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ASYMPTOTIC BEHAVIOR OF THE SOCLE OF FROBENIUS POWERS

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ABSTRACT. Let $(R,\mathfrak{m})$ be a local ring of prime characteristic p and q a varying power of p. We study the asymptotic behavior of the socle of $R/I^{[q]}$ where I is an $\mathfrak{m}$-primary ideal of R. In the graded case, we define the notion of diagonal F-threshold of R as the limit of the top socle degree of $R/\mathfrak{m}^{[q]}$ over q when $q \to \infty$. Diagonal F-threshold exists as a positive number (rational number in the latter case) when: (1) R is either a complete intersection or R is F-pure on the punctured spectrum; (2) R is a two dimensional normal domain. In the latter case, we also discuss its geometric interpretation and apply it to determine the strong semistability of the syzygy bundle of (x^d, y^d, z^d) over the smooth projective curve in $\mathbb{P}^2$ defined by $x^n + y^n + z^n = 0$. The rest of this paper concerns a different question about how the length of the socle of $R/I^{[q]}$ vary as q varies. We give explicit calculations of the length of the socle of $R/\mathfrak{m}^{[q]}$ for a class of hypersurface rings which attain the minimal Hilbert–Kunz function. We finally show, under mild conditions, the growth of such length function and the growth of the second Betti numbers of $R/\mathfrak{m}^{[q]}$ differ by at most a constant, as $q \to \infty$.

1. Introduction

We first review some notation and definitions used throughout the paper. In general, for a commutative ring R of prime characteristic $p > 0$, the Frobenius endomorphism $f : R \to R$ is defined by $f(r) = r^p$ for $r \in R$; its self-compositions are given by $f^n(r) = r^{p^n}$. Restriction of scalars along each iteration f^n endows R with a new R-module structure, denoted by ${}^{f^n}R$. For simplicity, we use q to denote p^n. If I is an ideal of R, the qth Frobenius power

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of I is the ideal generated by the qth powers of the generators of I, denoted by $I^{[q]}$. We use $F^n(-)$ to denote the functor from the category of R-modules to itself, given by base change along the Frobenius endomorphism $R \to {}^{f^n}R$. It is easy to see that $F^n(R/I) \cong R/I^{[q]}$. Also, the derived functors of $F^n(-)$ are $\mathrm{Tor}_i^R(-, {}^{f^n}R)$. For an R-module M, we use $\lambda(M)$ (resp. $\mathrm{pd}\, M$) to denote the length (resp. projective dimension) of M. When R is local with maximal ideal $\mathfrak{m}$, the socle of M is $(0 : \mathfrak{m})_M$, which is isomorphic to $\mathrm{Hom}_R(R/\mathfrak{m}, M)$. For an $\mathfrak{m}$-primary ideal I, the Hilbert–Kunz function of R with respect to I is the length function $\lambda(R/I^{[q]})$ (as a function of q); the Hilbert–Kunz multiplicity of R with respect to I is the limit of $\lambda(R/I^{[q]})/q^{\dim R}$ as $q \to \infty$. Such a limit always exists [18].

In this paper, we investigate questions related to the following general question: how does the socle of $R/I^{[q]}$ vary as q varies? While these questions are fairly well-understood when I has finite projective dimension (see [14]), they remain quite mysterious when the projective dimension of I is infinite and this is where our original motivation came from. In addition to that, our studies on these questions are also motivated by their relations with Hilbert–Kunz function and tight closure theory from many aspects. We refer to [8], [2], [23] for work along those lines.

The organization of this paper is as follows. In Section 2, for a standard graded local algebra $(R, \mathfrak{m})$ over a field of characteristic p, we define the notion of diagonal F-threshold $c^I(R)$ of R (with respect to a homogeneous $\mathfrak{m}$-primary ideal I). It is the limit of the top socle degree of $R/I^{[q]}$ over q as $q \to \infty$. Such a definition agrees with the definition of F-threshold in the literature, which is defined under a more general set-up. The existence of the diagonal F-threshold of complete intersection rings follows from recent work of Kustin and Vraciu (see Proposition 2.3). In general, it is not easy to calculate this invariant unless I has finite projective dimension. However, in graded dimension two case (smooth projective curve case), we can use the geometric tools developed in [5] to study it and the rest of Section 2 is devoted to this task. In graded dimension 2, the diagonal F-thresholds are all rational numbers. Specifically, we prove in Theorem 2.5 that if the syzygy bundle of I is strongly semistable and the degrees $d_1, \ldots, d_s$ of the generators of I satisfy certain condition, then the diagonal F-threshold $c^I(R)$ is just $\frac{d_1+\cdots+d_s}{s-1}$, a rational number independent of the characteristic p. The case that the syzygy bundle of I is not strongly semistable will be discussed in Theorem 2.6. We show that in this case, under certain conditions, $c^I(R)$ is equal to the rational number ν_t appeared originally in [5], whose definition (in general) relies on a result of Langer about the existence of the *Strong Harder–Narashimhan filtrations*. As a result, we can use a numerical condition on the diagonal F-threshold to characterize the strong semistability of the syzygy bundle of I when I is generated by homogeneous elements of the same degree and R is of the form $k[x, y, z]/(f)$ (Corollary 2.7).

