Illinois Journal of Mathematics
Volume 57, Number 3, Fall 2013, Pages 883–918
S 0019-2082

GEOMETRIC RSK AND THE TODA LATTICE

NEIL O'CONNELL

Abstract. We relate a continuous-time version of the geometric RSK correspondence to the Toda lattice, in a way which can be viewed as a semi-classical limit of a recent result by the author which relates the continuous-time geometric RSK mapping, with Brownian motion as input, to the quantum Toda lattice.

1. Introduction

The geometric RSK correspondence is a geometric lifting of the classical RSK correspondence. It was introduced by Kirillov [25] and further studied by Noumi and Yamada [33]. There is also a continuous-time version of the geometric RSK mapping, which was introduced in [34] and substantially developed in [5], [6] in the context of Littelmann's path model. In this setting, an important role is a played by a mapping Π_n (defined in the next section) which takes as input a continuous path $\eta(t), t \geq 0$ in $\mathbb{R}^n$ with $\eta(0) = 0$ and returns a path $\Pi_n\eta(t), t > 0$, also in $\mathbb{R}^n$. In the paper [37] it was shown that, if $\eta(t) = \sqrt{\varepsilon}B(t) + t\lambda$, where $\lambda \in \mathbb{R}^n$ and B is a standard Brownian motion in $\mathbb{R}^n$, then $\Pi_n\eta(t), t > 0$ is a diffusion process in $\mathbb{R}^n$ with infinitesimal generator given in terms of the Hamiltonian of the open quantum Toda chain with n particles. The aim of the present paper is to understand this result from the point of view of the classical Toda lattice.

By considering the semiclassical limit ($\varepsilon \to 0$) of this result, it can be seen (heuristically) from a result of Givental [20] (at least in the case $\lambda = 0$) that, if $\eta(t) = t\lambda$, then $\Pi_n\eta(t), t > 0$ should define a solution to the classical Toda flow (with opposite sign), and indeed this is the case. We will show it directly in the classical ($\varepsilon = 0$) setting by considering the continuous-time geometric RSK mapping with deterministic input. The approach is very elementary and mostly self-contained. Starting with the definition of the geometric RSK mapping, we soon arrive at familiar objects in the general theory of the Toda

Received September 17, 2013; received in final form January 23, 2014.
2010 *Mathematics Subject Classification.* Primary 37K10, 60K35.

lattice, thus providing further insight into the results of [37] from an integrable systems point of view.

The main conclusion is that there is a precise sense in which

$$\text{classical Toda} + \text{noise} = \text{quantum Toda}. \tag{1.1}$$

This statement requires some qualification, however. First, we consider both the classical and quantum system in imaginary time. For the classical system this means that the potential has a minus sign, that is, the Hamiltonian is given by

$$\frac{1}{2}\sum_{i=1}^{n} p_i^2 - \sum_{i=1}^{n-1} e^{x_{i+1}-x_i}.$$

For the quantum system, it means we consider normalised Hamiltonian

$$\mathcal{L}_\lambda = -\frac{1}{2}\psi_\lambda(x)^{-1}\left(H + \varepsilon\sum_i \lambda_i^2\right)\psi_\lambda(x) = \frac{\varepsilon}{2}\Delta + \varepsilon\nabla\log\psi_\lambda\cdot\nabla,$$

where

$$H = -\varepsilon\Delta + \frac{2}{\varepsilon}\sum_{i=1}^{n-1} e^{x_{i+1}-x_i}$$

is the Hamiltonian of the quantum Toda lattice and ψ_λ is a particular eigenfunction of H known as a (class one) $GL(n,\mathbb{R})$-Whittaker function. Moreover, in the 'equation' (1.1), the noise must be added in a very particular way. It is not simply a random perturbation of the classical Toda flow, but rather a random perturbation of a particular construction of the Toda flow which is closely related to the geometric RSK correspondence. Within this construction, the perturbation is simply:

Add noise to the constants of motion.

It will be interesting to investigate to what extent this relation extends to other integrable systems.

Along the way we observe the following curious fact. Consideration of the geometric RSK mapping with Brownian motion as input gives rise to a particular stochastic dynamics on triangles (the analogue of Gelfand–Tsetlin patterns in this setting), as discussed in [37]. In that paper, another quite different stochastic dynamics on triangles was also considered and shown to have the same fixed-time distributions, and to bear the same relation to the quantum Toda lattice. This latter dynamics can be interpreted as a geometric lifting of Warren's process [46], which in turn can be interpreted as a continuous version of a *shuffling algorithm* which has played an important role in the random tilings literature [32]. Similar dynamics on Gelfand–Tsetlin patterns, constructed using a general prescription of Diaconis and Fill [14], have been studied by Borodin and co-workers, see, for example, [7], [8]. It turns out that, in the semi-classical limit we consider in this paper, both the 'RSK type' and 'shuffling type' of dynamics are equivalent.

The outline of the paper is as follows. In the next section, we recall some relevant background material on factorisations of matrices. In Section 3, we define and recall some basic properties of the continuous-time geometric RSK mapping. In Section 4, we recall some facts about Whittaker functions and the quantum Toda lattice, which continue to play a role in the classical setting. In Section 5, we recall some of the main results of [37] and in Section 6 we briefly outline, at a heuristic level, what happens to these results in the semiclassical limit. In Section 7, we recall some basic definitions and properties of the opposite sign Toda lattice and, in Section 8, we formulate and prove the main results of the paper.

2. Preliminaries

Let $G = GL(n, \mathbb{C})$ and denote by B, N (resp. B_-, N_-) the subgroups of upper (resp. lower) triangular and uni-triangular matrices in G. Throughout this paper, an important role will be played by the totally positive part, which we will denote by $\mathcal{P}$, of the double Bruhat cell $B \cap B_- \bar{w}_0 B_-$, where $\bar{w}_0$ is a representative in G of the longest element $w_0 \in S_n$, as in [17], [4]. Concretely,

$$\mathcal{P} = \big\{ b \in B : \Delta_k^m(b) > 0, 1 \le k \le m \le n \big\}, \tag{2.1}$$

where

$$\Delta_k^m(b) = \det[b_{ij}]_{1 \le i \le k, m-k+1 \le j \le m}. \tag{2.2}$$

In the following, we adopt the convention that $\Delta_0^m(b) = 1$. The quantities $\Delta_k^m(b), 1 \le k \le m \le n$ uniquely determine $b \in \mathcal{P}$, as follows. (See, for example, [4] or [33, Proposition 1.5].) For $a \in \mathbb{C}^k$, define

$$\varepsilon^k(a) = \begin{pmatrix} a_1 & 1 & 0 & \dots & 0 \\ 0 & a_2 & 1 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ & & & a_{k-1} & 1 \\ 0 & & \dots & & a_k \end{pmatrix},$$

and denote by I_k the identity matrix of dimension k. For $1 \le m \le n$ and $w \in \mathbb{C}^{n-m+1}$, define

$$E_m(w) = \begin{pmatrix} I_{m-1} & 0 \\ 0 & \varepsilon^{n-m+1}(w) \end{pmatrix}.$$

Proposition 2.1. *Each element $b \in \mathcal{P}$ can be represented uniquely as a product of the form*

$$b = E_1\big(w^1\big) \cdots E_n\big(w^n\big),$$

where $w^m \in (\mathbb{R}_{>0})^{n-m+1}$ for each m. The w^m are given by

$$w_1^m = \Delta_1^m(b); \qquad w_i^m = \frac{\Delta_i^{m+i-1}(b)\Delta_{i-2}^{m+i-2}(b)}{\Delta_{i-1}^{m+i-1}(b)\Delta_{i-1}^{m+i-2}(b)}, \quad 1 < i \le n-m+1.$$

This provides a natural (Gelfand–Tsetlin) parameterization of $\mathcal{P}$ by the set of *triangles*

$$\mathcal{T} = \{X = (x_i^m) \in \mathbb{R}^{n(n+1)/2} : 1 \le i \le m \le n\}, \tag{2.3}$$

obtained by setting

$$x_1^m + \cdots + x_k^m = \log \Delta_k^m(b). \tag{2.4}$$

We will denote the corresponding bijective mapping by $f : \mathcal{P} \to \mathcal{T}$. Note that if we write $X = (x_i^m) = f(b)$, the w^m of Proposition 2.1 are given, in terms of the x_i^m, by

$$w_1^m = e^{x_1^m}; \qquad w_i^m = e^{x_i^{m+i-1} - x_{i-1}^{m+i-2}}, \quad 1 < i \le n-m+1.$$

The Weyl group associated with G is the symmetric group S_n. Each element $w \in S_n$ has a representative $\bar{w} \in G$ defined as follows. Denote the standard generators for $\mathfrak{gl}_n$ by h_i, e_i and f_i. For example, for $n = 3$,

$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad h_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$f_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad f_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

For adjacent transpositions $s_i = (i, i+1)$, define

$$\bar{s}_i = \exp(-e_i)\exp(f_i)\exp(-e_i) = (I - e_i)(I + f_i)(I - e_i).$$

In other words, $\bar{s}_i = \varphi_i \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ where φ_i is the natural embedding of $SL(2)$ into $GL(n)$ given by h_i, e_i and f_i. For example, when $n = 3$,

$$\bar{s}_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \bar{s}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Now let $w = s_{i_1} \cdots s_{i_r}$ be a reduced decomposition and define $\bar{w} = \bar{s}_{i_1} \cdots \bar{s}_{i_r}$. Note that $\overline{uv} = \bar{u}\bar{v}$ whenever $l(uv) = l(u) + l(v)$. Denote the longest element of S_n by

$$w_0 = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}.$$

For $n = 2$, $w_0 = s_1$ and

$$\bar{w}_0 = \bar{s}_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

For $n=3$, $w_0 = s_1s_2s_1 = s_2s_1s_2$ is represented by

$$\bar{w}_0 = \bar{s}_1\bar{s}_2\bar{s}_1 = \bar{s}_2\bar{s}_1\bar{s}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Denote the elementary lower uni-triangular Jacobi matrices by

$$l_i(a) = I_n + af_i, \quad 1 \le i \le n-1.$$

These matrices play a central role in parameterising of the set $(N_-)_{>0}$ of totally positive lower triangular matrices, see, for example, [4]. For $u \in \mathbb{C}^m$, $1 \le m < n$, define

$$L_m(u) = l_m(u_m)l_{m-1}(u_{m-1})\cdots l_1(u_1).$$

PROPOSITION 2.2. *Each $L \in (N_-)_{>0}$ can be written uniquely as a product*

$$L = L_1\big(u^1\big)L_2\big(u^2\big)\cdots L_{n-1}\big(u^{n-1}\big),$$

where $u^m \in (\mathbb{R}_{>0})^m$ for each m.

The next proposition is due to Berenstein, Fomin and Zelevinsky [4].

PROPOSITION 2.3. *Let $b \in \mathcal{P}$. Then $b\bar{w}_0$ has a Gauss decomposition $b\bar{w}_0 = LDU$ where*

$$D_{ii} = \frac{\Delta_i^n(b)}{\Delta_{i-1}^n(b)}, \quad 1 \le i \le n,$$

and $L \in (N_-)_{>0}$ is given by

$$L = L_1\big(u^1\big)L_2\big(u^2\big)\cdots L_{n-1}\big(u^{n-1}\big),$$

$$u_i^m = \frac{\Delta_{i-1}^m(b)\Delta_{i+1}^{m+1}(b)}{\Delta_i^m(b)\Delta_i^{m+1}(b)}, \quad 1 \le i \le m < n.$$

Note that, if $X = (x_i^m) = f(b)$, then the D_{ii} and u_i^m of Proposition 2.3 are given by

$$D_{ii} = e^{x_i^n}, \quad 1 \le i \le n$$

and

$$u_i^m = e^{x_{i+1}^{m+1} - x_i^m}, \quad 1 \le i \le m < n.$$

3. Geometric RSK in continuous time

In this section, we recall the definition and some basic properties of the continuous-time geometric RSK mapping. Many of the results of this section are essentially contained in the papers [5], [6], see also [37], [38]. For completeness, we include direct proofs of all the main statements, which are adapted to the present setting.

Let $\eta : [0,\infty) \to \mathbb{R}^n$ be a continuous path with $\eta(0) = 0$. Denote the coordinates of η by $\eta_1, \ldots, \eta_n$, so that

$$\eta(t) = \big(\eta_1(t), \ldots, \eta_n(t)\big), \quad t > 0.$$

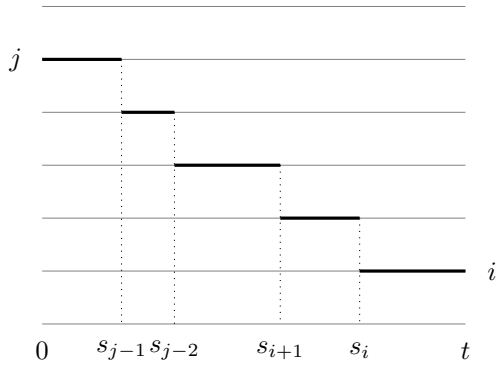


FIGURE 1. A down-right path $\phi = (s_{j-1}, \dots, s_i) \in \Omega_{ij}(t)$.

