbb{P}^n}, f(0) = x\right\}.$$

In Section 4, we apply this result to get a characterization of the projective hull $\hat{K}$ of connected compact sets $K \subset \mathbb{P}^n$ by existence of analytic discs with specific properties. More specifically, for $\Lambda > 0$ and a connected compact subset $K \subset \mathbb{P}^n$ the following is equivalent for a point $x \in \mathbb{P}^n$:

(A) $x \in \hat{K}(\Lambda)$.

(B) For every $\varepsilon > 0$ and every neighborhood $U$ of $K$ there exists a disc $f \in \mathcal{A}^U_{\mathbb{P}^n}$ such that $f(0) = x$ and

$$-\int_{\mathbb{D}} \log|\cdot| f^*\omega < \Lambda + \varepsilon.$$

Here, $\hat{K}(\Lambda)$, $\Lambda > 0$, are specific subsets of $\hat{K}$ such that $\hat{K} = \bigcup_\Lambda \hat{K}(\Lambda)$. These sets are defined using the *best constant function* for $K$ (see [4, Section 4]).

Finally, in Section 5, we see how the methods presented in Sections 2 and 3 also work for other currents, in particular for the current of integration for the hyperplane at infinity $H_\infty \subset \mathbb{P}^n$. This gives rise to the disc formula for the Siciak–Zahariuta extremal function which was first proved in [9] and [12].

We start by establishing the notation. We assume $X$ is a complex manifold. Here $X$ will be a subset of either affine space or projective space. Let $\mathcal{A}_X$ denote the space of closed analytic discs in $X$, that is continuous maps $f : \overline{\mathbb{D}} \to X$ which are holomorphic on the unit disc $\mathbb{D}$, and endow $\mathcal{A}_X$ with the compact-open topology. Assume $W \subset X$, then $\mathcal{A}^W_X$ is the subset of discs in $\mathcal{A}_X$ which map the unit circle $\mathbb{T}$ into $W$.

If $H$ is a *disc functional*, that is, a function from a subset of $\mathcal{A}_X$ to $[-\infty, +\infty]$, then its *envelope* with respect to the family $\mathcal{C} \subset \mathcal{A}_X$ is defined as the function

$$E_{\mathcal{C}} H(x) = \inf\{H(f); f \in \mathcal{C}, f(0) = x\}.$$

The domain of $E_{\mathcal{C}} H$ is all the points $x \in X$ such that $\{f \in \mathcal{C}; f(0) = x\}$ is non-empty.

For convenience, we write $E$ for $E_{\mathcal{A}_X}$ and $E_W$ for $E_{\mathcal{A}^W_X}$.

The standard example of a disc functional is the *Poisson disc functional* $H_\varphi : \mathcal{A}_X \to [-\infty, +\infty]$ associated with a function $\varphi : X \to \mathbb{R} \cup \{-\infty\}$. It is defined by $H_\varphi(f) = \int_{\mathbb{T}} \varphi \circ f\, d\sigma$, where the measure $\sigma$ is the arclength measure on $\mathbb{T}$ normalized to one. Another disc functional which we will use is the Poisson disc functional for the class of $\omega$-plurisubharmonic functions

$$H_{\omega,\varphi}(f) = -\int_{\mathbb{D}} \log|\cdot| f^*\omega + \int_{\mathbb{T}} \varphi \circ f\, d\sigma,$$

where $f^*\omega$ is the pullback of $\omega$ by $f$ (see Section 3). In our case, the $(1,1)$-current $\omega$ will be the Fubini–Study Kähler form on $\mathbb{P}^n$. However, in Section 5, we look briefly at other currents on $\mathbb{P}^n$, especially the case when $\omega$ is the current of integration for the hyperplane at infinity.

If $\varphi$ is a function defined on $W \subset X$, then we let

$$\mathcal{F}_\varphi = \{u \in \mathcal{PSH}(X); u|_W \leq \varphi\},$$
$$\mathcal{F}_{\omega,\varphi} = \{u \in \mathcal{PSH}(X,\omega); u|_W \leq \varphi\}.$$

Remark. Although it is more traditional to look at analytic discs which are holomorphic in a neighborhood of the closed unit disc we are only assuming the discs are continuous to the boundary. This does in fact not alter the results obtained here since every disc holomorphic in a neighborhood of $\overline{\mathbb{D}}$ is clearly in $\mathcal{A}_X$, and conversely if $f \in \mathcal{A}_X$ then $f(r\cdot)$, $r < 1$ is a family of discs holomorphic in a neighborhood of $\mathbb{D}$ such that $H_{\omega,\varphi}(f(r\cdot)) \to H_{\omega,\varphi}(f)$, when $r \to 1^-$. The reason for the approach here is that the authors of [8] applied a result of Forstnerič [2] which uses discs which are only continuous up to the boundary.

## 2. Plurisubharmonic subextensions

We start by looking at domains in $\mathbb{C}^n$ and prove a small generalization of a theorem of Lárusson and Poletsky.