In Section 3, we apply Corollary 2.7 to study the strong semistability of the syzygy bundle of $I = (x^d, y^d, z^d)$ over the smooth projective curve $\operatorname{Proj} k[x,y,z]/(x^n + y^n + z^n)$ in prime characteristic p. Our work here relies heavily on a very recent paper [15] of Kustin, Rahmati and Vraciu, in which they completely determine how the property $\operatorname{pd} I < \infty$ depends on the parameters p, n and d. We are able to transfer their results to determine, in quite many cases, how the strong semistability of the syzygy bundle of I depends on the parameters p, n and d. In particular, with some restrictions on p, n or d, the strong semistability of the syzygy bundle of I can be characterized by the condition $\operatorname{pd} I^{[q]} = \infty$ for all $q \gg 0$.

In Section 4 and Section 5, we study the asymptotic behavior of length of socle of $R/I^{[q]}$. Such a length function had been preliminarily investigated by the author in his Ph.D. thesis, for the purpose of answering a question of Dutta related to the nonnegativity conjecture of intersection multiplicity in the non-regular case, we refer to [9] and [17] for more details in that direction. Nevertheless, it is in general quite challenging to explicitly calculate this length function; even for the less complicated Hilbert–Kunz functions, the explicit calculations could be quite none trivial (for examples, see [10]). The main result here is an explicit calculation of this length function for a special class of hypersurface rings which attain the minimal Hilbert–Kunz function (see Definition 4.3). What makes this calculation possible in such a case is the observation that the entire socle of $R/\mathfrak{m}^{[q]}$ lives in the top degree spot. We do not know how to calculate this length function, or merely determine a leading term, when the socle contains elements of different degrees. We also point out a two-dimensional example in which the limit of $\lambda(\operatorname{soc}(R/\mathfrak{m}^{[q]})/q^{\max\{0, \dim R - 2\}})$ fail to exist as $q \to \infty$.

In Section 6, we use Gorenstein duality and some spectral sequence arguments to prove Theorem 6.1. In particular, it shows under mild conditions, the lengths of socle of $R/I^{[q]}$ and the second Betti numbers of $R/I^{[q]}$ differ only by a constant for q sufficiently large.

2. Asymptotic behavior of top socle degree of Frobenius powers

Throughout this section, we assume $(R, \mathfrak{m})$ is a standard graded Noetherian local algebra over a field of positive characteristic p. Let I be a homogeneous $\mathfrak{m}$-primary ideal of R. Recall the a-invariant $a(R)$ of R is the largest integer m such that $(H^{\dim R}_{\mathfrak{m}}(R))_m \neq 0$, where $H^{\dim R}_{\mathfrak{m}}(R)$ is the top local cohomology module of R. When R is complete intersection of the form S/C where S is the polynomial ring $k[x_1, \dots, x_n]$ and C is the ideal generated by homogeneous regular sequence $f_1, \dots, f_t$, $a(R) = \sum_i \deg f_i - n$. For every standard graded Artinian R-module $M = \bigoplus M_n$, we use t. s. d(M) to denote the top socle degree of M, which is equal to $\max\{n \mid M_n \neq 0\}$. Recently, Kustin and Vraciu (see [14], Proposition 7.1) established the following lower bound for the top socle degree of $R/I^{[q]}$, in the case either R is complete intersection or R is Gorenstein and F-pure:

THEOREM 2.1 (Kustin–Vraciu). *Assume either R is complete intersection or R is Gorenstein and F-pure. If* t. s. $\mathrm{d}(R/I)=s$, *then for every q*

$$\text{t. s. d}\big(R/I^{[q]}\big) \geq \big(s-a(R)\big)q+a(R).$$

We remark here that since R is standard graded, the powers $\mathfrak{m}^r$ are exactly $\bigoplus_{n\geq r} R_n$. Therefore the top socle degree of $R/I^{[q]}$ is nothing but the invariant

$$\nu_{\mathfrak{m}}^I(q) := \max\big\{r \in \mathbb{N} | \mathfrak{m}^r \not\subseteq I^{[q]}\big\}.$$

The limit (when exists) of $\{\nu_{\mathfrak{m}}^I(q)/q\}$ as $q\to\infty$ is a special case of an invariant called *F-threshold*. More generally, for any ideals $\mathfrak{a}$ and J of R (not necessarily graded) with $\mathfrak{a}\subseteq\sqrt{J}$, one can define $\nu_{\mathfrak{a}}^J(q)=\max\{r\in\mathbb{N}|\mathfrak{a}^r\not\subseteq J^{[q]}\}$ and study the convergence of the sequence $\nu_{\mathfrak{a}}^J(q)/q$ as $q\to\infty$. We refer to [1], [11], [12], [21] for details on that direction. F-thresholds are known to exist for rings which is F-pure on the punctured spectrum. Here we focus on a special case of F-threshold which we will call *diagonal F-threshold*.

DEFINITION 2.2. The *diagonal F-threshold* of R with respect to an $\mathfrak{m}$-primary ideal I, denoted $c^I(R)$, is defined as

$$c^I(R)=\lim_{q\to\infty}\frac{\text{t. s. d}(R/I^{[q]})}{q}$$

whenever such a limit exists. We also use $c(R)$ to denote $c^{\mathfrak{m}}(R)$, and simply call it the diagonal F-threshold of R.

PROPOSITION 2.3. *The diagonal F-threshold $c^I(R)$ exists when R is complete intersection or is F-pure on the punctured spectrum. Moreover, if R is complete intersection of the form S/C where S is the polynomial ring $k[x_1,\dots,x_n]$ and C is the ideal generated by a homogeneous regular sequence $f_1,\dots,f_t$, assuming J is the lift of I in S, then*

$$c^I(R)\geq c^J(S)-\sum \deg f_i. \tag{2.1}$$

In particular, $c(R)\geq -a(R)$, where $a(R)$ is the a-invariant.

Proof. The argument for the existence of diagonal F-threshold (or more generally, the F-threshold) in the case of F-pure on the punctured spectrum can be found in [12]. Assume R is complete intersection. By Theorem 2.1, the sequence

$$\frac{\text{t. s. d}(R/I^{[q]})-a(R)}{q} \tag{2.2}$$

is increasing as q increases. On the other hand, this sequence is bounded up by a very simple argument contained in [2]. We include that argument here for the sake of convenience. Choose K large enough such that $\mathfrak{m}^K \subseteq I$ and let L be the number of generators of $\mathfrak{m}^K$, then we have trivial inclusion $(\mathfrak{m}^K)^{Lq} \subseteq (\mathfrak{m}^K)^{[q]} \subseteq I^{[q]}$. This means

$$\text{t. s. d}\big(R/I^{[q]}\big) < (KL)q.$$

Therefore the sequence (2.2) converges as $q \to \infty$, which implies the diagonal F-threshold $c^I(R)$ exists.

For the proof of the lower bound (2.1), we write I as $I_1 \cap I_2 \cap \cdots \cap I_b$ with each I_i irreducible. Let J_i be the lift of I_i in S. It is easy to check that $c^I(R) \geq c^{I_i}(R)$ for each i. Since the Frobenius endomorphism on S is flat, $c^J(S) = \max\{c^{J_i}(S)|i\}$. Thus, we reduce (2.1) to the case that I is irreducible. In such a case, we have (see [14], p. 206)

$$\text{t. s. d}\left(\frac{S}{J^{[q]}+C}\right) = \text{t. s. d}\left(\frac{S}{J^{[q]}}\right) - M_q, \tag{2.3}$$

where M_q is the least degree among homogeneous nonzero elements of $(J^{[q]} : C)/J^{[q]}$, which is less than or equal to $(q-1)\sum \deg f_i$. Then the lower bound (2.1) follows from dividing both sides of (2.3) by q and taking the limit. In particular, when I is the maximal ideal $\mathfrak{m}$ of R, J is the maximal ideal of S. In this case, since $c^J(S) = \dim S = n$, the right-hand side of (2.1) is $n - \sum \deg f_i = -a(R)$. □

If R is Gorenstein and F-pure, Theorem 2.1 also provides a lower bound for diagonal F-thresholds, namely,

$$c^I(R) \geq \text{t. s. d}(R/I) - a(R).$$

In particular, when $I = \mathfrak{m}$, one has $c(R) \geq -a(R)$. Such a bound is achievable. For example, let $R = k[x,y,z,w]/(xy - zw)$, then $c(R) = 2$ and $a(R) = -2$.