For $1 \le i < j \le n$, set

$$\Omega_{ij}(t) = \{0 < s_{j-1} < \cdots < s_i < t\}.$$

In the following, it will be helpful to think of elements $\phi \in \Omega_{ij}(t)$ as 'down-right paths' in the semi-lattice $\mathbb{R} \times \mathbb{Z}$ starting at $(0, j)$ and ending at (t, i), as shown in Figure 1. Write $d\phi = ds_i \cdots ds_{j-1}$ for the Euclidean measure on $\Omega_{ij}(t)$. For $\phi = (s_{j-1}, \dots, s_i) \in \Omega_{ij}(t)$, we define

$$E_\eta(\phi) = \eta_j(s_{j-1}) + \eta_{j-1}(s_{j-2}) - \eta_{j-1}(s_{j-1}) + \cdots + \eta_i(t) - \eta_i(s_i).$$

For $1 \le i < j \le n$ and $t \ge 0$, set

$$b_{ii}(t) = e^{\eta_i(t)}, \qquad b_{ij}(t) = \int_{\Omega_{ij}(t)} e^{E_\eta(\phi)}\, d\phi. \tag{3.1}$$

Setting $b_{ij} = 0$ for $i > j$, this defines a path in the subgroup B of upper triangular matrices in $GL(n, \mathbb{C})$. If η is smooth, then $b(t) = (b_{ij}(t))$, $t \ge 0$, satisfies the evolution equation

$$\dot{b} = \varepsilon(\dot{\eta}) b, \tag{3.2}$$

with initial condition $b(0) = I_n$, where $\varepsilon(\lambda) = \varepsilon_\lambda$ is defined for $\lambda \in \mathbb{C}^n$ by

$$\varepsilon_\lambda = \begin{pmatrix} \lambda_1 & 1 & 0 & \dots & 0 \\ 0 & \lambda_2 & 1 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ & & & \lambda_{n-1} & 1 \\ 0 & & \dots & & \lambda_n \end{pmatrix}. \tag{3.3}$$

For $n = 1$, we define $\varepsilon_\lambda = \lambda_1$.

The next proposition is a special case of [5, Proposition 3.11]. It shows that $b(t) \in \mathcal{P}$ for each $t > 0$ and also makes clear the connection with Kirillov's

original definition of the geometric RSK mapping in terms on non-intersecting lattice paths. For completeness, we include a direct proof which is adapted to the present setting.

PROPOSITION 3.1.

$$\Delta_k^m\big(b(t)\big) = \int_{\Omega_k^m(t)} e^{E_\eta(\phi_1)+\cdots+E_\eta(\phi_k)}\, d\phi_1 \cdots d\phi_k, \tag{3.4}$$

where the integral is with respect to the Euclidean measure on the set $\Omega_k^m(t)$ *of* k*-tuples of non-intersecting down-right paths* $\phi_1, \ldots, \phi_k$ *starting at* $(0, m-k+1), \ldots, (0, m)$*, respectively, and ending at* $(t, 1), \ldots, (t, k)$.

Proof. This is a straightforward variation of the Karlin–McGregor/Lindström–Gessel–Viennot formula, and is proved by a standard path-switching argument, see, for example, [24, Section 1.2]. Let

$$Y(t) = \big(Y_1(t), \ldots, Y_k(t)\big), \quad t \geq 0$$

be a collection of independent, unit-rate Poisson processes started at positions $(1, \ldots, k)$. Then Y is a continuous time Markov chain with state space $\mathbb{N}^k$. Denote the transition probabilities of Y by

$$p_t(y, y') = P\big(Y(s+t) = y' | Y(s) = y\big), \quad s, t \geq 0.$$

For $\sigma \in S_k$ and $y \in \mathbb{N}^k$, write $\sigma y = (y_{\sigma(1)}, \ldots, y_{\sigma(k)})$. We note that Y enjoys the strong Markov property and its law is S_k-invariant, that is, $p_t(\sigma y, \sigma y') = p_t(y, y')$ for all $\sigma \in S_k$. Let

$$T_i = \inf\big\{t > 0 : Y_i(t) = Y_{i+1}(t)\big\}, \quad 1 \leq i \leq n-1,$$

and $T = \min_i T_i$. Fix $t > 0$ and let Z be an integrable, measurable function of $(Y(r), 0 \leq r \leq t)$ which is invariant under the substitutions $Y \to \tilde{Y}^{(i)}$, where for each i,

$$\tilde{Y}^{(i)}(r) = \begin{cases} Y(r), & r \leq T_i, \\ s_i Y(r), & r > T_i, \end{cases}$$

with s_i denoting the adjacent transposition $(i, i+1)$.

We will first show that, for $y \in \mathbb{N}^k$ with $y_1 < y_2 < \cdots < y_k$,

$$E\big[Z; T > t; Y(t) = y\big] = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) E\big[Z; Y(t) = \sigma y\big]. \tag{3.5}$$

Note that, since $E[Z; T > t; Y(t) = \sigma y] = 0$ unless σ is the identity, this is equivalent to

$$\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) E\big[Z; T \leq t; Y(t) = \sigma y\big] = 0,$$

or indeed

$$\sum_i \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) E\big[Z; T = T_i \leq t; Y(t) = \sigma y\big] = 0.$$

It therefore suffices to show that, for each i,

$$\sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)E\big[Z;T=T_i\le t;Y(t)=\sigma y\big]=0.$$

Now fix i and write $\tilde{Y}=\tilde{Y}^{(i)}$. Then, by the strong Markov property and S_k-invariance of Y, $\tilde{Y}$ has the same law as Y. Moreover,

$$T_j=\inf\big\{t>0:\tilde{Y}_j(t)=\tilde{Y}_{j+1}(t)\big\},\quad 1\le j\le n-1.$$

Thus,

$$\begin{aligned}E\big[Z;T=T_i\le t;Y(t)=\sigma y\big]&=E\big[Z;T=T_i\le t;\tilde{Y}(t)=s_i\sigma y\big]\\&=E\big[Z;T=T_i\le t;Y(t)=s_i\sigma y\big],\end{aligned}$$

and we conclude that

$$\begin{aligned}&\sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)E\big[Z;T=T_i\le t;Y(t)=\sigma y\big]\\&\quad=\sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)E\big[Z;T=T_i\le t;Y(t)=s_i\sigma y\big]\\&\quad=-\sum_{\sigma\in S_k}\operatorname{sgn}(s_i\sigma)E\big[Z;T=T_i\le t;Y(t)=s_i\sigma y\big]\\&\quad=-\sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)E\big[Z;T=T_i\le t;Y(t)=\sigma y\big],\end{aligned}$$

as required.

To see that (3.5) implies the formula (3.4), we take

$$Z=e^{E_\eta(\hat{Y}_1)+\cdots+E_\eta(\hat{Y}_k)},$$

where $\hat{Y}(s)=Y(t-s)$, and $y=(m-k+1,\dots,m)$. Then, using the fact that a collection of independent, identically distributed exponential random variables, conditioned on the value of their sum is uniformly distributed in the corresponding simplex, we can write:

$$\begin{aligned}&E\big[Z;T>t;Y(t)=y\big]\\&\quad=P\big(Y(t)=y\big)\big|\Omega_{k,m}(t)\big|^{-k}\int_{\Omega_k^m(t)}e^{E_\eta(\phi_1)+\cdots+E_\eta(\phi_k)}\,d\phi_1\cdots d\phi_k\end{aligned}$$

and

$$\begin{aligned}&\sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)E\big[Z;Y(t)=\sigma y\big]\\&\quad=\det\big[E\big[e^{E_\eta(\hat{Y}_i)};Y_i(t)=y_j\big]\big]_{1\le i,j\le k}\\&\quad=\det\big[P\big(Y_i(t)=y_j\big)\big|\Omega_{i,m-k+j}(t)\big|^{-1}b_{i,m-k+j}(t)\big]_{1\le i,j\le k}.\end{aligned}$$

Now, for each $i \le j$,

$$P\big(Y_i(t) = j\big) = e^{-t}\frac{t^{j-i}}{(j-i)!} = e^{-t}\big|\Omega_{ij}(t)\big|,$$

hence

$$P\big(Y_i(t) = y_j\big)\big|\Omega_{i,m-k+j}(t)\big|^{-1} = e^{-t}$$

and

$$P\big(Y(t) = y\big)\big|\Omega_{k,m}(t)\big|^{-k} = e^{-kt},$$

which concludes the proof. □

In particular, $X(t) = (x_i^m(t)) = f(b(t)), t > 0$ defines a path in the set of triangles $\mathcal{T}$. The mapping $\Pi : \eta \mapsto (X(t), t > 0)$ was introduced and studied in the papers [5], [6] and can be thought of as a continuous time version of the geometric RSK correspondence introduced by Kirillov [25]. It also appeared, in a different form (see below) in the paper [34]. For readers familiar with the usual RSK correspondence, for each fixed $t > 0$, the path $(\eta(s), 0 \le s \le t)$ should be interpreted as the input 'word', the triangle $X(t) = (x_i^m(t), 1 \le i \le m \le n)$ as the 'P-tableau', the path $(x^n(s), 0 < s \le t)$ as the 'Q-tableau' and the vector $x^n(t)$ as their common 'shape'.

The mapping $\Pi : \eta \mapsto (X(t), t > 0)$ defined above admits the following alternative formulation which is, in fact, equivalent to the original definition given in [34]. For $i = 1, \dots, n-1$, and continuous $\eta : (0, \infty) \to \mathbb{R}^n$, define

$$(P_i\eta)(t) = \eta(t) + \left(\log \int_0^t e^{\eta_{i+1}(s) - \eta_i(s)}\, ds\right)(e_i - e_{i+1}),$$

where $e_1, \dots, e_n$ denote the standard basis vectors in $\mathbb{R}^n$. Let Π_1 denote the identity mapping ($\Pi_1\eta = \eta$) and, for $2 \le m \le n$, define

$$\Pi_m = P_1 \circ \cdots \circ P_{m-1} \circ \Pi_{m-1}.$$

Now, for continuous $\eta : [0, \infty) \to \mathbb{R}^n$ with $\eta(0) = 0$, define $X(t) = (x_i^m(t))$, $t > 0$, by

$$x_i^m(t) = (\Pi_m\eta)_i(t), \quad 1 \le i \le m \le n. \tag{3.6}$$

Then it holds that $\Pi\eta = (X(t), t > 0)$. This follows from a more general result [5, Theorem 3.5], which states that $(e^{x_1^n}, \dots, e^{x_n^n})$ is the diagonal part in the Gauss decomposition of $b\bar{w}_0$, as well as the recursive nature of the construction. For completeness, we will include a self-contained proof of this fact in the following, see Proposition 3.4 below.

In [5], it was shown that the P_i satisfy the braid relations, that is

$$P_iP_{i+1}P_i = P_{i+1}P_iP_{i+1}, \quad i = 1, \dots, n-1.$$

It follows that, for each $w \in S_n$,

$$P_w := P_{i_r} \cdots P_{i_1}$$

is well defined, where $w = s_{i_1} \cdots s_{i_r}$ is any reduced decomposition of w as a product of adjacent transpositions, where s_i denotes the transposition $(i, i+1)$. The above-defined Π_n is in fact P_{w_0}, as can be seen using the reduced decomposition $1\,21\,321 \cdots n-1 \cdots 21$.

It is a straightforward consequence of (3.6) that, for smooth η, the triangle $X = (x_i^m)$ evolves according to the dynamics

$$\begin{aligned}(3.7)\quad &\dot{x}_1^1 = \dot{\eta}_1;\\ &\dot{x}_1^m = \dot{x}_1^{m-1} + e^{x_2^m - x_1^{m-1}}, \qquad \dot{x}_m^m = \dot{\eta}_m - e^{x_m^m - x_{m-1}^{m-1}}, \quad 2 \le m \le n;\\ &\dot{x}_i^m = \dot{x}_i^{m-1} + e^{x_{i+1}^m - x_i^{m-1}} - e^{x_i^m - x_{i-1}^{m-1}}, \quad 1 < i < m \le n.\end{aligned}$$

(For details, see Proposition 3.3.) Note that the initial value is singular.

We now will explain how, using the evolution equations (3.7), one can 'insert' a path η into an arbitrary initial triangle $\xi \in \mathcal{T}$. In the language of RSK, this corresponds to inserting a word into an arbitrary initial P-tableau. For a smooth path η, the dynamic (3.7) defines a flow on triangles which we denote by S_t^η. In other words, if we set $X(0) = \xi$ and allow $X(t)$ to evolve according to (3.7), then $X(t) = S_t^\eta \xi$. Similarly, we denote by R_t^η the flow on $\mathcal{P}$ defined by $R_t^\eta b_0 = b(t)$, where $b(t)$ is the solution to (3.2) with initial condition $b(0) = b_0$. We will explain shortly how to extend the definitions of the flows S_t^η and R_t^η to continuous paths η, but first we make a note of the important relation between them.

PROPOSITION 3.2. *For smooth $\eta : [0,\infty) \to \mathbb{R}^n$ with $\eta(0) = 0$,*

$$(3.8)\qquad R_t^\eta = f^{-1} \circ S_t^\eta \circ f.$$

Proof. Let $X(0) \in \mathcal{T}$ and, for $t \ge 0$, $X(t) = S_t^\eta X(0)$ and $b(t) = f^{-1}(X(t))$. Let us write $\lambda = \dot{\eta}$. We are required to show that $\dot{b} = \varepsilon_\lambda b$. We will prove this by induction. Write $\mathcal{T} = \mathcal{T}^n$, $f = f^n$, $\mathcal{P} = \mathcal{P}^n$, $\varepsilon = \varepsilon^n$ to emphasize their dependence on n and for $m < n$ denote by S_t^π the flow defined on $\mathcal{T}^m$ defined in the same way as above by a smooth path $\pi : [0,\infty) \to \mathbb{R}^m$.

For $n = 1$, $b = e^{x_1^1}$ so $\dot{x}_1^1 = \lambda_1$ implies $\dot{b} = \lambda_1 b$, as required.