The settings are the following. For domains $W$ and $X$ in $\mathbb{C}^n$, $W \subset X$, and an upper semicontinuous function $\varphi : W \to \mathbb{R} \cup \{-\infty\}$, consider the function

$$\sup \mathcal{F}_\varphi(x) = \sup\{u(x); u \in \mathcal{F}_\varphi\}.$$

If $\mathcal{F}_\varphi$ is locally bounded above, then this is the largest plurisubharmonic function on $X$ dominated by $\varphi$ on $W$. The goal is to find sufficient conditions on $W$ and $X$ so that we can prove a disc formula for this function, namely that $\sup F_\varphi = E_W H_\varphi$, or if we write it out

$$\sup\{u(x); u \in \mathcal{PSH}(X), u|_W \leq \varphi\} = \inf\left\{\int_{\mathbb{T}} \varphi \circ f \, d\sigma; f \in \mathcal{A}_X^W, f(0) = x\right\}.$$

We will need the following definitions.

Definition 2.1. We say that two discs $f_0$ and $f_1$ in $\mathcal{A}_X^W$ with $f_0(0) = f_1(0)$ are *centre-homotopic* if there is a continuous map $f : \overline{\mathbb{D}} \times [0,1] \to X$ such that

- $f(\cdot,t) \in \mathcal{A}_X^W$ for all $t \in [0,1]$,
- $f(\cdot,0) = f_0$ and $f(\cdot,1) = f_1$,
- $f(0,t) = f_0(0) = f_1(0)$ for all $t \in [0,1]$.

Definition 2.2. If $W \subset X$, then a $W$*-disc structure* on $X$ is a family $\beta = (\beta_\nu)_\nu$ of continuous maps $\beta_\nu : U_\nu \to \mathcal{A}_X^W$, where $(U_\nu)_\nu$ is an open covering of $X$, such that

- $\beta_\nu(x)(0) = x$ for all $x \in U_\nu$ (i.e. $x$ is mapped to a disc centred at $x$),

- if $x \in U_\nu \cap U_\mu$, then $\beta_\nu(x)$ and $\beta_\mu(x)$ are centre-homotopic.

Furthermore, if there is a $\mu$ such that $U_\mu = W$ and $\beta_\mu(w)(\cdot) = w$ for every $w \in W$ (i.e., $\beta_\mu(w)$ is the constant disc), then we say that the disc structure is *schlicht*.

For a $W$-disc structure $\beta$ we let $\mathcal{B} \subset \mathcal{A}_X^W$ denote the family of discs in $\beta$, $\mathcal{B} = \bigcup_\nu \beta_\nu(U_\nu)$.

Lárusson and Poletsky showed in [8, Theorem 3] that if $W$ has a schlicht disc structure then $\sup \mathcal{F}_\varphi = E_W H_\varphi$. By looking closely at their proof, one sees that it is sufficient to have a disc structure $\beta$ such that $EH_{E_\mathcal{B} H_\varphi} \le \varphi$ on $W$. From this small observation, we get the following theorem.

THEOREM 2.3. *Let $W \subset X$ be domains in $\mathbb{C}^n$, let $\varphi : W \to \mathbb{R} \cup \{-\infty\}$ be an upper semicontinuous function, and assume $\beta$ is a $W$-disc structure on $X$ such that $E_\mathcal{B} H_\varphi \le \varphi$ on $W$. Then*

$$\sup \mathcal{F}_\varphi = E_W H_\varphi.$$

*Proof.* The formula follows from the inequalities

$$\sup \mathcal{F}_\varphi \le E_W H_\varphi \le EH_{E_\mathcal{B} H_\varphi} \le \sup \mathcal{F}_\varphi.$$

The first inequality follows from the subaverage property of the subharmonic function $u \circ f$. If $u \in \mathcal{F}_\varphi$ and $f \in \mathcal{A}_X^W$, $f(0) = x$, then

$$u(x) = (u \circ f)(0) \le \int_\mathbb{T} u \circ f \, d\sigma \le \int_\mathbb{T} \varphi \circ f \, d\sigma = H_\varphi(f).$$

Taking supremum on the left-hand side over $u \in \mathcal{F}_\varphi$ and infimum on the right-hand side over $f \in \mathcal{A}_X^W$ gives the inequality.

Lemma 2 in [8] gives the second inequality.

A fundamental property of the envelops of the Poisson disc functional is that $EH_\psi \le \psi$ and therefore, with $\psi = E_\mathcal{B} H_\varphi$,

$$EH_{E_\mathcal{B} H_\varphi} \le E_\mathcal{B} H_\varphi \le \varphi. \tag{1}$$

The last inequality then follows from the fact that the function $EH_{E_\mathcal{B} H_\varphi}$ is plurisubharmonic by Poletsky's theorem [13], [14] and not greater than $\varphi$ by (1). It is therefore in the class $\mathcal{F}_\varphi$ we are taking supremum over. □

The difference between having a schlicht disc structure and a disc structure as in Theorem 2.3 is that we do not necessarily have to have the constant discs on $W$, it is for example sufficient to have discs which map the unit circle "close" to the center of the disc. This can be seen in detail from the construction in Lemma 2.6 where we define a disc structure on a certain class of sets in $X = \mathbb{C}^m \setminus \{0\}$ which satisfy the condition in Theorem 2.3. In Sections 3 and 4, we let $m = n + 1$ and apply the result of Theorem 2.3 in $\mathbb{P}^n$ using homogeneous coordinates $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$.