Remark 2.4. For a Cohen–Macaulay normal domain R, Brenner gave an upper bound of t. s. d$(R/I^{[q]})$ which is better than the trivial upper bound used in the proof of Proposition 2.3. We refer to [2] for details. When $I = \mathfrak{m}$, there is a lower bound $m(q)$ for t. s. d$(R/\mathfrak{m}^{[q]})$ given by Buchweitz and Chen, see Theorem 4.2 below. This is a uniform lower bound for all graded hypersurfaces of the form $k[x_0, x_1, \ldots, x_n]/(f)$, where f runs through all homogeneous polynomials of degree d. In particular, this implies $c(R) \geq \frac{n+1}{2}$.

What information is encoded by the diagonal F-threshold $c^I(R)$ (or more generally, by F-thresholds)? Questions of this kind have been studied by many authors from many different angles (for example, see [11], [12]). In the remaining part of this section, we investigate this question in the case of smooth projective curves. We first briefly recall some basic definitions.

Harder–Narasimhan filtrations. Let Y be a smooth projective curve over an algebraically closed field. For any vector bundle $\mathcal{V}$ on Y of rank r, the degree of $\mathcal{V}$ is defined as the degree of the line bundle $\wedge^r\mathcal{V}$. The slope of $\mathcal{V}$, denoted $\mu(\mathcal{V})$, is defined as the fraction $\deg(\mathcal{V})/r$. Slope is additive on tensor products of bundles: $\mu(\mathcal{V}\otimes\mathcal{W})=\mu(\mathcal{V})+\mu(\mathcal{W})$. If $f:Y'\to Y$ is a finite map of degree q, then $\deg(f^*(\mathcal{V}))=q\deg(\mathcal{V})$ and so $\mu(f^*(\mathcal{V}))=q\mu(\mathcal{V})$.

A bundle $\mathcal{V}$ is called *semistable* if for every subbundle $\mathcal{W}\subseteq\mathcal{V}$ one has $\mu(\mathcal{W})\le\mu(\mathcal{V})$. Clearly, bundles of rank 1 are always semistable, and duals and twists of semistable bundles are semistable.

Any bundle $\mathcal{V}$ has a filtration by subbundles

$$0=\mathcal{V}_0\subset\mathcal{V}_1\subset\cdots\subset\mathcal{V}_t=\mathcal{V}$$

such that $\mathcal{V}_k/\mathcal{V}_{k-1}$ is semistable and $\mu(\mathcal{V}_k/\mathcal{V}_{k-1})>\mu(\mathcal{V}_{k+1}/\mathcal{V}_k)$ for each k. This filtration is unique, and it is called the *Harder–Narasimhan (or HN) filtration* of $\mathcal{V}$.

In positive characteristic, we use F to denote the absolute Frobenius morphism $F:Y\to Y$. Pulling back a vector bundle under F does not necessarily preserve semistability. Therefore, the pullback under F^n of an HN filtration of $\mathcal{V}$ does not always give an HN filtration of $(F^*)^n(\mathcal{V})$. However, by the work of Langer [16], there always exists a so called strong NH filtration, i.e., for some n_0, the HN filtration of $(F^*)^{n_0}(\mathcal{V})$ has the property that all its Frobenius pullbacks are the HN filtrations of $(F^*)^n(\mathcal{V})$, for all $n>n_0$.