For general n, by Proposition 2.1 we can write

$$b = E_1\big(w^1\big) \cdots E_n\big(w^n\big),$$

where

$$w_1^m = e^{x_1^m}; \qquad w_i^m = e^{x_i^{m+i-1} - x_{i-1}^{m+i-2}}, \quad 1 < i \le n-m+1.$$

Define $Y = (y_i^m) \in \mathcal{T}^{n-1}$ by $y_i^m = x_i^{m+1}$, $1 \le i \le m \le n-1$. In other words, Y is the triangle of size $n-1$ obtained from X by removing $x_1^1, \ldots, x_n^n$. Define a smooth path $\pi : [0,\infty) \to \mathbb{R}^{n-1}$ by setting $\pi(0) = 0$ and $\dot{\pi} = \nu$, where

$$\begin{aligned}(3.9)\qquad &\nu_1 = \lambda_1 + a_1, \qquad \nu_2 = \lambda_2 - a_1 + a_2, \qquad \ldots\\ &\nu_{n-1} = \lambda_{n-1} - a_{n-2} + a_{n-1},\\ &a_i = e^{x_{i+1}^{i+1} - x_i^i}, \qquad 1 \le i \le n-1.\end{aligned}$$

Then, by (3.7), $Y(t) = S_t^{\pi} Y(0)$. Moreover, from the definition of the w^m above, we can write

$$b = E_1(w^1) \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & b^{n-1} & \\ 0 & & & \end{pmatrix},$$

where $b^{n-1} = (f^{n-1})^{-1}(Y)$. We note that $w^1 = (e^{x_1^1}, a_1, \dots, a_{n-1})$ and so, by (3.7),

$$\dot{w}^1 = \big(\lambda_1 w_1^1, (\lambda_2 - \lambda_1 - a_1) w_2^1, \dots, (\lambda_n - \lambda_{n-1} - a_{n-1}) w_n^1\big). \tag{3.10}$$

By the induction hypothesis, $\dot{b}^{n-1} = \varepsilon^{n-1}(\nu) b^{n-1}$. Thus,

$$\begin{aligned} \dot{b} &= \dot{E}_1(w^1) \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & b^{n-1} & \\ 0 & & & \end{pmatrix} + E_1(w^1) \begin{pmatrix} 0 & 0 & & \dots & 0 \\ 0 & & & & \\ \vdots & & \varepsilon^{n-1}(\nu) b^{n-1} & & \\ 0 & & & & \end{pmatrix} \\ &= \left[\dot{E}_1(w^1) + E_1(w^1) \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & \varepsilon^{n-1}(\nu) & \\ 0 & & & \end{pmatrix} \right] \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & b^{n-1} & \\ 0 & & & \end{pmatrix}. \end{aligned}$$

Note that $\dot{E}_1(w^1) = \operatorname{diag}(\dot{w}^1)$. It therefore suffices to show that

$$\operatorname{diag}(\dot{w}^1) + E_1(w^1) \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & \varepsilon^{n-1}(\nu) & \\ 0 & & & \end{pmatrix} = \varepsilon^n(\lambda) E_1(w^1),$$

which is readily verified using (3.9) and (3.10). □

We will now explain how to extend the definition of the flow S_t^{η} to continuous paths $\eta : [0, \infty) \to \mathbb{R}^n$ with $\eta(0) = 0$, by simply solving the equations (3.7) in terms of η and then observing that η need not be smooth in order for the solution to make sense.

Let $\xi = (\xi_i^m) \in \mathcal{T}$ and $\eta : [0, \infty) \to \mathbb{R}^n$ continuous with $\eta(0) = 0$. Denote by $\Pi^{\xi}\eta$ the path $X(t) = (x_i^m(t)), t \geq 0$ in $\mathcal{T}$, defined as follows. Set $\mu_1 = \xi_1^1$,

$$\mu_m = \sum_{i=1}^{m} \xi_i^m - \sum_{i=1}^{m-1} \xi_i^{m-1},$$

and define

$$\pi(t) = \eta(t) + \sum_{i=1}^{n} \mu_i e_i. \tag{3.11}$$

For $2 \le k \le m \le n$, define

$$r_k^m = \sum_{i=1}^{k-1} \xi_i^{m-1} - \sum_{i=1}^{k-1} \xi_i^m. \tag{3.12}$$

Set

$$x_1^1(t) = \pi_1(t), \tag{3.13}$$

and, for $2 \le m \le n$,

$$x_m^m(t) = \pi_m(t) - \log\left[e^{-r_m^m} + \int_0^t e^{\pi_m(s) - x_{m-1}^{m-1}(s)}\, ds\right]. \tag{3.14}$$

For $2 \le m \le n$ and $1 < k < m$,

$$x_k^m(t) = y_k^m(t) - \log\left[e^{-r_k^m} + \int_0^t e^{y_k^m(s) - x_{k-1}^{m-1}(s)}\, ds\right], \tag{3.15}$$

where

$$y_k^m(t) = x_k^{m-1}(t) + \log\left[e^{-r_{k+1}^m} + \int_0^t e^{y_{k+1}^m(s) - x_k^{m-1}(s)}\, ds\right]; \tag{3.16}$$

for $2 \le m \le n$, writing $y_2^2 = \pi_2$,

$$x_1^m(t) = x_1^{m-1}(t) + \log\left[e^{-r_2^m} + \int_0^t e^{y_2^m(s) - x_1^{m-1}(s)}\, ds\right]. \tag{3.17}$$

The mapping Π^ξ can be written more compactly as follows. For $r \in \mathbb{R}$, $i = 1, \dots, n-1$, and continuous $\eta : [0, \infty) \to \mathbb{R}^n$, define

$$\big(P_i^r \eta\big)(t) = \eta(t) + \left(\log\left[e^{-r} + \int_0^t e^{\eta_{i+1}(s) - \eta_i(s)}\, ds\right]\right)(e_i - e_{i+1}).$$

Define Π_1^ξ by $\Pi_1^\xi \eta(t) = \pi(t)$, where π is defined as above by (3.11). For $1 < m \le n$, set

$$\Pi_m^\xi = P_1^{r_2^m} \circ \cdots \circ P_{m-1}^{r_m^m} \circ \Pi_{m-1}^\xi.$$

Then $\Pi^\xi \eta = ((\Pi_m^\xi \eta)_i, 1 \le i \le m \le n)$.

PROPOSITION 3.3. *Let $\eta : [0, \infty) \to \mathbb{R}^n$ be a smooth path with $\eta(0) = 0$. Then, for $t \ge 0$,*

$$S_t^\eta \xi = \Pi^\xi \eta(t).$$

Also, for $t > 0$, $\Pi\eta(t)$ evolves according to (3.7).

Proof. Let $X(t) = (x_i^m(t))$ be defined by (3.13)–(3.17). For convenience, write $y_m^m(t) = \pi_m(t)$ and observe that, for $2 \le k \le m \le n$,

$$\frac{d}{dt} \log\left[e^{-r_k^m} + \int_0^t e^{y_k^m(s) - x_{k-1}^{m-1}(s)}\, ds\right] = e^{x_k^m - x_{k-1}^{m-1}}. \tag{3.18}$$

First note that $\dot{x}_1^1 = \dot{\eta}_1$ and, using (3.18),

$$\dot{x}_m^m = \dot{\eta}_m - e^{x_m^m - x_{m-1}^{m-1}}$$

for $2 \le m \le n$. For $2 \le m \le n$ and $1 < k < m$, by (3.15), (3.16) and (3.18),

$$\dot{x}_k^m = \dot{y}_k^m - e^{x_k^m - x_{k-1}^{m-1}} = \dot{x}_k^{m-1} + e^{x_{k+1}^m - x_k^{m-1}} - e^{x_k^m - x_{k-1}^{m-1}}.$$

For $2 \le m \le n$, by (3.17) and (3.18),

$$\dot{x}_1^m = \dot{x}_1^{m-1} + e^{x_2^m - x_1^{m-1}}.$$

We have thus shown that $X(t)$ satisfies (3.7), and it remains to check that $X(0) = \xi$.

It follows immediately from the definitions (3.11), (3.13) and (3.14) that

$$x_m^m(0) = \mu_m + r_m^m = \xi_m^m$$

for $1 \le m \le n$. For $2 \le m \le n$ and $1 < k < m$, from (3.15) and (3.16),

$$x_k^m(0) = y_k^m(0) + r_k^m = x_k^{m-1}(0) - r_{k+1}^m + r_k^m = x_k^{m-1}(0) + \xi_k^m - \xi_k^{m-1};$$

for $2 \le m \le n$, from (3.17),

$$x_1^m(0) = x_1^{m-1}(0) - r_2^m = x_1^{m-1}(0) + \xi_1^m - \xi_1^{m-1}.$$

It follows that $x_i^m(0) = \xi_i^m$ for $1 \le i \le m \le n$, as required.

The second claim also follows from the above argument, taking $e^{-r_k^m} = 0$ for $2 \le k \le m \le n$. □

Thus, for $\xi \in \mathcal{T}$, $b \in \mathcal{P}$ and continuous $\eta : [0, \infty) \to \mathbb{R}^n$ with $\eta(0) = 0$, we define, for $t \ge 0$,

$$S_t^\eta \xi = \Pi^\xi \eta(t), \qquad R_t^\eta b = f^{-1}\big(S_t^\eta f(b)\big). \tag{3.19}$$

Let $\eta : [0, \infty) \to \mathbb{R}^n$ be a continuous path with $\eta(0) = 0$. Set $b(t) = R_t^\eta b(0)$ with either $b(0) \in \mathcal{P}$ or $b(0) = I$. Then, for each $t > 0$, $b(t) \in \mathcal{P}$ and we can define $X(t) = (x_i^m(t)) = f(b(t))$. If $b(0) = I$ then $X(t) = \Pi\eta(t)$. If $b(0) \in \mathcal{P}$, then $X(t) = S_t^\eta \xi = \Pi^\xi \eta(t)$ where $\xi = f(b(0))$. By Proposition 2.3, for each $t > 0$, $b(t)\bar{w}_0$ has a Gauss decomposition

$$b(t)\bar{w}_0 = L(t)D(t)U(t),$$

where

$$D_{ii} = e^{x_i^n}, \quad 1 \le i \le n$$

and $L \in (N_-)_{>0}$ is given by

$$L = L_1\big(u^1\big)L_2\big(u^2\big)\cdots L_{n-1}\big(u^{n-1}\big),$$
$$u_i^m = e^{x_{i+1}^{m+1} - x_i^m}, \quad 1 \le i \le m < n.$$

For a square matrix A denote by $\Pi_-(A)$ the strictly lower triangular part of A. The next proposition is essentially a special case of [6, Proposition 6.4].

PROPOSITION 3.4. *If η is smooth, then $L(t)$ and $R(t) = D(t)U(t)$ satisfy*

$$\dot{L} = L\Pi_-\big(L^{-1}\varepsilon(\dot{\eta})L\big), \qquad \dot{R} = \varepsilon\big(\dot{x}^n\big)R.$$

Proof. We have $\dot{b} = \varepsilon(\dot{\eta})b$, hence

$$\dot{L}R + L\dot{R} = \varepsilon(\dot{\eta})LR,$$

or, equivalently,

$$L^{-1}\dot{L} + \dot{R}R^{-1} = L^{-1}\varepsilon(\dot{\eta})L.$$

Now, since $\dot{R}R^{-1}$ is upper triangular and $L^{-1}\dot{L}$ is strictly lower triangular, this implies

$$L^{-1}\dot{L} = \Pi_-\big(L^{-1}\dot{L} + \dot{R}R^{-1}\big) = \Pi_-\big(L^{-1}\varepsilon(\dot{\eta})L\big),$$

proving the first claim. Similarly, the strictly upper triangular part of $\dot{R}R^{-1}$ must equal the strictly upper triangular part of $L^{-1}\varepsilon(\dot{\eta})L$, which is just the shift matrix $\varepsilon(0)$; also, since $R = DU$, the diagonal part of $\dot{R}R^{-1}$ is $\dot{D}D^{-1} = \operatorname{diag}(\dot{x}^n)$. Hence, $\dot{R}R^{-1} = \operatorname{diag}(\dot{x}^n) + \varepsilon(0) = \varepsilon(\dot{x}^n)$, as required. □

4. Whittaker functions and the quantum Toda lattice

Following [20], [23], [18], for $X = (x_i^m) \in \mathcal{T}$ and $\lambda \in \mathbb{R}^n$, we define, for $n \geq 2$,

$$\mathcal{F}(X) = \sum_{1 \leq i \leq m < n} e^{x_{i+1}^{m+1} - x_i^m} + e^{x_i^m - x_i^{m+1}} \tag{4.1}$$

and

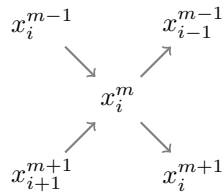
$$\mathcal{F}_\lambda(X) = \sum_{m=1}^{n} \lambda_m \left(\sum_{i=1}^{m-1} x_i^{m-1} - \sum_{i=1}^{m} x_i^m \right) + \mathcal{F}(X). \tag{4.2}$$

If $n = 1$, we set $\mathcal{F}(X) = 0$ and $\mathcal{F}_\lambda(X) = -\lambda_1 x_1^1$.

For some readers, the following graphical representation may be helpful for understanding the above definition, and also for following some of the proofs which will be given later. We view a triangle $X = (x_i^m)$ as an array:

$$X = \begin{array}{ccccccc} & & & x_1^1 & & & \\ & & x_2^2 & & x_1^2 & & \\ & \iddots & & & & \ddots & \\ x_n^n & & \cdots\cdots & & & & x_1^n \end{array}$$

The elements of this array are connected by arrows as shown, with the obvious omissions at the boundary:



To an arrow $a \to b$, we associated the weight e^{a-b}. Then $\mathcal{F}(X)$ is just the sum of the weights associated with the arrows in the diagram of X.

For $x \in \mathbb{R}^n$, denote by $\mathcal{T}(x)$ the set of triangles $X = (x_i^m) \in \mathcal{T}$ with bottom row $x^n = x$. Let $\varepsilon > 0$ and define

$$\psi_\lambda(x) = \int_{\mathcal{T}(x)} e^{-\mathcal{F}_\lambda(X)/\varepsilon} \prod_{1 \le i \le m < n} dx_i^m.$$

These are eigenfunctions of the quantum Toda lattice with Hamiltonian

$$H = -\varepsilon\Delta + \frac{2}{\varepsilon}\sum_{i=1}^{n-1} e^{x_{i+1}-x_i},$$

also known as $GL(n,\mathbb{R})$-Whittaker functions [28], [20], [23], [18], [22].

In [45], it was shown that, for each $x \in \mathbb{R}^n$, the function $\mathcal{F}(X)$ is strictly convex and has a unique critical point on $\mathcal{T}(x)$, which is a minimum. This property extends trivially to $\mathcal{F}_\lambda(X)$, as we are simply adding a linear functional. Denote the corresponding critical point by $X_\lambda^*(x)$. Note that, since $\mathcal{F}(X)$ is strictly convex, $X_\lambda^*(x)$ is a continuous function of λ.