DEFINITION 2.4. A set $W \subset \mathbb{C}^m$ is a *complex cone* if $\lambda x \in W$ for every $\lambda \in \mathbb{C}$ and $x \in W$. A complex cone in $\mathbb{C}^m \setminus \{0\}$ is simply a complex cone in $\mathbb{C}^m$ with 0 removed.

DEFINITION 2.5. Assume $W$ is a complex cone. A function $\varphi : W \to \mathbb{R} \cup \{-\infty\}$ is called *logarithmically homogeneous* if

$$\varphi(\lambda x) = \varphi(x) + \log|\lambda|,$$

for every $\lambda \in \mathbb{C} \setminus \{0\}$ and $x \in W$.

Note that every function on a complex cone in $\mathbb{C}^m \setminus \{0\}$ which is logarithmically homogeneous extends automatically over 0 and takes the value $-\infty$ there.

LEMMA 2.6. *Assume $W \subset \mathbb{C}^m \setminus \{0\}$, $m \geq 2$ is a complex cone and a domain, and assume that $\varphi : W \to \mathbb{R} \cup \{-\infty\}$ is an upper semicontinuous function which is logarithmically homogeneous. Then there exists a $W$-disc structure $\mathcal{B}$ in $\mathbb{C}^m \setminus \{0\}$ such that $E_{\mathcal{B}} H_\varphi \leq \varphi$.*

*Proof.* For each $w \in W$ let $U_w = \mathbb{C}^m \setminus \{\lambda w; \lambda \in \mathbb{C}\}$. Then for $x \in U_w$ define the analytic discs $\beta_w(x)$ by

$$f_{x,w}(t) = \beta_w(x)(t) = \left(\frac{\|x-w\|}{r} - \frac{r}{\|x-w\|}\right) tw + \left(1 + \frac{r}{\|x-w\|} t\right) x,$$

where

$$r = \min\left\{\frac{\|x-w\|}{1+\|x-w\|}, \frac{d(w, W^c)}{2}\right\}$$

and $W^c$ is the complement of $W$ in $\mathbb{C}^m \setminus \{0\}$.

It is more convenient to write the formula for these discs in the following way

$$f_{x,w}(t) = \begin{cases} (1 + \frac{\|x-w\|}{r} t) \underbrace{[w + (\frac{\|x-w\|+rt}{r+\|x-w\|t}) \frac{r}{\|x-w\|}(x-w)]}_{(\star)} & \\ \qquad\qquad\qquad\qquad\qquad \text{if } t \neq -\frac{r}{\|x-w\|}, & \\ (1 - \frac{r^2}{\|x-w\|^2})(x-w) \quad \text{if } t = -\frac{r}{\|x-w\|}. & \end{cases}$$

Then we see that 0 is mapped to $x$. Furthermore, the factor in the brackets, $(\star)$, maps the closed unit disc into the complex line through $x$ and $w$ and maps the unit circle into a circle with centre $w$ and radius $r$. This can be seen from the fact that if $t \in \mathbb{T}$ then $|\frac{\|x-w\|+rt}{r+\|x-w\|t}| = 1$ and

$$\big\|w - (\star)\big\| = r < d\big(w, W^c\big).$$

This implies that for $t \in \mathbb{T}$ the point $f_{x,w}(t) = (1 + \frac{\|x-w\|}{r} t)(\star)$ is also in $W$ since $W$ is a complex cone.

Note also that 0 is not in the image of $f_{x,w}$ since, by the definition of $U_w$, the complex line through $x$ and $w$ does not include 0.

To show that $(\beta_\nu)_\nu$ is a $W$-disc structure in $\mathbb{C}^m \setminus \{0\}$ we need to show that every two discs with the same centre are centre-homotopic, that is $f_{x,w}$ and $f_{x,w'}$ are centre-homotopic for every $w, w' \in W$. Since $W$ is connected the set $W \setminus \{\lambda x; \lambda \in \mathbb{C}\}$ is also connected and path connected. Therefore, there is a path $\gamma : [0,1] \to W \setminus \{\lambda x; \lambda \in \mathbb{C}\}$ such that $\gamma(0) = w$ and $\gamma(1) = w'$. Define the map $f : \overline{\mathbb{D}} \times [0,1] \to \mathbb{C}^m \setminus \{0\}$ by

$$f(t,s) = f_{x,\gamma(s)}(t).$$

The function $f$ clearly satisfies all the conditions in Definition 2.1, which means that we have defined a $W$-disc structure $\mathcal{B} = \bigcup_{w \in W} \{f_{x,w}; x \in U_w\}$.