For $X = (x_i^m) \in \mathcal{T}$ and, for $1 \le i \le m < n$, define $l_i^m = l_i^m(X)$ and $r_i^m = r_i^m(X)$ as follows. For $1 \le i < m < n$,

$$l_i^m = e^{x_{i+1}^{m+1} - x_i^m} + e^{x_i^{m-1} - x_i^m},$$

and, for $1 \le m < n$,

$$l_m^m = e^{x_{m+1}^{m+1} - x_m^m}.$$

Similarly, for $1 < i \le m < n$,

$$r_i^m = e^{x_i^m - x_i^{m+1}} + e^{x_i^m - x_{i-1}^{m-1}},$$

and, for $1 \le m < n$,

$$r_1^m = e^{x_1^m - x_1^{m+1}}.$$

For each $x \in \mathbb{R}^n$, the critical point $X = X_\lambda^*(x)$ of $\mathcal{F}_\lambda$ on $\mathcal{T}(x)$ satisfies

$$\lambda_m + l_i^m(X) = \lambda_{m+1} + r_i^m(X), \quad 1 \le i \le m < n. \tag{4.3}$$

For $\lambda \in \mathbb{R}^n$, denote by $\mathcal{T}_\lambda$ the set of $X = (x_i^m) \in \mathcal{T}$ which satisfy the critical point equations (4.3). Note that each element $X = (x_i^m) \in \mathcal{T}_\lambda$ is uniquely determined by its bottom row x^n, via $X = X_\lambda^*(x^n)$.

REMARK 4.1. The critical point equations (4.3) are closely related to the geometric Bender–Knuth transformations introduced in [25]. These are birational involutions b_i^m, $1 \le i \le m < n$ defined on $\mathcal{T}$ as follows: if $X = (x_i^m) \in \mathcal{T}$ then $b_i^m(X)$ is obtained from X by replacing x_i^m with

$$\tilde{x}_i^m = x_i^m + \log \frac{l_i^m(X)}{r_i^m(X)},$$

and leaving the other entries unchanged. Thus, $X \in \mathcal{T}_0$ if, and only if, it is invariant under all of the geometric Bender–Knuth transformations b_i^m, $1 \le i \le m < n$. We note that this implies, in particular, that each $X \in \mathcal{T}_0$ is a fixed point of the geometric lifting of Schutzenberger's involution, defined in [25] as a composition of geometric Bender–Knuth transformations.

5. Geometric RSK and Brownian motion

In this section, we recall some of the main results from [37] which relate the continuous-time geometric RSK mapping, with Brownian motion as input, to the quantum Toda lattice.

THEOREM 5.1 ([37]). *If $\eta(t)$ is a Brownian motion in $\mathbb{R}^n$ with $\eta(0) = 0$, infinitesimal variance ε and drift $\lambda \in \mathbb{R}^n$, then $x(t) = \Pi_n \eta(t)$ is a diffusion process in $\mathbb{R}^n$ with infinitesimal generator given by*

$$\mathcal{L}_\lambda = -\frac{1}{2}\psi_\lambda(x)^{-1}\left(H + \frac{1}{\varepsilon}\sum_i \lambda_i^2\right)\psi_\lambda(x) = \frac{\varepsilon}{2}\Delta + \varepsilon \nabla \log \psi_\lambda \cdot \nabla.$$

We note that in [37] this statement was proved in the case $\varepsilon = 1$, but this is minor modification. In the case $n = 2$, it is equivalent to a Theorem of Matsumoto and Yor [31]. The diffusion process with generator $\mathcal{L}_\lambda$ was first introduced in [3], in the context of more general root systems.

In the paper [37], more general initial conditions were also considered, and two quite different stochastic dynamics on triangles, namely:

$$\begin{aligned}
(5.1)\qquad & dx_1^1 = dB_1 + \lambda_1\, dt;\\
& dx_m^m = dB_m + \left(\lambda_m - e^{x_m^m - x_{m-1}^{m-1}}\right) dt, \quad 2 \le m \le n;\\
& dx_1^m = dx_1^{m-1} + e^{x_2^m - x_1^{m-1}}\, dt, \quad 2 \le m \le n;\\
& dx_i^m = dx_i^{m-1} + \left(e^{x_{i+1}^m - x_i^{m-1}} - e^{x_i^m - x_{i-1}^{m-1}}\right) dt, \quad 1 < i < m \le n
\end{aligned}$$

and

$$\begin{aligned}
(5.2)\qquad & dx_1^1 = dW_1^1 + \lambda_1\, dt;\\
& dx_m^m = dW_m^m + \left(\lambda_m - e^{x_m^m - x_{m-1}^{m-1}}\right) dt, \quad 2 \le m \le n;\\
& dx_1^m = dW_1^m + \left(\lambda_m + e^{x_1^{m-1} - x_1^m}\right) dt, \quad 2 \le m \le n;\\
& dx_i^m = dW_i^m + \left(\lambda_m + e^{x_i^{m-1} - x_i^m} - e^{x_i^m - x_{i-1}^{m-1}}\right) dt, \quad 1 < i < m \le n,
\end{aligned}$$

where $B_i, W_i^m, 1 \le i \le m \le n$ are independent one-dimensional Brownian motions (without drift) and, for the purposes of this discussion, each with infinitesimal variance ε.

The first of these, (5.1), describes the (stochastic) evolution of $\Pi\eta(t)$, or $\Pi^\xi\eta(t)$ any initial condition $\xi \in \mathcal{T}$, where $\eta(t) = B(t) + t\lambda$, as can be seen, for example, from the proof of Proposition 3.3 above. The second dynamic (5.2) is a geometric lifting of Warren's process [46].

THEOREM 5.2 ([37]). *For each $x \in \mathbb{R}^n$, if the initial condition $X(0)$ is chosen at random according to the probability measure on $\mathcal{T}(x)$ with density proportional to $e^{-\mathcal{F}_\lambda(X)/\varepsilon}$ and $X(t)$ evolves according to (5.1) or (5.2), then $x^n(t), t \ge 0$ is a diffusion process with infinitesimal generator $\mathcal{L}_\lambda$, started at x. Moreover, for each $t > 0$, the conditional law of $X(t)$ given $x^n(s), 0 \le s \le t$ is supported on $\mathcal{T}(x^n(t))$ with density proportional to $e^{-\mathcal{F}_\lambda(X)/\varepsilon}$.*

This theorem can be represented as a commutative diagram, as follows. Denote by Q_t^λ the Markov semigroup associated with the diffusion with infinitesimal generator $\mathcal{L}_\lambda$, that is, $Q_t^\lambda = e^{t\mathcal{L}_\lambda}$, by P_t^λ the Markov semigroup associated with the Markov process defined by *either* (5.1) or (5.2), and define Markov kernels Σ_λ from $\mathbb{R}^n$ to $\mathcal{T}$ and π from $\mathcal{T}$ to $\mathbb{R}^n$, by

$$(\Sigma_\lambda f)(x) = \psi_\lambda(x)^{-1} \int_{\mathcal{T}(x)} e^{-\mathcal{F}_\lambda(X)/\varepsilon} f(X) \prod_{1 \le i \le m < n} dx_i^m$$

for suitable $f : \mathcal{T} \to \mathbb{R}$ and, writing $X = (x_i^m) \in \mathcal{T}$,

$$(\pi g)(X) = g\big(x^n\big).$$

Then, according to Theorem 5.2, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{T} & \xleftarrow{\Sigma_\lambda} & \mathbb{R}^n \\ {\scriptstyle P_t^\lambda}\downarrow & & \downarrow{\scriptstyle Q_t^\lambda} \\ \mathcal{T} & \xrightarrow{\pi} & \mathbb{R}^n \end{array}$$

Theorem 5.1 is a generalisation of Pitman's '$2M - X$' theorem [43], which states that, if X_t is a standard one-dimensional Brownian motion and $M_t = \max_{0 \le s \le t} X_s$, then $2M - X$ is a three-dimensional Bessel process. Pitman's theorem was generalised to the type A_{n-1} case (from A_1) in [10], [41] and to arbitrary finite Coxeter groups in [5], [6]. For discrete versions, see [35], [36], [30]. These generalisations are closely related to the RSK correspondence, and also longest increasing subsequences, percolation and queues [1], [2], [21]. Pitman's theorem was extended to the geometric setting by Matsumoto and Yor [31], and Theorem 5.1 can be regarded as a geometric lifting of the generalisations of Pitman's theorem for type A_{n-1} given in [10], [41].

It has been generalised to other types in [12]. A discrete-time version is given in [13] (see also [40]) in the context of Kirillov's geometric RSK mapping on matrices, and a fully discrete q-version, in the context of Ruijenaars q-Toda difference operators and q-Whittaker functions, in [39] (see also [9], [42]).

The second dynamics (5.2) is a geometric lifting of a process on Gelfand–Tsetlin patterns introduced by Warren [46]. Nordenstam [32] showed that a discrete version of Warren's process is in fact closely related to a *shuffling algorithm* which was previously studied in the random tilings literature [15], [44], see also [8], [47], [7]. For some time now, it has been a natural question to understand the relationship between these two types of dynamics. As we will see, there is one setting in which the answer is simple: in the semi-classical ($\varepsilon \to 0$) limit of Theorem 5.2, they are, in fact, equivalent!

6. A semi-classical limit

Theorem 5.1 can be restated as follows: if $\eta(t)$ is a Brownian motion in $\mathbb{R}^n$ with $\eta(0) = 0$, infinitesimal variance ε and drift $\lambda \in \mathbb{R}^n$, then $x(t) = \Pi_n \eta(t)$ is a weak solution to the stochastic differential equation

$$dx = \sqrt{\varepsilon}\, dW + \varepsilon \nabla \log \psi_\lambda(x)\, dt, \tag{6.1}$$

where W is a standard Brownian motion in $\mathbb{R}^n$. From the definition of ψ_λ, formally taking the limit in (6.1) as $\varepsilon \to 0$ yields

$$\dot{x} = -\nabla u_\lambda(x), \tag{6.2}$$

where, in the notation of Section 4,

$$u_\lambda(x) = \mathcal{F}_\lambda\big(X_\lambda^*(x)\big). \tag{6.3}$$

In the case $\lambda = 0$, this gradient flow is discussed by Givental [20], where it is shown to be equivalent to the Toda flow (with opposite sign) on its most degenerate iso-spectral manifold on which the eigenvalues of the Lax matrix are all equal to zero. In fact, the corresponding statement holds true for any $\lambda \in \mathbb{R}^n$, namely, that the gradient flow (6.2) describes the Toda flow on the iso-spectral manifold corresponding to λ. For more details, see Section 7 and Theorem 8.5 below.

On the other hand, when $\varepsilon = 0$, $\eta(t) = t\lambda$ almost surely. This suggests that the image of the path $\eta(t) = t\lambda$ under Π_n defines a solution to the Toda flow on the iso-spectral manifold corresponding to $\lambda_1, \dots, \lambda_n$, and indeed this is the case, as we will show in Theorem 8.5.

For more general initial conditions, in the context of Theorem 5.2, note that when $\varepsilon = 0$, (5.1) becomes

$$\begin{aligned} &\dot{x}_1^1 = \lambda_1; \\ &\dot{x}_1^m = \dot{x}_1^{m-1} + e^{x_2^m - x_1^{m-1}}, \qquad \dot{x}_m^m = \lambda_m - e^{x_m^m - x_{m-1}^{m-1}}, \quad 2 \le m \le n; \\ &\dot{x}_i^m = \dot{x}_i^{m-1} + e^{x_{i+1}^m - x_i^{m-1}} - e^{x_i^m - x_{i-1}^{m-1}}, \quad 1 < i < m \le n \end{aligned} \tag{6.4}$$

and (5.2) becomes

$$
\begin{aligned}
(6.5)\qquad & \dot{x}_1^1 = \lambda_1; \\
& \dot{x}_1^m = \lambda_m + e^{x_1^{m-1} - x_1^m}, \qquad \dot{x}_m^m = \lambda_m - e^{x_m^m - x_{m-1}^{m-1}}, \quad 2 \le m \le n; \\
& \dot{x}_i^m = \lambda_m + e^{x_i^{m-1} - x_i^m} - e^{x_i^m - x_{i-1}^{m-1}}, \quad 1 < i < m \le n.
\end{aligned}
$$

Moreover, the fixed time marginals of the process $X(t)$, which are the same in either case, are concentrated on $\mathcal{T}_\lambda$. As we will see, $\mathcal{T}_\lambda$ is stable under both of the flows defined by (6.4) and (6.5), and on $\mathcal{T}_\lambda$, they are in fact equivalent. Moreover, we will show that if $X(0) \in \mathcal{T}_\lambda$ and $X(t)$ evolves according to either/both, then $x^n(t)$ defines a solution to the Toda flow with opposite sign on the iso-spectral manifold corresponding to λ. This last statement can be interpreted as a semi-classical limit of Theorem 5.2. From this, we will also deduce the semi-classical limit of Theorem 5.1. For precise statements, see Theorems 8.4 and 8.5 below.

7. The Toda lattice

The Toda lattice is completely integrable Hamiltonian system which has been extensively studied in the literature, see, for example, the survey [26]. We will consider the Toda lattice with opposite sign, with Hamiltonian

$$
(7.1)\qquad \frac{1}{2}\sum_{i=1}^{n} p_i^2 - \sum_{i=1}^{n-1} e^{x_{i+1} - x_i}.
$$

This is a special case of the *indefinite* Toda lattice [26], [27]. The equations of motion are

$$
\begin{aligned}
(7.2)\qquad & \ddot{x}_1 = -e^{x_2 - x_1}, \qquad \ddot{x}_n = e^{x_n - x_{n-1}}; \\
& \ddot{x}_i = -e^{x_{i+1} - x_i} + e^{x_i - x_{i-1}}, \quad i = 2, \dots, n-1.
\end{aligned}
$$

Writing $q_i = e^{x_{i+1} - x_i}$ for $1 \le i \le n-1$, set

$$
(7.3)\qquad M = \begin{pmatrix}
p_1 & 1 & 0 & \cdots & 0 \\
-q_1 & p_2 & 1 & \cdots & 0 \\
0 & -q_2 & \ddots & \ddots & \vdots \\
 & & & & 0 \\
\vdots & & & \ddots & 1 \\
0 & \dots & 0 & -q_{n-1} & p_n
\end{pmatrix}
$$

and

$$Q = \Pi_-(M) = \begin{pmatrix} 0 & 0 & \dots & 0 \\ -q_1 & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ & & 0 & 0 \\ 0 & \dots & -q_{n-1} & 0 \end{pmatrix}. \tag{7.4}$$

Then (M, Q) form a Lax pair, that is,

$$\dot{M} = [M, Q] \tag{7.5}$$

and (7.5) is equivalent to the equations of motion (7.2) or, equivalently,

$$\begin{aligned} &\dot{q}_i = (p_{i+1} - p_i)q_i, \quad i = 1, \dots, n-1; \\ &\dot{p}_1 = -q_1; \qquad \dot{p}_n = q_{n-1}; \\ &\dot{p}_i = q_{i+1} - q_i, \quad i = 2, \dots, n-1. \end{aligned} \tag{7.6}$$

In particular, the eigenvalues $\lambda_1, \dots, \lambda_n$ of M form a complete set of integrals of motion for the system. Denote by $\mathcal{M}$ the set of complex, tridiagonal, Hessenberg matrices and by $\mathcal{M}_\lambda$ the subset of those matrices with eigenvalues given by $\lambda = (\lambda_1, \dots, \lambda_n)$.

The relation to the usual Toda lattice with Hamiltonian

$$\frac{1}{2}\sum_{i=1}^{n} \pi_i^2 + \sum_{i=1}^{n-1} e^{\xi_{i+1} - \xi_i}$$

is as follows. If $x(t)$ is a solution to the opposite sign Toda lattice which can be analytically continued in the time variable t then, formally at least, $\xi(t) = x(\iota t)$ defines a solution to the usual Toda lattice, that is

$$\begin{aligned} &\ddot{\xi}_1 = e^{\xi_2 - \xi_1}, \qquad \ddot{\xi}_n = -e^{\xi_n - \xi_{n-1}}; \\ &\ddot{\xi}_i = e^{\xi_{i+1} - \xi_i} - e^{\xi_i - \xi_{i-1}}, \quad i = 2, \dots, n-1. \end{aligned} \tag{7.7}$$

To solve (7.5) for a given initial condition $M_0 \in \mathcal{M}$ we write, for each $t \geq 0$,

$$e^{tM_0} = n(t)r(t),$$

where $n(t) \in N_-$ and $r(t) \in B$, assuming for the moment that such a factorisation is possible. Then

$$M(t) = n(t)^{-1} M_0 n(t) = r(t) M_0 r(t)^{-1}$$

defines a solution to (7.5) with $M(0) = M_0$. The matrices $n(t)$ and $r(t)$ evolve according to

$$\dot{n} = nQ, \qquad \dot{r} = Pr,$$

where $P = M - Q$. Denote the corresponding flow on $\mathcal{M}$ by T_t, so that

$$M(t) = T_t M_0.$$

There may exist times at which the solution blows up, as discussed, for example, in [16], [26], [27].

Finally, we recall an important result of Kostant [29], namely that for any $M \in \mathcal{M}_\lambda$, there is a unique $L \in N_-$ such that

$$M = L^{-1}\varepsilon_\lambda L.$$

Thus, if $M_0 \in \mathcal{M}_\lambda$ with $M_0 = L_0^{-1}\varepsilon_\lambda L_0$, then we can write

$$M(t) = L(t)^{-1}\varepsilon_\lambda L(t),$$

where $L(t) = L_0 n(t)$ evolves according to $\dot{L} = LQ$ with $L(0) = L_0$.

8. Flows on triangles and upper triangular matrices

We consider two flows on $\mathcal{T}$, which we denote by S_t^λ and $\tilde{S}_t^\lambda$, and define by

$$\begin{aligned}
&\dot{x}_1^1 = \lambda_1;\\
&\dot{x}_1^m = \dot{x}_1^{m-1} + e^{x_2^m - x_1^{m-1}}, \qquad \dot{x}_m^m = \lambda_m - e^{x_m^m - x_{m-1}^{m-1}}, \quad 2 \le m \le n;\\
&\dot{x}_i^m = \dot{x}_i^{m-1} + e^{x_{i+1}^m - x_i^{m-1}} - e^{x_i^m - x_{i-1}^{m-1}}, \quad 1 < i < m \le n
\end{aligned} \tag{8.1}$$

and

$$\begin{aligned}
&\dot{x}_1^1 = \lambda_1;\\
&\dot{x}_1^m = \lambda_m + e^{x_1^{m-1} - x_1^m}, \qquad \dot{x}_m^m = \lambda_m - e^{x_m^m - x_{m-1}^{m-1}}, \quad 2 \le m \le n;\\
&\dot{x}_i^m = \lambda_m + e^{x_i^{m-1} - x_i^m} - e^{x_i^m - x_{i-1}^{m-1}}, \quad 1 < i < m \le n
\end{aligned} \tag{8.2}$$

respectively.

PROPOSITION 8.1. *For each $\lambda \in \mathbb{R}^n$, $\mathcal{T}_\lambda$ is invariant under the flows S_t^λ and $\tilde{S}_t^\lambda$ and, moreover, on $\mathcal{T}_\lambda$ these flows are equivalent.*

Proof. We will first show that $\mathcal{T}_\lambda$ is stable under the dynamics (8.1). By (4.3), this is equivalent to showing that $\mathcal{T}_\lambda$ is stable under the dynamics (8.2). Suppose $X \in \mathcal{T}_\lambda$. Then it follows from (8.2), using (4.3), that

$$\dot{x}_i^m = \lambda_{m+1} + e^{x_i^m - x_i^{m+1}} - e^{x_{i+1}^{m+1} - x_i^m}, \quad 1 \le i \le m < n. \tag{8.3}$$

Using (8.2) and (8.3) we have, for $1 < i < m < n$,

$$\begin{aligned}
\dot{l}_i^m &= \big(\dot{x}_i^{m-1} - \dot{x}_i^m\big)e^{x_i^{m-1} - x_i^m} + \big(\dot{x}_{i+1}^{m+1} - \dot{x}_i^m\big)e^{x_{i+1}^{m+1} - x_i^m}\\
&= \big(e^{x_i^m - x_{i-1}^{m-1}} - e^{x_{i+1}^m - x_i^{m-1}}\big)e^{x_i^{m-1} - x_i^m}\\
&\quad + \big(e^{x_{i+1}^m - x_{i+1}^{m+1}} - e^{x_i^m - x_i^{m+1}}\big)e^{x_{i+1}^{m+1} - x_i^m}\\
&= e^{x_i^{m-1} - x_{i-1}^{m-1}} - e^{x_{i+1}^{m+1} - x_i^{m+1}}
\end{aligned}$$

and

$$\begin{aligned}
\dot{r}_i^m &= \big(\dot{x}_i^m - \dot{x}_{i-1}^{m-1}\big)e^{x_i^m - x_{i-1}^{m-1}} + \big(\dot{x}_i^m - \dot{x}_i^{m+1}\big)e^{x_i^m - x_i^{m+1}}\\
&= \big(e^{x_i^{m-1} - x_i^m} - e^{x_{i-1}^{m-1} - x_{i-1}^m}\big)e^{x_i^m - x_{i-1}^{m-1}}
\end{aligned}$$

$$+\left(e^{x_i^{m+1}-x_{i-1}^m}-e^{x_{i+1}^{m+1}-x_i^m}\right)e^{x_i^m-x_i^{m+1}}$$
$$=e^{x_i^{m-1}-x_{i-1}^{m-1}}-e^{x_{i+1}^{m+1}-x_i^{m+1}}.$$

Similarly, for $1\le m<n$, we obtain

$$\dot{l}_1^m=\dot{r}_1^m=-e^{x_2^{m+1}-x_1^{m+1}},\qquad \dot{l}_m^m=\dot{r}_m^m=-e^{x_{m+1}^{m+1}-x_m^{m+1}}.$$

We conclude that $\dot{l}_i^m=\dot{r}_i^m$ for all $1\le i\le m<n$, which shows that $\mathcal{T}_\lambda$ is stable, as required. □

The flow S_t^λ has a convenient representation in terms of the simple linear flow on B defined by

$$R_t^\lambda b=e^{t\varepsilon_\lambda}b.$$

PROPOSITION 8.2. *On* $\mathcal{T}$, $S_t^\lambda=f\circ R_t^\lambda\circ f^{-1}$.

Proof. This follows from Proposition 3.2, taking $\eta(t)=t\lambda$. □

We will now explain how the restriction of the flow S_t^λ to $\mathcal{T}_\lambda$ is related to the Toda flow on $\mathcal{M}_\lambda$.

Define a map $h:\mathcal{T}\to(N_-)_{>0}$, $X=(x_i^m)\mapsto L$, by

$$L=L_1\left(u^1\right)L_2\left(u^2\right)\cdots L_{n-1}\left(u^{n-1}\right),$$

where

$$u_i^m=e^{x_{i+1}^{m+1}-x_i^m},\quad 1\le i\le m<n.$$

Define another map $g_\lambda:\mathcal{T}\to\mathcal{M}$, $X=(x_i^m)\mapsto M$, where M is defined by (7.3) with

$$q_i=e^{x_{i+1}^n-x_i^n},\quad 1\le i\le n-1,$$

and $p=p^n$, where p_i^m, $1\le i\le m\le n$ are defined by

$$p_1^m=p_1^{m-1}+e^{x_2^m-x_1^{m-1}};\qquad p_m^m=\lambda_m-e^{x_n^n-x_{n-1}^{n-1}};$$
$$p_i^m=p_i^{m-1}+e^{x_{i+1}^m-x_i^{m-1}}-e^{x_i^m-x_{i-1}^{m-1}},\quad 1<i<m.$$

Note that, if $X\in\mathcal{T}_\lambda$, then by (4.3), we can write

$$p_1=\lambda_n+e^{x_1^{n-1}-x_1^n};\qquad p_n=\lambda_n-e^{x_n^n-x_{n-1}^{n-1}};$$
$$p_i=\lambda_n+e^{x_i^{n-1}-x_i^n}-e^{x_i^n-x_{i-1}^{n-1}},\quad 1<i<n.$$

PROPOSITION 8.3. *Let* $X=(x_i^m)\in\mathcal{T}$, $M=g_\lambda(X)$ *and* $L=h(X)$. *If* $X\in\mathcal{T}_\lambda$, *then*

$$M=L^{-1}\varepsilon_\lambda L. \tag{8.4}$$

Proof. For $u\in\mathbb{C}^m$, $1\le m<n$, define

$$K_m(u)=\begin{pmatrix}\delta^m(u) & 0\\ 0 & I_{n-m-1}\end{pmatrix},$$

where

$$\delta^m(u) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ u_1 & 1 & 0 & \dots & 0 \\ 0 & u_2 & 1 & \dots & 0 \\ & & \ddots & & \\ 0 & \dots & 0 & u_m & 1 \end{pmatrix}.$$

Note that

$$L_m(u)^{-1} = K_m(-u).$$

Let $X \in \mathcal{T}$ and define

$$\begin{aligned} u_i^m &= e^{x_{i+1}^{m+1} - x_i^m}, \quad 1 \le i \le m < n; \\ v_i^m &= e^{x_i^m - x_i^{m+1}}, \quad 1 \le i \le m < n; \\ q_i^m &= e^{x_{i+1}^m - x_i^m}, \quad 1 \le i < m \le n; \\ p_1^m &= p_1^{m-1} + e^{x_2^m - x_1^{m-1}}; \qquad p_m^m = \lambda_m - e^{x_n^n - x_{n-1}^{n-1}}; \\ p_i^m &= p_i^{m-1} + e^{x_{i+1}^m - x_i^{m-1}} - e^{x_i^m - x_{i-1}^{m-1}}, \quad 1 < i < m. \end{aligned}$$

For $2 \le m \le n$, write $\lambda^m = (\lambda_1, \dots, \lambda_m)$, $L_m(u) = L_m^n(u)$, $K_m(u) = K_m^n(u)$,

$$L^{(m)} = L^m\big(u^1\big) \cdots L_{m-1}^m\big(u^{m-1}\big)$$

and

$$M^{(m)} = \begin{pmatrix} p_1^m & 1 & 0 & \cdots & 0 \\ -q_1^m & p_2^m & 1 & \cdots & 0 \\ 0 & -q_2^m & \ddots & \ddots & \vdots \\ & & & & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \dots & 0 & -q_{m-1}^m & p_m^m \end{pmatrix}.$$

For $v \in \mathbb{C}^m$ write

$$\varepsilon^m(v) = \begin{pmatrix} v_1 & 1 & 0 & \dots & 0 \\ 0 & v_2 & 1 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ & & & v_{m-1} & 1 \\ 0 & & \dots & & v_m \end{pmatrix}.$$

First, we will show that $X \in \mathcal{T}_\lambda$ implies (8.4). We prove this by induction.

For $m=2$, write $q=q_1^2$, $p_i=p_i^2$, $u=u_1^1$, $v=v_1^1$. Note that $q=uv$, $p_1=\lambda_1+u$, $p_2=\lambda_2-u$ and, by (4.3), $\lambda_1+u=\lambda_2+v$. Then

$$\begin{aligned}\bigl(L^{(2)}\bigr)^{-1}\varepsilon^2(\lambda)L^{(2)} &= \begin{pmatrix}1 & 0\\ -u & 1\end{pmatrix}\begin{pmatrix}\lambda_1 & 1\\ 0 & \lambda_2\end{pmatrix}\begin{pmatrix}1 & 0\\ u & 1\end{pmatrix}\\ &= \begin{pmatrix}\lambda_1+u & 1\\ -u\lambda_1+\lambda_2(\lambda_2-u) & \lambda_2-u\end{pmatrix}\\ &= \begin{pmatrix}\lambda_1+u & 1\\ -q & \lambda_2-u\end{pmatrix}=M^{(2)},\end{aligned}$$

as required.