We will now show that $E_{\mathcal{B}} H_\varphi \leq \varphi$ on $W$. Fix $x \in W$ and $\varepsilon > 0$. Since $\varphi$ is upper semicontinuous there is an open neighborhood $U$ of $x$ such that $\varphi|_U \leq \varphi(x) + \varepsilon/2$. Then select $w$ close enough to $x$ so that

- $\frac{1}{2\pi} \log(1 + \|x - w\|) < \varepsilon/2$,
- $r = \min\{\frac{\|x-w\|}{1+\|x-w\|}, \frac{d(w,W^c)}{2}\}$ is equal to $\frac{\|x-w\|}{1+\|x-w\|}$,
- the disc on the complex line through $x$ and $w$ with centre $w$ and radius $r$ (defined as above) is in $U$.

Then, by using the properties of $\varphi$, the properties of the term $(\star)$ and the Riesz representation formula [5, Theorem 3.3.6] we see that

$$\begin{aligned}
E_{\mathcal{B}} H_\varphi(x) &\leq \int_{\mathbb{T}} \varphi \circ f_{x,w} \, d\sigma \\
&= \int_{\mathbb{T}} \varphi\Big(\Big(1 + \frac{\|x-w\|}{r} t\Big)(\star)\Big) \, d\sigma \\
&= \int_{\mathbb{T}} \varphi\big((\star)\big) \, d\sigma + \int_{\mathbb{T}} \log\Big|1 + \frac{\|x-w\|}{r} t\Big| \, d\sigma \\
&\leq \sup_U \varphi - \frac{1}{2\pi} \log\Big| - \frac{r}{\|x-w\|}\Big| \\
&\leq \varphi(x) + \frac{\varepsilon}{2} + \frac{1}{2\pi} \log\big(1 + \|x-w\|\big) \\
&\leq \varphi(x) + \varepsilon.
\end{aligned}$$

This holds for every $\varepsilon > 0$, hence $E_{\mathcal{B}} H_\varphi \leq \varphi$. □

It should be noted that the disc structure above is under heavy influence from the set of “good discs” used in [9] and [12] for the original proof of Equation (7).

## 3. Disc formula for the global relative extremal function in $(\mathbb{P}^n, \omega)$

We let $\omega$ be the Fubini–Study Kähler form for $\mathbb{P}^n$. Recall that an upper semicontinuous function $u$ on $\mathbb{P}^n$ is called $\omega$-plurisubharmonic (or quasiplurisubharmonic) if $dd^c u + \omega \geq 0$. We denote the family of $\omega$-plurisubharmonic function on $\mathbb{P}^n$ by $\mathcal{PSH}(\mathbb{P}^n, \omega)$.

If $f \in \mathcal{A}_{\mathbb{P}^n}$, then there is a well-defined pullback of $\omega$ by $f$, denoted $f^*\omega$. It is defined locally by $\Delta\psi \circ f$ where $\psi$ is a local potential of $\omega$, that is, $\psi$ is a plurisubharmonic function such that $dd^c\psi = \omega$. For more details about $\omega$-plurisubharmonic functions and analytic discs, see [11, Section 2].

The pullback of the current $\omega$ to $\mathbb{C}^{n+1} \setminus \{0\}$ satisfies

$$\pi^*\omega = dd^c \log \|\cdot\|,$$

where $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is the projection

$$\pi(z_0, z_1, \dots, z_n) = [z_0 : z_1 : \cdots : z_n].$$

This implies that if $\tilde{f} \in \mathcal{A}_{\mathbb{C}^{n+1}\setminus\{0\}}$ and we let $f = \pi \circ \tilde{f} \in \mathcal{A}_{\mathbb{P}^n}$, then

$$f^*\omega = \Delta \log \|\tilde{f}\|.$$

Note that there is a one to one correspondence between the family $\mathcal{PSH}(\mathbb{P}^n, \omega)$ and $\{u \in \mathcal{PSH}(\mathbb{C}^{n+1} \setminus \{0\}); u \text{ logarithmically homogeneous}\}$. See [3, Example 2.2].

Lemma 3.1. *Assume $X$ is a domain and a complex cone in $\mathbb{C}^{n+1} \setminus \{0\}$. If $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ is upper semicontinuous and logarithmically homogeneous then the function* $\sup \mathcal{F}_\varphi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{R} \cup \{-\infty\}$,

$$\sup \mathcal{F}_\varphi(x) = \sup\{u(x); u \in \mathcal{PSH}\big(\mathbb{C}^{n+1} \setminus \{0\}\big), u|_X \leq \varphi\}$$

*is also logarithmically homogeneous.*

*Proof.* We will first show that $\mathcal{F}_\varphi$ is locally bounded above to ensure that $\sup \mathcal{F}_\varphi$ is plurisubharmonic. Fix $w \in X$, then for $x \in \mathbb{C}^{n+1} \setminus \{\lambda w; \lambda \in \mathbb{C}\}$ there exists by Lemma 2.6 a disc $f_{w,x}$ centered at $x$ which maps $\mathbb{T}$ into $X$. Then for every $u \in \mathcal{F}_\varphi$, by the subaverage property,

$$u(x) = u\big(f_{x,w}(0)\big) \leq \int_{\mathbb{T}} u \circ f_{x,w}\, d\sigma \leq \sup_{|y-w|=r} \varphi(y).$$

By the continuity of $x \mapsto r = \min\{\frac{\|x-w\|}{1+\|x-w\|}, \frac{d(w,W^c)}{2}\}$ and the upper semicontinuity of $\varphi$ we see that $\mathcal{F}_\varphi$ is locally bounded above.