Now fix $2\le m<n$. For $A\in\mathbb{C}^{m\times m}$, write

$$H_m(A)=\begin{pmatrix} & & 0\\ & A & \vdots\\ & & 0\\ & & 1\\ 0 & \dots & 0\end{pmatrix}.$$

Note that

$$L_m^{m+1}\bigl(u^m\bigr)^{-1}=K_m^{m+1}\bigl(-u^m\bigr)=\delta^m\bigl(-u^m\bigr)$$

and

$$u_i^m v_i^m=q_i^{m+1},\quad 1\le i<m.$$

Also, since $X\in\mathcal{T}_\lambda$, (4.3) implies

$$\begin{aligned}p_1^{m+1}&=\lambda_{m+1}+v_1^m;\qquad p_{m+1}^{m+1}=\lambda_{m+1}-u_m^m;\\ p_i^{m+1}&=\lambda_{m+1}+v_i^m-u_{i-1}^m,\quad 1<i<m+1.\end{aligned}$$

Hence,

$$M^{(m+1)}-\lambda_{m+1}I_{m+1}=L_m^{m+1}\bigl(u^m\bigr)^{-1}H_m\bigl(\varepsilon^m\bigl(v^m\bigr)\bigr),$$

and so

$$\begin{aligned}L_m^{m+1}\bigl(u^m\bigr)\bigl[M^{(m+1)}-\lambda_{m+1}I_{m+1}\bigr]L_m^{m+1}\bigl(u^m\bigr)^{-1}\\ =H_m\bigl(\varepsilon^m\bigl(v^m\bigr)\bigr)L_m^{m+1}\bigl(u^m\bigr)^{-1}.\end{aligned}\tag{8.5}$$

Now, by (4.3),

$$p_i^m=\lambda_{m+1}+v_i^m-u_i^m,\quad 1\le i\le m.$$

Using this, and

$$u_i^m v_{i+1}^m=q_i^m,\quad 1\le i<m,$$

the right-hand side of (8.5) becomes

$$H_m\bigl(\varepsilon^m\bigl(v^m\bigr)\bigr)L_m^{m+1}\bigl(u^m\bigr)^{-1}=H_m\bigl(M^{(m)}-\lambda_{m+1}I_m\bigr).$$

By the induction hypothesis,

$$M^{(m)}=\bigl(L^{(m)}\bigr)^{-1}\varepsilon^m\bigl(\lambda^m\bigr)L^{(m)}.$$

Hence, using

$$H_m\big(\varepsilon^m\big(\lambda^m\big)-\lambda_{m+1}I_m\big)=\varepsilon^{m+1}\big(\lambda^{m+1}\big)-\lambda_{m+1}I_{m+1}$$

and

$$\begin{aligned}&H_m\big(\big(L^{(m)}\big)^{-1}AL^{(m)}\big)\\&\quad=\big[L_1^{m+1}\big(u^1\big)\cdots L_{m-1}^{m+1}\big(u^{m-1}\big)\big]^{-1}H_m(A)L_1^{m+1}\big(u^1\big)\cdots L_{m-1}^{m+1}\big(u^{m-1}\big),\end{aligned}$$

we conclude that

$$M^{(m+1)}=\big(L^{(m+1)}\big)^{-1}\varepsilon^{m+1}\big(\lambda^{m+1}\big)L^{(m+1)},$$

as required.

Now we will show that (8.4) implies $X\in\mathcal{T}_\lambda$, again by induction. Write $\mathcal{T}=\mathcal{T}^n$, $\mathcal{T}_\lambda=\mathcal{T}_\lambda^n$ to emphasize their dependence on n. For $1\le m\le n$, write $X^m=(x^1,\dots,x^m)$. □

REMARK 8.1. It can be shown that, in fact, $X\in\mathcal{T}_\lambda$ if, and only if,

$$M^{(m)}=\big(L^{(m)}\big)^{-1}\varepsilon^m\big(\lambda^m\big)L^{(m)},$$

for each $2\le m\le n$.

In the next theorem, in the case $\lambda=0$, the formula (8.7) and the gradient flow representation (8.8) are due to Givental [20].

THEOREM 8.4. *Let $x,\lambda\in\mathbb{R}^n$ and define $X(t)=(x_i^m(t))=S_t^\lambda X(0)$ where $X(0)\in\mathcal{T}_\lambda$. Let $M(t)=g_\lambda(X(t))$, $b(t)=f^{-1}(X(t))$, and*

$$L=L_1\big(u^1\big)L_2\big(u^2\big)\cdots L_{n-1}\big(u^{n-1}\big),$$

where

$$u_i^m=e^{x_{i+1}^{m+1}-x_i^m},\quad 1\le i\le m<n.$$

Let $Q=\Pi_-(M)$ and $P=M-Q$. Then:

(i) *For all $t\ge0$,*

$$M(t)=L(t)^{-1}\varepsilon_\lambda L(t)$$

and we have the Gauss decompositions

$$b(t)\bar{w}_0=L(t)R(t),\qquad e^{tM(0)}=n(t)r(t),$$

where $n(t)=L(0)^{-1}L(t)$, $R(t)\in B$, $r(t)=R(t)R(0)^{-1}$ and these satisfy

$$\dot{L}=LQ,\qquad \dot{R}=PR,\qquad \dot{n}=nQ,\qquad \dot{r}=Pr.$$

In particular, $M(t)$ defines a solution to the Toda flow on $\mathcal{M}_\lambda$.

(ii) *The eigenvalues $\lambda_1,\dots,\lambda_n$ are given by*

$$\lambda_1=\dot{x}_1^1,\qquad \lambda_m=\sum_{i=1}^m\dot{x}_i^m-\sum_{i=1}^{m-1}\dot{x}_i^{m-1},\quad 2\le m\le n. \tag{8.6}$$

(iii) *Writing* $x = x(t) = x^n(t)$,

$$\mathcal{F}(X) = (n-1)\dot{x}_1 + (n-3)\dot{x}_2 + \cdots + (1-n)\dot{x}_n \tag{8.7}$$

and

$$\dot{x} = -\nabla_x u_\lambda(x). \tag{8.8}$$

Proof. (i) First note that, since $b(t) \in \mathcal{P}$ for all $t \geq 0$, by Proposition 2.3, we have the Gauss decomposition $b(t)\bar{w}_0 = L(t)R(t)$ where $L(t)$ is defined as in the statement of the Theorem. It follows from Propositions 8.1 and 8.3 that $M(t) = L(t)^{-1}\varepsilon_\lambda L(t)$ for all $t \geq 0$. Thus, by Proposition 3.4, $\dot{L} = LQ$, which implies the Lax equation $\dot{M} = [M, Q]$. We also have from Proposition 3.4 that $\dot{R} = PR$. Applying Proposition 8.3 at $t = 0$, we have $M(0) = L(0)^{-1}\varepsilon_\lambda L(0)$; it follows, using

$$L(t)R(t) = e^{t\varepsilon_\lambda} L(0)R(0),$$

that

$$e^{tM(0)} = L(0)^{-1}e^{t\varepsilon_\lambda}L(0) = L(0)^{-1}L(t)R(t)R(0)^{-1}.$$

Moreover, defining $n(t) = L(0)^{-1}L(t)$ and $r(t) = R(t)R(0)^{-1}$, we have $\dot{n} = nQ$ and $\dot{r} = Pr$, as required.

Here is another, simple direct proof that $x = x^n$ satisfies the Toda equations (7.2). For convenience, write $y = x^{n-1}$. First suppose $1 < i < n$. By (8.2),

$$\dot{x}_i = \lambda_n + e^{y_i - x_i} - e^{x_i - y_{i-1}}$$

and, by (8.3),

$$\dot{y}_i = \lambda_n + e^{y_i - x_i} - e^{x_{i+1} - y_i}, \qquad \dot{y}_{i-1} = \lambda_n + e^{y_{i-1} - x_{i-1}} - e^{x_i - y_{i-1}}.$$

Hence,

$$\begin{aligned}
\ddot{x}_i &= (\dot{y}_i - \dot{x}_i)e^{y_i - x_i} + (\dot{y}_{i-1} - \dot{x}_i)e^{x_i - y_{i-1}} \\
&= \left(e^{x_i - y_{i-1}} - e^{x_{i+1} - y_i}\right)e^{y_i - x_i} + \left(e^{y_{i-1} - x_{i-1}} - e^{y_i - x_i}\right)e^{x_i - y_{i-1}} \\
&= -e^{x_{i+1} - x_i} + e^{x_i - x_{i-1}}.
\end{aligned}$$

For $i = 1$, we have

$$\dot{x}_1 = \lambda_n + e^{y_1 - x_1}, \qquad \dot{y}_1 = \lambda_n + e^{y_1 - x_1} - e^{x_2 - y_1},$$

and hence

$$\ddot{x}_1 = (\dot{y}_1 - \dot{x}_1)e^{y_1 - x_1} = -e^{x_2 - x_1};$$

for $i = n$,

$$\dot{x}_n = \lambda_n - e^{x_n - y_{n-1}}, \qquad \dot{y}_{n-1} = \lambda_n - e^{x_n - y_{n-1}} + e^{y_{n-1} - x_{n-1}},$$

and we obtain

$$\ddot{x}_n = (\dot{y}_{n-1} - \dot{x}_n)e^{x_n - y_{n-1}} = e^{x_n - x_{n-1}},$$

as required.

Part (ii) follows immediately from (8.1).

(iii) We will prove (8.7) and (8.8) by induction. Write Y_λ^n, $\mathcal{T} = \mathcal{T}^n$, $\mathcal{T}_\lambda = \mathcal{T}_\lambda^n$, $\mathcal{F}_\lambda^n$ and u_λ^n to emphasize their dependence on n. For $1 \le m \le n$, write $\lambda^m = (\lambda_1, \dots, \lambda_m)$ and $X^m = (x^1, \dots, x^m) \in \mathcal{T}_{\lambda^m}$. We will show that, for each $1 \le m \le n$,

$$\mathcal{F}^m\big(X^m\big) = (m-1)\dot{x}_1^m + (m-3)\dot{x}_2^m + \cdots + (1-m)\dot{x}_m^m \tag{8.9}$$

and

$$\dot{x}^m = -\nabla_{x^m} u_{\lambda^m}^m\big(x^m\big). \tag{8.10}$$

First, we show (8.9). For $m = 1$, $\mathcal{F}(X) = 0$ and the result holds trivially. Let $m > 1$ and write $x = x^m$ and $y = x^{m-1}$.

$$\mathcal{F}^2\big(X^2\big) = e^{x_2 - y_1} + e^{y - x_1}.$$

By (8.2), $\dot{x}_1 = \lambda_2 + e^{y-x_1}$ and $\dot{x}_2 = \lambda_2 - e^{x_2 - y}$, and so $\mathcal{F}^2(X^2) = \dot{x}_1 - \dot{x}_2$, as required. Now suppose $m > 2$. Note that

$$\mathcal{F}^m\big(X^m\big) = \mathcal{F}^{m-1}\big(X^{m-1}\big) + \mathcal{E}^m,$$

where

$$\mathcal{E}^m = e^{y_1 - x_1} + e^{x_2 - y_1} + \cdots + e^{x_m - y_{m-1}}.$$

By the induction hypothesis,

$$\mathcal{F}^{m-1}\big(X^{m-1}\big) = (m-2)\dot{y}_1 + (m-4)\dot{y}_2 \cdots (2-m)\dot{y}_{m-1}.$$

Adding $\mathcal{E}_m$ and using (8.3), then (8.2), gives

$$\begin{aligned}\mathcal{F}^m\big(X^m\big) &= (m-1)e^{y_1 - x_1} \\ &\quad + (m-3)\big(e^{y_2 - x_2} - e^{x_2 - y_1}\big) + \cdots + (1-m)\big(-e^{x_m - y_{m-1}}\big) \\ &= (m-1)\dot{x}_1 + (m-3)\dot{x}_2 + \cdots + (1-m)\dot{x}_m,\end{aligned}$$

as required.

Now we will prove (8.10). For $m = 1$, $u_\lambda(x) = -\lambda_1 x_1^1$ and the result holds trivially. Let $m > 1$ and write $x = x^m$ and $y = x^{m-1}$. Set

$$\mathcal{E}_a^m = \mathcal{E}^m + a\left(\sum_{j=1}^{m-1} y_j - \sum_{i=1}^{m} x_i\right),$$

and note that

$$u_{\lambda^m}^m(x) = u_{\lambda^{m-1}}^{m-1}(y) + \mathcal{E}_{\lambda^m}^m.$$

First, suppose $1 < i < m$. Note that

$$\frac{\partial \mathcal{E}_{\lambda^m}^m}{\partial x_i} = \sum_{j=1}^{m-1} \frac{\partial \mathcal{E}_{\lambda^m}^m}{\partial y_j} \frac{\partial y_j}{\partial x_i} + e^{x_i - y_{i-1}} - e^{y_i - x_i} - \lambda_m.$$

On the other hand, by the induction hypothesis and (8.3),

$$\frac{\partial u_{\lambda^{m-1}}^{m-1}(y)}{\partial y_j} = -\dot{y}_j = -\lambda_m - e^{y_i - x_i} + e^{x_{i+1} - y_i} = -\frac{\partial \mathcal{E}_{\lambda^m}^m}{\partial y_j}.$$

Thus,

$$\begin{aligned}\partial_{x_i} u_\lambda(x) &= \sum_{j=1}^{m-1} \frac{\partial u_{\lambda^{m-1}}^{m-1}(y)}{\partial y_j} \frac{\partial y_j}{\partial x_i} + \sum_{j=1}^{m-1} \frac{\partial \mathcal{E}_{\lambda^m}^m}{\partial y_j} \frac{\partial y_j}{\partial x_i} + e^{x_i - y_{i-1}} - e^{y_i - x_i} - \lambda_m \\ &= e^{x_i - y_{i-1}} - e^{y_i - x_i} - \lambda_m = -\dot{x}_i,\end{aligned}$$

as required. The cases $i = 1$ and $i = m$ are similar. □

We note that, as this is a recursive construction, implicit in the statement of Theorem 8.4 is the statement that, for each $m \le n$, $x^m(t)$ defines a solution to the m-particle Toda flow on the iso-spectral manifold corresponding to $\lambda_1, \dots, \lambda_m$.

Note, in particular, the above shows that $T_t \circ g_\lambda = g_\lambda \circ S_t^\lambda$ on $\mathcal{T}_\lambda$. To summarise, if we let $\mathcal{P}_\lambda = f^{-1}\mathcal{T}_\lambda$, then the following diagram commutes:

$$\begin{array}{ccccc} \mathcal{P}_\lambda & \xrightarrow{f} & \mathcal{T}_\lambda & \xrightarrow{g_\lambda} & \mathcal{M}_\lambda \\ \Big\downarrow{\scriptstyle R_t^\lambda} & & \Big\downarrow{\scriptstyle S_t^\lambda} & & \Big\downarrow{\scriptstyle T_t} \\ \mathcal{P}_\lambda & \xrightarrow{f} & \mathcal{T}_\lambda & \xrightarrow{g_\lambda} & \mathcal{M}_\lambda \end{array}$$

In particular, the semi-classical limit of the commutative diagram shown at the end of Section 6 is:

$$\begin{array}{ccc} \mathcal{T}_\lambda & \xleftarrow{X_\lambda^*} & \mathbb{R}^n \\ \Big\downarrow{\scriptstyle S_t^\lambda} & & \Big\downarrow{\scriptstyle T_t^\lambda} \\ \mathcal{T}_\lambda & \xrightarrow{\pi} & \mathbb{R}^n \end{array}$$

where now T_t^λ denotes the gradient flow defined by (8.8) and, with a slight abuse of notation, π denotes the projection $\pi : X = (x_i^m) \mapsto x^n$.

Example 8.1. Suppose $n = 2$ and $x = \lambda = 0$. Then, on $\mathcal{T}(x)$,

$$\mathcal{F}_0(X) = \mathcal{F}(X) = e^{-x_1^1} + e^{x_1^1}.$$

This has its unique critical point at $x_1^1 = 0$, and so

$$X(0) = \begin{matrix} x_1^1(0) & \\ x_2^2(0) & \end{matrix} \quad \begin{matrix} & \\ x_1^2(0) & \end{matrix} = X_0^*(0,0) = \begin{matrix} 0 & \\ 0 & \end{matrix} \quad \begin{matrix} & \\ & 0 \end{matrix}.$$

Setting $b(0) = f^{-1}(X_0^*(0,0))$, this implies

$$\log \Delta_1^2\big(b(0)\big) = \log \Delta_2^2\big(b(0)\big) = \log \Delta_1^1\big(b(0)\big) = 0$$

and hence

$$b(0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Now,

$$\varepsilon_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad e^{t\varepsilon_0} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

and hence

$$b(t) = e^{t\varepsilon_0} b(0) = \begin{pmatrix} 1 & 1+t \\ 0 & 1 \end{pmatrix}.$$

Now, applying the map f again gives

$$X(t) = \begin{matrix} & 0 \\ -\log(1+t) & \end{matrix} \qquad \log(1+t),$$

and hence

$$x^2(t) = \big(\log(1+t), -\log(1+t)\big).$$

Note that this gives

$$p_1 = \frac{1}{1+t}, \qquad p_2 = -\frac{1}{1+t}, \qquad q = e^{x_2^2 - x_1^2} = \frac{1}{(1+t)^2}.$$

For the usual Toda lattice this gives the solution

$$\xi(t) = \big(\log(1+\iota t), -\log(1+\iota t)\big).$$

The symmetric form of the Lax matrix in this case is given by

$$\Lambda = \begin{pmatrix} \pi_1 & e^{(\xi_2 - \xi_1)/2} \\ e^{(\xi_2 - \xi_1)/2} & \pi_2 \end{pmatrix} = \begin{pmatrix} \frac{\iota}{1+\iota t} & \frac{1}{1+\iota t} \\ \frac{1}{1+\iota t} & -\frac{\iota}{1+\iota t} \end{pmatrix}$$

which, at $t = 0$, is given by

$$\Lambda(0) = \begin{pmatrix} \iota & 1 \\ 1 & -\iota \end{pmatrix}.$$

EXAMPLE 8.2. Suppose $n = 2$, $x = (x, -x)$ and $\lambda = (\lambda, -\lambda)$. Then

$$X_\lambda^*(x) = \begin{matrix} & y \\ -x & x \end{matrix},$$

where $y \in \mathbb{R}$ is the unique solution to

$$\lambda + e^{-x-y} = -\lambda + e^{y-x},$$

that is

$$e^{-y} = \sqrt{\lambda^2 e^{2x} + 1} - \lambda e^x.$$

Now

$$b(0) = \begin{pmatrix} e^y & e^x \\ 0 & e^{-y} \end{pmatrix}, \qquad \varepsilon_\lambda = \begin{pmatrix} \lambda & 1 \\ 0 & -\lambda \end{pmatrix},$$

$$e^{t\varepsilon_\lambda} = \begin{pmatrix} e^{\lambda t} & \frac{1}{\lambda}\sinh(\lambda t) \\ 0 & e^{-\lambda t} \end{pmatrix},$$

and hence

$$b(t) = e^{t\varepsilon_\lambda} b(0) = \begin{pmatrix} e^{y+\lambda t} & e^{x+\lambda t} + e^{-y}\frac{1}{\lambda}\sinh(\lambda t) \\ 0 & e^{-y-\lambda t} \end{pmatrix}.$$

The solution is thus given by

$$x_1^2(t) = \log\left(e^{x+\lambda t} + e^{-y}\frac{1}{\lambda}\sinh(\lambda t)\right),$$

and $x_2^2(t) = -x_1^2(t)$. Note that

$$q(t) = e^{x_2^2(t) - x_1^2(t)} = \left(e^{x+\lambda t} + e^{-y}\frac{1}{\lambda}\sinh(\lambda t)\right)^{-2}.$$

Note that, when $x \to -\infty$, $y \to 0$ and $b(0) \to I$, and the solution becomes

$$x_1^2(t) = \log\left(\frac{1}{\lambda}\sinh(\lambda t)\right),$$

as discussed in Example 8.3 below.

Finally, we consider the flow R_t^λ on B started from the identity.

THEOREM 8.5. *Let $\lambda \in \mathbb{R}^n$ and $b(t) = e^{t\varepsilon_\lambda}$. Note that $f(b) = \Pi\eta$, where $\eta(t) = t\lambda$. Then $\Pi\eta(t) \in \mathcal{T}_\lambda$ for $t > 0$ and $g_\lambda(\Pi\eta(t)), t > 0$ defines a solution to the Toda flow on $\mathcal{M}_\lambda$ which is singular at $t = 0$. All of the other conclusions of Theorem 8.4 also hold for $t > 0$ with $X(t) = (x_i^m(t)) = \Pi\eta(t)$. In this case, $b(t)$ is given explicitly by $b_{ii}(t) = e^{\lambda_i t}$ and, for $i < j$,*

$$b_{ij}(t) = \sum_{i \le k \le j} \left[\prod_{l=i,\dots,j; l\ne k} (\lambda_k - \lambda_l)^{-1}\right] e^{\lambda_k t} = \frac{1}{2\pi\iota}\oint \frac{e^{tz}\,dz}{\prod_{i\le k\le j}(z-\lambda_k)}, \tag{8.11}$$

where the integration is anti-clockwise around a circle containing $\lambda_i, \dots, \lambda_j$. Alternatively, for $i < j$, we can write

$$b_{ij}(t) = \left[\prod_{i\le k<l\le j} (\lambda_k - \lambda_l)\right]^{-1} \begin{vmatrix} e^{\lambda_i t} & \lambda_i^{j-i-1} & \dots & \lambda_i & 1 \\ e^{\lambda_{i+1} t} & \lambda_{i+1}^{j-i-1} & \dots & \lambda_{i+1} & 1 \\ & & \vdots & & \\ e^{\lambda_j t} & \lambda_j^{j-i-1} & \dots & \lambda_j & 1 \end{vmatrix}. \tag{8.12}$$

We recall that, by definition,

$$x_1^n + \cdots + x_k^n = \log \tau_k, \quad 1 \le k \le n,$$

where $\tau_1 = b_{1n}$ *and, for* $2 \le k \le n$,

$$\tau_k = \begin{vmatrix} b_{1,n-k+1} & \dots & b_{1,n-1} & b_{1n} \\ b_{2,n-k+1} & \dots & b_{2,n-1} & b_{2n} \\ & \vdots & & \\ b_{k,n-k+1} & \dots & b_{k,n-1} & b_{kn} \end{vmatrix}. \tag{8.13}$$

Writing $\tau_1 = \tau$, *this solution can also be expressed in the more familiar form*

$$\tau_k = \begin{vmatrix} \tau^{(k-1)} & \dots & \tau' & \tau \\ \tau^{(k)} & \dots & \tau'' & \tau' \\ & \vdots & & \\ \tau^{(2k-2)} & \dots & & \tau^{(k-1)} \end{vmatrix}. \tag{8.14}$$

As matrix integrals, assuming $|\lambda_i| < 1$ *for each* i,

$$\tau_k = \int_{U(k)} \frac{(\det M)^{k-1} e^{t \operatorname{tr} M}}{\prod_{i=1}^n \det(M - \lambda_i I)}\, dM. \tag{8.15}$$

When $\lambda = 0$,

$$\tau_k = \frac{(k-1)!(k-2)! \cdots 1}{(n-1)!(n-2)! \cdots (n-k)!} t^{k(n-k)}. \tag{8.16}$$

Proof. Let $\xi(N) = X_\lambda^*(-N\rho^n)$, where $\rho^n = (n-1, n-3, \dots, 1-n)$, and set $X^N(t) = S_t^\lambda \xi(N)$. By Propositions 3.3 and 8.1, $X^N(t) = \Pi^{\xi(N)}\eta(t) \in \mathcal{T}_\lambda$ for each $t > 0$.

For $2 \le k \le m \le n$, define

$$r_k^m(N) = \sum_{i=1}^{k-1} \xi_i^{m-1}(N) - \sum_{i=1}^{k-1} \xi_i^m(N).$$

We will show that, as $N \to \infty$, $r_i^m(N) \to +\infty$ for each $2 \le i \le m \le n$. Note that this implies that $\Pi^{\xi(N)}(\eta)(t)$ converges to $\Pi(\eta)(t)$ for each $t > 0$, and hence that $\Pi(\eta)(t) \in \mathcal{T}_\lambda$ for each $t > 0$. But $\Pi(\eta)(t)$ also evolves according to (8.1) for $t > 0$, so by Theorem 8.4, $g_\lambda(\Pi\eta(t)), t > 0$ defines a solution to the Toda flow on $\mathcal{M}_\lambda$, as claimed. For convenience, write $X^N(0) = X = (x_i^m)$. For $1 \le m \le n$, define $\tilde{X} = (\tilde{x}_i^m) \in \mathcal{T}$ by $\tilde{x}^m = x^m + N\rho^m$. Note that $\tilde{x}^n = 0$. Then $\mathcal{F}_\lambda(X) = e^N \mathcal{F}_{e^{-N}\lambda}(\tilde{X})$ and $\tilde{X} = X^*_{e^{-N}\lambda}(0)$. Thus, as $N \to \infty$, $\tilde{X} \to X_0^*(0)$. It follows that, as $N \to \infty$, for each $1 \le i \le m \le n$, $x_i^m \sim -N\rho_i^m$ and hence $r_k^m(N) \sim N(k-1) \to \infty$, as required.

We verify the formula (8.11) by induction. Without loss of generality, we only need to show that the formula holds for b_{1n}, $n \ge 2$. As $b(t)$ is a continuous function of λ we can also assume, for convenience, that the λ_i are distinct. Write $b_n(t) = b_{1n}(t)$. First, we note that

$$b_2(t) = \int_0^t e^{\lambda_2 s + \lambda_1(t-s)}\, ds = \frac{1}{\lambda_1 - \lambda_2}\left(e^{\lambda_1 t} - e^{\lambda_2 t}\right).$$

Now assume that, for each $t>0$,

$$b_{n-1}(t)=\sum_{1\le k\le n-1}\Bigg[\prod_{l=1,\dots,n-1;l\neq k}(\lambda_k-\lambda_l)^{-1}\Bigg]e^{\lambda_k t}.$$

From the definition (3.1), we can write

$$\begin{aligned}
b_{1n}(t)&=\int_0^t b_{n-1}(s)e^{\lambda_n(t-s)}\,ds\\
&=\sum_{k=1}^{n-1}\Bigg[\prod_{l=1,\dots,n-1;l\neq k}(\lambda_k-\lambda_l)^{-1}\Bigg]\Bigg[\int_0^t e^{(\lambda_k-\lambda_n)s}\,ds\Bigg]e^{\lambda_n t}\\
&=\sum_{k=1}^{n-1}\Bigg[\prod_{l=1,\dots,n;l\neq k}(\lambda_k-\lambda_l)^{-1}\Bigg]\big[e^{(\lambda_k-\lambda_n)t}-1\big]e^{\lambda_n t}\\
&=\sum_{k=1}^{n-1}\Bigg[\prod_{l=1,\dots,n;l\neq k}(\lambda_k-\lambda_l)^{-1}\Bigg]\big[e^{\lambda_k t}-e^{\lambda_n t}\big].
\end{aligned}$$

It therefore suffices to show that

$$\sum_{k=1}^{n-1}\Bigg[\prod_{l=1,\dots,n;l\neq k}(\lambda_k-\lambda_l)^{-1}\Bigg]=-\prod_{1\le k\le n-1}(\lambda_n-\lambda_k)^{-1}$$

or, equivalently,

$$\sum_{k=1}^{n}\Bigg[\prod_{l=1,\dots,n;l\neq k}(\lambda_k-\lambda_l)^{-1}\Bigg]=0. \tag{8.17}$$

To see that this holds, denote

$$\Delta_m(a_1,\dots,a_m)=\prod_{1\le i<j\le m}(a_i-a_j)=\det\big[a_i^{n-j}\big]_{i,j=1,\dots,m}$$

and note that

$$\begin{aligned}
\Delta_n(\lambda)\sum_{k=1}^{n}\Bigg[\prod_{l=1,\dots,n;l\neq k}(\lambda_k-\lambda_l)^{-1}\Bigg]&=\sum_{i=1}^{n}(-1)^{i-1}\Delta_{n-1}(\lambda_1,\dots,\widehat{\lambda_i},\dots,\lambda_n)\\
&=\begin{vmatrix}1&\lambda_1^{n-2}&\dots&\lambda_1&1\\1&\lambda_2^{n-2}&\dots&\lambda_2&1\\&&\vdots&&\\1&\lambda_n^{n-2}&\dots&\lambda_n&1\end{vmatrix}=0,
\end{aligned}$$

which implies (8.17). Note that essentially the same calculation yields the alternative formula (8.12).