If $u \in \mathcal{F}$ then $u \leq \varphi$ on $X$, which implies that for $\lambda \in \mathbb{C}^*$ we have

$$u(\lambda x) + \log(\lambda) \leq \varphi(\lambda x) + \log(\lambda) = \varphi(x).$$

This shows that the following is equivalent for $\lambda \in \mathbb{C}^*$

- $u \in \mathcal{F}_\varphi$,
- $u(\lambda\cdot) + \log(\lambda) \in \mathcal{F}_\varphi$,
- $u(\cdot/\lambda) - \log(\lambda) \in \mathcal{F}_\varphi$.

Finally, we see that $\sup \mathcal{F}_\varphi$ logarithmically homogeneous since

$$\begin{aligned}\sup \mathcal{F}_\varphi(x) &= \sup\bigl\{u(x); u \in \mathcal{F}_\varphi\bigr\}\\ &= \sup\bigl\{u(x); u(\cdot/\lambda) - \log(\lambda) \in \mathcal{F}_\varphi\bigr\}\\ &= \sup\bigl\{u(\lambda x) + \log(\lambda); u \in \mathcal{F}_\varphi\bigr\}\\ &= \sup \mathcal{F}_\varphi(\lambda x) + \log(\lambda).\end{aligned}$$ □

Now we prove the main result of this section.

THEOREM 3.2. *Let $\omega$ by the Fubini–Study Kähler form. If $W \subset \mathbb{P}^n$ is a domain and $\varphi : W \to \mathbb{R} \cup \{-\infty\}$ is an upper semicontinuous function then for $x \in \mathbb{P}^n$,*

$$\begin{aligned}&\sup\bigl\{u(x); u \in \mathcal{PSH}\bigl(\mathbb{P}^n, \omega\bigr), u|_W \le \varphi\bigr\}\\ &\qquad = \inf\left\{-\int_{\mathbb{D}} \log|\cdot|\, f^*\omega + \int_{\mathbb{T}} \varphi \circ f\, d\sigma; f \in \mathcal{A}^W_{\mathbb{P}^n}, f(0) = x\right\}\end{aligned} \tag{2}$$

*or, using the notation from Section* 1, $\sup \mathcal{F}_{\omega,\varphi} = E_W H_{\omega,\varphi}$.

*Proof.* Define the complex cone $\tilde{W} = \pi^{-1}(W)$ and define the logarithmically homogeneous function $\tilde{\varphi} : \tilde{W} \to \mathbb{R} \cup \{-\infty\}$ by

$$\tilde{\varphi}(z) = \varphi\bigl([z_0 : z_1 : \cdots : z_n]\bigr) + \log\|z\|.$$

Fix $\tilde{f} \in \mathcal{A}^{\tilde{W}}_{\mathbb{C}^{n+1}\setminus\{0\}}$ and define $f \in \mathcal{A}^W_{\mathbb{P}^n}$ by $f = \pi \circ \tilde{f}$. Let $z = \tilde{f}(0)$, which implies $\pi(z) = f(0)$. Then

$$\tilde{\varphi} \circ \tilde{f} = \varphi \circ f + \log\|\tilde{f}\|.$$

By the Riesz representation formula for the function $\log\|\tilde{f}\|$ at the point 0,

$$\begin{aligned}\log\bigl\|\tilde{f}(0)\bigr\| &= \int_{\mathbb{D}} \log|\cdot|\,\Delta\log\|\tilde{f}\| + \int_{\mathbb{T}} \log\|\tilde{f}\|\, d\sigma\\ &= \int_{\mathbb{D}} \log|\cdot|\,\tilde{f}^*\bigl(\pi^*\omega\bigr) + \int_{\mathbb{T}} \log\|\tilde{f}\|\, d\sigma.\end{aligned}$$

Since $\tilde{f}^*(\pi^*\omega) = (\pi \circ \tilde{f})^*\omega = f^*\omega$, this shows that

$$\int_{\mathbb{T}} \log\|\tilde{f}\|\, d\sigma = -\int_{\mathbb{D}} \log|\cdot|\, f^*\omega + \log\|z\|.$$

Using the three previous equalities, we derive the following

$$
\begin{aligned}
H_{\omega,\varphi}(f) &= \int_{\mathbb{T}} \varphi\circ f\,d\sigma - \int_{\mathbb{D}} \log|\cdot|\,f^*\omega \\
&= \int_{\mathbb{T}} \tilde\varphi\circ\tilde f\,d\sigma - \int_{\mathbb{T}} \log\|\tilde f\|\,d\sigma - \int_{\mathbb{D}} \log|\cdot|\,f^*\omega \\
&= \int_{\mathbb{T}} \tilde\varphi\circ\tilde f\,d\sigma - \log\|z\| \\
&= H_{\tilde\varphi}(\tilde f) - \log\|z\|.
\end{aligned} \tag{3}
$$

Now note that every disc $\tilde f\in\mathcal{A}^{\tilde W}_{\mathbb{C}^{n+1}\setminus\{0\}}$ gives a disc $f=\pi\circ\tilde f\in\mathcal{A}^W_{\mathbb{P}^n}$, and conversely for every disc $f\in\mathcal{A}^W_{\mathbb{P}^n}$ there is a disc $\tilde f\in\mathcal{A}^{\tilde W}_{\mathbb{C}^{n+1}\setminus\{0\}}$ such that $f=\pi\circ\tilde f$. In particular $f(0)=\pi(z)=\pi(\tilde f(0))$.

Hence, taking the infimum over $f\in\mathcal{A}^W_{\mathbb{P}^n}$, $f(0)=\pi(z)$, on the left-hand side of (3) corresponds to taking infimum over $\tilde f\in\mathcal{A}^{\tilde W}_{\mathbb{C}^n\setminus\{0\}}$, $\tilde f(0)=z$, on the right-hand side. This shows that

$$
E_W H_{\omega,\varphi}\big(\pi(z)\big) = E_{\tilde W} H_{\tilde\varphi}(z) - \log\|z\|. \tag{4}
$$

By Lemma 2.6, $\tilde W$ admits a $\tilde W$-disc structure $\beta$ such that $E_{\mathcal{B}}H_{\tilde\varphi}\le\tilde\varphi$, and by Theorem 2.3 we have

$$
\sup\mathcal{F}_{\tilde\varphi} = E_{\tilde W} H_{\tilde\varphi} \tag{5}
$$

in $\mathbb{C}^{n+1}\setminus\{0\}$.

Finally, by (4), (5), and Lemma 3.1, we can finish the proof,

$$
E_W H_{\omega,\varphi}\big(\pi(z)\big) = E_{\tilde W} H_{\tilde\varphi}(z) - \log\|z\| = \sup\mathcal{F}_{\tilde\varphi}(z) - \log\|z\| = \sup\mathcal{F}_{\omega,\varphi}\big(\pi(z)\big).
$$

□

## 4. Applications to projective hulls

The case when $\varphi=0$ in Theorem 3.2 is interesting in its own, since it gives a disc formula for the global extremal function defined in [3, Section 5 and 6].

Definition 4.1. Let $E$ be a Borel subset in $\mathbb{P}^n$. The *global extremal function* for $E$ is defined as

$$
\Lambda_E(x) = \sup\big\{u(x); u\in\mathcal{PSH}\big(\mathbb{P}^n,\omega\big), u|_E\le 0\big\}.
$$

In the case when $E$ is a domain, Theorem 3.2 (with $\varphi=0$) gives the following formula

$$
\Lambda_E(x) = \inf\Big\{-\int_{\mathbb{D}} \log|\cdot|\,f^*\omega; f\in\mathcal{A}^E_{\mathbb{P}^n}, f(0)=x\Big\}.
$$

DEFINITION 4.2. Let $K$ be a compact subset of $\mathbb{P}^n$. The *projective hull* of $K$, denoted $\hat{K}$, is defined as all the points $x \in \mathbb{P}^n$ for which there exists a constant $C_x$ such that

$$\big\|P(x)\big\| \le C_x^d \sup_K \|P\| \quad \text{for all } P \in H^0\big(\mathbb{P}^n, \mathcal{O}(d)\big), \tag{6}$$

where $H^0(\mathbb{P}^n, \mathcal{O}(d))$ are the holomorphic sections of $\mathcal{O}_{\mathbb{P}^n}(d)$.

Just as the polynomial hull can be characterized by the Siciak–Zahariuta extremal function, the projective hull can be characterized using the global extremal function.

PROPOSITION 4.3 ([4], Section 4). *If $K \subset \mathbb{P}^n$ is compact, then*

$$\hat{K} = \big\{x \in \mathbb{P}^n; \Lambda_K(x) < +\infty\big\}.$$

*Furthermore, for each $x$ the value* $\exp(\Lambda_K(x))$ *is equal to the infimum of all $C_x$ such that* (6) *holds.*

For a constant $\Lambda \ge 0$ we let

$$\hat{K}(\Lambda) = \big\{x \in \mathbb{P}^n; \Lambda_K(x) \le \Lambda\big\}.$$

The projective hull can then be written as a union of the sets $\hat{K}(\Lambda)$.

The disc formula proved in Section 3 is only for domains in $\mathbb{P}^n$ but not compact sets. This forces us to take a decreasing sequence of open neighborhoods of $K$. The following proposition then allows us to take the limit to obtain $\Lambda_K$.