Set $\tau_1=b_{1n}$ and, for $2\le k\le n$,

$$\tau_k=\det[b_{ij}]_{1\le i\le k,m-k+1\le j\le m}.$$

Then

$$x_1^n + \cdots + x_k^n = \log \tau_k, \quad 1 \le k \le n,$$

and since x^n satisfies the Toda equations (7.2), this implies

$$(\log \tau_k)'' = -\frac{\tau_{k+1}\tau_{k-1}}{\tau_k^2}$$

for each $1 \le k \le n$ with the conventions $\tau_0 = 1$ and $\tau_{n+1} = 0$.

On the other hand, the tau functions $\tilde{\tau}_k$ defined by $\tilde{\tau}_0 = 1$, $\tilde{\tau}_1 = \tau_1$,

$$\tilde{\tau}_k = \det\left[\tau_1^{(k+i-j-1)}\right]_{1 \le i,j \le k}, \quad k \ge 2$$

also satisfy

$$(\log \tilde{\tau}_k)'' = -\frac{\tilde{\tau}_{k+1}\tilde{\tau}_{k-1}}{\tilde{\tau}_k^2}.$$

This is well known and is easily verified using basic properties of Wronskians and Sylvester's identity. It follows, by induction, that $\tilde{\tau}_k = \tau_k$ for $1 \le k \le n$, and in fact $\tilde{\tau}_k = 0$ for $k > n$.

To obtain the last formula (8.16), note that when $\lambda = 0$, $b_{ij}(t) = t^{j-i}/(j-i)!$ for $i < j$, hence

$$\begin{aligned}\tau_k &= \det\left[\frac{t^{n-k+j-i}}{(n-k+j-i)!}1_{n-k+j-i\ge 0}\right]_{i,j=1,\dots,k} \\ &= \frac{(k-1)!(k-2)!\cdots 1}{(n-1)!(n-2)!\cdots(n-k)!}t^{k(n-k)}\det\left[\binom{n-i}{k-j}\right]_{i,j=1,\dots,k},\end{aligned}$$

with the convention that $\binom{a}{b} = 0$ if $b > a$. Now, by a theorem of Gessel and Viennot [19],

$$\det\left[\binom{n-i}{k-j}\right]_{i,j=1,\dots,k} = 1,$$

so we are done. □

EXAMPLE 8.3. Suppose $n = 2$ and $\lambda = (\lambda, -\lambda)$. Then

$$x(t) = \left(\log\left[\frac{1}{\lambda}\sinh(\lambda t)\right], -\log\left[\frac{1}{\lambda}\sinh(\lambda t)\right]\right).$$

This yields the solution

$$\xi(t) = \left(\log\left[\frac{\iota}{\lambda}\sin(\lambda t)\right], -\log\left[\frac{\iota}{\lambda}\sin(\lambda t)\right]\right)$$

of the usual Toda lattice. Note that

$$e^{\xi_2 - \xi_1} = -\frac{\lambda^2}{\sin^2(\lambda t)}.$$

We can also take λ to purely imaginary, $\lambda = \iota\gamma$ say, where $\gamma \in \mathbb{R}$. Then

$$e^{\xi_2 - \xi_1} = -\frac{\gamma^2}{\sinh^2(\gamma t)}.$$

This particular singular solution of the usual Toda lattice is discussed, for example, in [11]. When $\lambda = 0$,

$$x(t) = (\log t, -\log t), \qquad \xi(t) = (\iota\pi/2 + \log t, -\iota\pi/2 - \log t),$$

and

$$e^{\xi_2 - \xi_1} = -1/t^2.$$

Acknowledgments. Thanks to Mark Adler, Percy Deift, Nick Ercolani, Govind Menon and Pierre van Moerbeke for helpful discussions. This research was supported in part by EPSRC grant number EP/I014829/1.

References

[1] J. Baik, P. Deift and K. Johansson, *On the distribution of the length of the longest increasing subsequence of random permutations*, J. Amer. Math. Soc. **12** (1999), 1119–1178. MR 1682248

[2] Y. Baryshnikov, *GUEs and queues*, Probab. Theory Related Fields **119** (2001), 256–274. MR 1818248

[3] F. Baudoin and N. O'Connell, *Exponential functionals of Brownian motion and class one Whittaker functions*, Ann. Inst. Henri Poincaré Probab. Stat. **47** (2011), 1096–1120. MR 2884226

[4] A. Berenstein, S. Fomin and A. Zelevinsky, *Parameterizations of canonical bases and totally positive matrices*, Adv. Math. **122** (1996), 49–149. MR 1405449

[5] Ph. Biane, Ph. Bougerol and N. O'Connell, *Littelmann paths and Brownian paths*, Duke Math. J. **130** (2005), 127–167. MR 2176549

[6] Ph. Biane, Ph. Bougerol and N. O'Connell, *Continuous crystals and Duistermaat–Heckman measure for Coxeter groups*, Adv. Math. **221** (2009), 1522–1583. MR 2522427

[7] A. Borodin and I. Corwin, *Macdonald processes*, Probab. Theory Related Fields **158** (2014), 225–400. MR 3152785

[8] A. Borodin and P. Ferrari, *Large time asymptotics of growth models on space-like paths. I. PushASEP*, Electron. J. Probab. **13** (2008), 1380–1418. MR 2438811

[9] A. Borodin and L. Petrov, *Nearest neighbor Markov dynamics on Macdonald processes*, available at `arXiv:1305.5501`.

[10] Ph. Bougerol and Th. Jeulin, *Paths in Weyl chambers and random matrices*, Probab. Theory Related Fields **124** (2002), 517–543. MR 1942321

[11] L. Casian and Y. Kodama, *Singular structure of Toda lattices and cohomology of certain compact Lie groups*, J. Comput. Appl. Math. **202** (2007), 56–79. MR 2301813

[12] R. Chhaibi, *Modèle de Littelmann pour cristaux géométriques, fonctions de Whittaker sur des groupes de Lie et mouvement brownien*, Ph.D. thesis, Université Paris VI—Pierre et Marie Curie, 2012.

[13] I. Corwin, N. O'Connell, T. Seppäläinen and N. Zygouras, *Tropical combinatorics and Whittaker functions*, Duke Math. J. **163** (2014), 513–563. MR 3165422

[14] P. Diaconis and J. Fill, *Strong stationary times via a new form of duality*, Ann. Probab. **18** (1990), 1483–1522. MR 1071805

[15] N. Elkies, G. Kuperberg, M. Larsen and J. Propp, *Alternating-sign matrices and domino tilings. II*, J. Algebraic Combin. **1** (1992), no. 3, 219–234. MR 1194076

[16] N. M. Ercolani, H. Flaschka and S. Singer, *The geometry of the full Toda lattice*, Progr. Math. **115** (1993), 181–226. MR 1279823

[17] S. Fomin and A. Zelevinsky, *Double Bruhat cells and total positivity*, J. Amer. Math. Soc. **12** (1999), 335–380. MR 1652878

[18] A. Gerasimov, S. Kharchev, D. Lebedev and S. Oblezin, *On a Gauss–Givental representation of quantum Toda chain wave equation*, Int. Math. Res. Not. IMRN (2006), 1–23. MR 2219213
[19] I. Gessel and G. Viennot, *Binomial determinants, paths, and hook length formulae*, Adv. in Math. **58** (1985), 300–321. MR 0815360
[20] A. Givental, *Stationary phase integrals, quantum Toda lattices, flag manifolds and the mirror conjecture*, Topics in singularity theory, AMS Transl. Ser. 2, vol. 180, Amer. Math. Soc., Providence, RI, 1997, 103–115. MR 1767115
[21] J. Gravner, C. A. Tracy and H. Widom, *Limit theorems for height fluctuations in a class of discrete space and time growth models*, J. Stat. Phys. **102** (2001), 1085–1132. MR 1830441
[22] T. Ishii and E. Stade, *New formulas for Whittaker functions on $GL(n,\mathbb{R})$*, J. Funct. Anal. **244** (2007), 289–314. MR 2294485
[23] D. Joe and B. Kim, *Equivariant mirrors and the Virasoro conjecture for flag manifolds*, Int. Math. Res. Not. IMRN **15** (2003), 859–882. MR 1952524
[24] L. Jones and N. O'Connell, *Weyl chambers, symmetric spaces and number variance saturation*, ALEA Lat. Am. J. Probab. Math. Stat. **2** (2006), 91–118. MR 2249664
[25] A. N. Kirillov, *Introduction to tropical combinatorics*, Physics and combinatorics, Proc. Nagoya 2000 2nd internat. workshop (A. N. Kirillov and N. Liskova, eds.), World Scientific, Singapore, 2001, pp. 82–150. MR 1872253
[26] Y. Kodama and B. A. Shipman, *The finite non-periodic Toda lattice: A geometric and topological viewpoint*, to appear in Modern Encyclopedia of Mathematical Physics (MEMPhys); available at `arXiv:0805.1389`.
[27] Y. Kodama and J. Ye, *Toda lattices with indefinite metric II: Topology of the isospectral manifolds*, Phys. D **121** (1998), 89–108. MR 1636697
[28] B. Kostant, *Quantisation and representation theory*, Representation theory of Lie groups, Proc. SRC/LMS Research Symposium, Oxford 1977, LMS Lecture Notes, vol. 34, Cambridge University Press, Cambridge, 1977, pp. 287–316.
[29] B. Kostant, *The solution to a generalized Toda lattice and representation theory*, Adv. Math. **34** (1979), 195–338. MR 0550790
[30] C. Lecouvey, E. Lesigne and M. Peigné, *Random walks in Weyl chambers and crystals*, Proc. Lond. Math. Soc. (3) **104** (2012), 323–358. MR 2880243
[31] H. Matsumoto and M. Yor, *A version of Pitman's $2M - X$ theorem for geometric Brownian motions*, C. R. Math. Acad. Sci. Paris **328** (1999), 1067–1074. MR 1696208
[32] E. Nordenstam, *On the shuffling algorithm for domino tilings*, Electron. J. Probab. **15** (2010), 75–95. MR 2578383
[33] M. Noumi and Y. Yamada, *Tropical Robinson–Schensted–Knuth correspondence and birational Weyl group actions*, Adv. Stud. Pure Math. **40** (2004), 371–442. MR 2074600
[34] N. O'Connell, *Random matrices, non-colliding processes and queues*, Séminaire de Probabilités XXXVI, Lecture Notes in Math., vol. 1801, Springer, Berlin, 2003, pp. 165–182. MR 1971584
[35] N. O'Connell, *A path-transformation for random walks and the Robinson–Schensted correspondence*, Trans. Amer. Math. Soc. **355** (2003), 3669–3697. MR 1990168
[36] N. O'Connell, *Conditioned random walks and the RSK correspondence*, J. Phys. A **36** (2003), no. 12, 3049–3066. MR 1986407
[37] N. O'Connell, *Directed polymers and the quantum Toda lattice*, Ann. Probab. **40** (2012), 437–458. MR 2952082
[38] N. O'Connell, *Whittaker functions and related stochastic processes*, Random matrix theory, interacting particle systems and integrable systems, MSRI Publications, vol. 65, 2014.
[39] N. O'Connell and Y. Pei, *A q-weighted version of the Robinson–Schensted algorithm*, Electron. J. Probab. **18** (2013), 1–25. MR 3126578

[40] N. O'Connell, T. Seppäläinen and N. Zygouras, *Geometric RSK correspondence, Whittaker functions and symmetrized random polymers*, Invent. Math. **197** (2014), 361–416. MR 3232009

[41] N. O'Connell and M. Yor, *A representation for non-colliding random walks*, Electron. Commun. Probab. **7** (2002), 1–12. MR 1887169

[42] Y. Pei, *A symmetry property of q-weighted Robinson–Schensted and other branching insertion algorithms*, J. Algebraic Combin. (2014), doi:10.1007/s10801-014-0505-x.

[43] J. W. Pitman, *One-dimensional Brownian motion and the three-dimensional Bessel process*, Adv. in Appl. Probab. **7** (1975), 511–526. MR 0375485

[44] J. Propp, *Generalized domino-shuffling*, Theoret. Comput. Sci. **303** (2003), 267–301. MR 1990768

[45] K. Rietsch, *A mirror construction for the totally nonnegative part of the Peterson variety*, Nagoya Math. J. **183** (2006), 105–142. MR 2253887

[46] J. Warren, *Dyson's Brownian motions, intertwining and interlacing*, Electron. J. Probab. **12** (2007), 573–590. MR 2299928

[47] J. Warren and P. Windridge, *Some examples of dynamics for Gelfand–Tsetlin patterns*. Electron. J. Probab. **14** (2009), 1745–1769. MR 2535012

Neil O'Connell, Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK

E-mail address: n.m.o-connell@warwick.ac.uk