PROPOSITION 4.4. *Assume $K \subset \mathbb{P}^n$ is a compact set and $(U_j)_j$ is a decreasing sequence of open subsets in $\mathbb{P}^n$ such that $\bigcap_j U_j = K$. Then*

$$\Lambda_K = \lim_{j\to\infty} \Lambda_{U_j}.$$

*Proof.* Note first that since $U_j$ is a decreasing sequence then the sequence $\Lambda_{U_j}$ is increasing, in particular $\lim_{j\to\infty} \Lambda_{U_j}$ exists.

Since each function $\Lambda_{U_j}$ is in $\mathcal{PSH}(\mathbb{P}^n, \omega)$ (see [3, Theorem 5.2 and Proposition 5.6]) and is 0 on $U_j \supset K$, then $\Lambda_K \ge \Lambda_{U_j}$ and therefore $\Lambda_K \ge \lim_{j\to\infty} \Lambda_{U_j}$.

Let $\varepsilon > 0$. Since each function $u \in \mathcal{PSH}(\mathbb{P}^n, \omega)$, $u|_K \le 0$ is upper semicontinuous there is a neighborhood $U$ of $K$ such that $u|_U \le \varepsilon$. Find $U_{j_0}$ such that $U_{j_0} \subset U$. Then for $x \in X$,

$$u(x) - \varepsilon \le \Lambda_{U_{j_0}}(x) \le \lim_{j\to\infty} \Lambda_{U_j}$$

which implies, by taking supremum over $u$ and letting $\varepsilon \to 0$, that $\Lambda_K \le \lim_{j\to\infty} \Lambda_{U_j}$. □

By combining the disc formula for the global extremal function with Proposition 4.3, we obtain a new characterization of the projective hull for a connected set. This characterization is quantitative, that is it uses $\hat{K}(\Lambda)$ just as the characterization by existence of currents [4, Theorem 11.1].

THEOREM 4.5. *Let $\Lambda > 0$. For a point $x$ in a connected compact subset $K \subset \mathbb{P}^n$ the following is equivalent*

(A) $x \in \hat{K}(\Lambda)$.

(B) *For every $\varepsilon > 0$ and every neighborhood $U$ of $K$ there exists a disc $f \in \mathcal{A}^U_{\mathbb{P}^n}$ such that $f(0) = x$ and*

$$-\int_{\mathbb{D}} \log|\cdot|f^*\omega < \Lambda + \varepsilon.$$

*Proof.* First, assume $x \in \hat{K}(\Lambda)$. By Proposition 4.4 , there is a domain $V$ such that $K \subset V \subset U$ and $\Lambda_V(x) < \Lambda_K(x) + \varepsilon/2$. By Theorem 3.2, there is a disc $f \in \mathcal{A}^V_{\mathbb{P}^n}$ such that $f(0) = x$ and

$$-\int_{\mathbb{D}} \log|\cdot|f^*\omega < \Lambda_V(x) + \frac{\varepsilon}{2}.$$

Then

$$-\int_{\mathbb{D}} \log|\cdot|f^*\omega \leq \Lambda_V(x) + \frac{\varepsilon}{2} \leq \Lambda_K(x) + \varepsilon \leq \Lambda + \varepsilon.$$

Conversely, assume (B) holds. Now let $U_j$ be a decreasing sequence of domains such that $\bigcap_j U_j = K$. This implies that $\lim_{j\to\infty} \Lambda_{U_j} = \Lambda_K$. The sets $U_j$ can be chosen to be connected because $K$ is always contained in one connected component of an open neighborhood of $K$ (otherwise $K$ would not be connected).

For each $j$ there is a disc $f_j \in \mathcal{A}^{U_j}_{\mathbb{P}^n}$ such that $f_j(0) = x$ and

$$\Lambda_{U_j}(x) \leq -\int_{\mathbb{D}} \log|\cdot|f_j^*\omega < \Lambda + \frac{1}{j}$$

which implies $\Lambda_K(x) = \lim_{j\to\infty} \Lambda_{U_j}(x) \leq \Lambda$, that is $x \in \hat{K}(\Lambda)$. □

**4.1. Characterizations of Drnovšek and Forstnerič.** In [1] Drnovšek and Forstnerič gave several characterizations of the projective and polynomial hulls using the existence of analytic disc, both for connected sets and sets which are not connected. They used analytic discs in $\mathbb{P}^n$ with the bounded lifting property obtained from Poletsky's disc formula and discs in $\mathbb{C}^{n+1} \setminus \{0\}$ derived from the disc formula of the Siciak–Zahariuta extremal function [9].

The result from [1] regarding connected sets which is best compatible with Theorem 4.5 is the following.

THEOREM 4.6 ([1], Theorem 5.1). *Let $K$ be a compact connected set in $\mathbb{P}^n$. A point $p \in \mathbb{C}^{n+1} \setminus \{0\}$ belongs to the polynomial hull of the set $S_K \subset \mathbb{C}^{n+1}$ and hence $x = \pi(p) \in \mathbb{P}^n$ belongs to the projective hull of $K$, if and only if there exists a sequence of analytic discs $F_j : \overline{\mathbb{D}} \to \mathbb{C}^{n+1} \setminus \{0\}$ such that $F_j(0) = p$ and*

$$\lim_{j\to\infty} \max_{t\in[0,2\pi]} \operatorname{dist}(F_j\big(e^{it}, S_K\big) = 0.$$

The set $S_K$ is the lifting of $K$ restricted to the unit sphere in $\mathbb{C}^{n+1}$, $S_K = \pi^{-1}(K) \cap \{z \in \mathbb{C}^{n+1}; \|z\| = 1\}$.

We can restate the theorem above without the limit and with focusing on the point $x \in \mathbb{P}^n$.

THEOREM 4.7. *For a point $x$ in a connected compact subset $K \subset \mathbb{P}^n$ the following is equivalent.*

(A′) $x \in \hat{K}$.

(B′) *There exists $p \in \pi^{-1}(x)$ such that for every neighborhood $\tilde{U}$ of $S_K$ there exists a disc $\tilde{f} \in \mathcal{A}^{\tilde{U}}_{\mathbb{C}^{n+1}\setminus\{0\}}$ such that $\tilde{f}(0) = p$.*

Note that this characterization is not quantitative, in other words it does not use $\Lambda$ and $\hat{K}(\Lambda)$ as Theorem 4.5 does. It is however no hard to show directly that (B) and (B′) are equivalent.

## 5. Siciak–Zahariuta extremal function and other global extremal functions

The Fubini–Study Kähler form is not the only current on $\mathbb{P}^n$ which we wish to consider. Another interesting class of quasiplurisubharmonic functions are those where the current $\omega$ is the current of integration $[H_\infty]$ for the hyperplane at infinity $H_\infty$. Then we look at $\mathbb{P}^n$ as the union of $\mathbb{C}^n$ and $H_\infty$. The class of quasiplurisubharmonic functions $\mathcal{PSH}(\mathbb{P}^n, [H_\infty])$ then becomes the Lelong-class

$$\mathcal{L} = \big\{u \in \mathcal{PSH}\big(\mathbb{C}^n\big); u(x) \le \log\|x\| + C_u\big\}.$$

The potential for the pullback of this current to $\mathbb{C}^{n+1} \setminus \{0\}$, denoted $\pi^*[H_\infty]$, has a global potential. That is $\pi^*[H_\infty] = dd^c \log|z_0|$, assuming $H_\infty = \pi(\{z \in \mathbb{C}^{n+1} \setminus \{0\}; z_0 = 0\})$.

The *Siciak–Zahariuta extremal function* for a set $W$ is defined as

$$\sup\big\{u(x); u \in \mathcal{L}, u|_W \le 0\big\}$$

and the weighted version as

$$\sup\big\{u(x); u \in \mathcal{L}, u|W \le \varphi\big\},$$

where $\varphi : W \to \overline{\mathbb{R}}$ is a function.

If $W \subset \mathbb{C}^n$ is a domain and $\varphi : W \to \mathbb{R} \cup \{-\infty\}$ is an upper semicontinuous function, then there is a disc formula for the Siciak–Zahariuta function [9], [12],

$$\begin{aligned}(7)\qquad &\sup\big\{u(x); u \in \mathcal{L}, u|W \le \varphi\big\}\\ &\quad = \inf\Big\{-\sum_{a \in f^{-1}(H_\infty)} \log|a| + \int_{\mathbb{T}} \varphi \circ f\, d\sigma; f \in \mathcal{A}^W_{\mathbb{P}^n}, f(0) = x\Big\}.\end{aligned}$$

There is also a formula when $W$ is not connected [10], which uses a different approach and is somewhat more complicated.

The formula above can be proven easily by the same methods as the formula in Theorem 3.2 by replacing $\log\|z\|$ with $\log|z_0|$.

We only have to note two things. First, a function $u \in \mathcal{L}$ extends to a plurisubharmonic and logarithmically homogeneous function $\tilde{u} : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{R} \cup \{-\infty\}$, by

$$\tilde{u}(z_0, z_1, \dots, z_n) = u\left(\frac{(z_1, \dots, z_n)}{z_0}\right) + \log|z_0|.$$

Secondly, if $f = [f_0 : f_1 : \cdots : f_n] \in \mathcal{A}_{\mathbb{P}^n}$, then

$$\int_D \log|\cdot|\, f^*[H_\infty] = \sum_{a \in f^{-1}(H_\infty)} m_a \log|a|,$$

where $m_a$ is the multiplicity of the zero of $f_0$ at $a$. However, by Proposition 1 in [9] the multiplicity $m_a$ can by omitted because for a disc $f$ with zero of order $m_a$ at $a$ there is a disc with $m_a$ different simple zeros sufficiently close to $a$. This implies that the multiplicity $m_a$ can be omitted in the disc formula above.

Remark. The methods described here actually apply to every current $\tilde{\omega}$ on $\mathbb{P}^n$ such that the pullback to $\mathbb{C}^{n+1} \setminus \{0\}$, $\pi^*\tilde{\omega}$, has a logarithmically homogeneous potential $\psi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{R} \cup \{-\infty\}$ such that $dd^c\psi = \pi^*\tilde{\omega}$.

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