Suppose R is a standard-graded two-dimensional normal domain and $I=(f_1,\dots,f_s)$ where f_i is homogeneous of degree d_i for $1\le i\le s$. Let $Y=\operatorname{Proj}R$. Consider the syzygy bundle $\mathcal{S}=\operatorname{Syz}(f_1,\dots,f_s)$ on Y given by the exact sequence

$$0\longrightarrow\mathcal{S}\longrightarrow\bigoplus_{i=1}^{s}\mathcal{O}(-d_i)\xrightarrow{f_1,\dots,f_s}\mathcal{O}\longrightarrow 0 \tag{2.4}$$

and the pullback of this exact sequence along F^n (with a subsequent twist by $m\in\mathbb{Z}$)

$$0\longrightarrow\mathcal{S}^q(m)\longrightarrow\bigoplus_{i=1}^{s}\mathcal{O}(m-qd_i)\xrightarrow{f_1^q,\dots,f_s^q}\mathcal{O}(m)\longrightarrow 0, \tag{2.5}$$

where $\mathcal{S}^q$ denotes the pullback $(F^*)^n(\mathcal{S})=\operatorname{Syz}(f_1^q,\dots,f_s^q)$. Applying the sheaf cohomology to (2.5), we have a long exact sequence

$$\begin{aligned}0\longrightarrow H^0\big(Y,\mathcal{S}^q(m)\big)\longrightarrow\bigoplus_{i=1}^{s}H^0\big(Y,\mathcal{O}(m-qd_i)\big)&\xrightarrow{f_1^q,\dots,f_s^q}H^0\big(Y,\mathcal{O}(m)\big)\\ \longrightarrow H^1\big(Y,\mathcal{S}^q(m)\big)\longrightarrow\bigoplus_{i=1}^{s}H^1\big(Y,\mathcal{O}(m-qd_i)\big)&\longrightarrow\cdots.\end{aligned} \tag{2.6}$$

As R is normal, the cokernel of the third map (from left) in the long exact sequence (2.6) is the mth graded piece of $R/I^{[q]}$.

Now we are ready to move back to the question we asked earlier in this section. Set $\deg Y = \deg \mathcal{O}_Y(1)$ and let ω_Y denote the dualizing sheaf on Y. We first treat the case where the syzygy bundle is strongly semistable.

THEOREM 2.5. *Suppose the syzygy bundle $\mathcal{S}$ is strongly semistable. Assume also the degrees d_i satisfy the condition $\frac{d_1+\cdots+d_s}{s-1} > \max_i\{d_i\}$, then*

$$c^I(R) = (d_1 + \cdots + d_s)/(s-1).$$

Proof. The rightmost term in (2.6) is zero for $m > \max_i\{qd_i\}$. This is because, by Serre duality, $h^1(\mathcal{O}(m-qd_i)) = h^0(\mathcal{O}(-m+qd_i)\otimes\omega_Y)$, which equals zero since the degree of $\mathcal{O}(-m+qd_i)\otimes\omega_Y$ is negative.

From [5], we know for $m > \lceil\frac{d_1+\cdots+d_s}{s-1}q\rceil + \frac{\deg\omega_Y}{\deg Y}$, $H^1(Y,\mathcal{S}^q(m))$ vanishes. It follows that

$$\text{t. s. d}\big(R/I^{[q]}\big) \leq \left\lceil \frac{d_1+\cdots+d_s}{s-1}q\right\rceil + \frac{\deg\omega_Y}{\deg Y}.$$

Also from [5], for $m \leq \lceil\frac{d_1+\cdots+d_s}{s-1}q\rceil - 1$, $H^0(Y,\mathcal{S}^q(m))$ vanishes. An easy calculation asserts that, for $m = \lceil\frac{d_1+\cdots+d_s}{s-1}q\rceil - 1$ and $q \gg 0$,

$$\sum_{i=1}^{s} h^1\big(\mathcal{O}(m-qd_i)\big) \neq h^0\big(\mathcal{O}(m)\big).$$

Thus

$$\text{t. s. d}\big(R/I^{[q]}\big) \geq \left\lceil \frac{d_1+\cdots+d_s}{s-1}q\right\rceil - 1 \quad \text{for } q \gg 0,$$

and the theorem follows. $\square$

We next discuss the case where the syzygy bundle is not strongly semistable. In such a case, using strong NH filtrations, Brenner defined rational numbers $\nu_1,\ldots,\nu_t$ for the syzygy bundle $\mathcal{S}$

$$\nu_i = -\frac{\mu(F^{*n}(\mathcal{S}_i))/\mu(F^{*n}(\mathcal{S}_{i-1}))}{q\deg\mathcal{O}(1)},$$

where $0 = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots \subset \mathcal{S}_t = \mathcal{S}$ is a HN filtration of $F^{*n_0}(\mathcal{S})$ which is strong and $q = p^{n_0+n}$. These ν_i's satisfy $\min\{d_i\} \leq \nu_1 < \cdots < \nu_t \leq \max\{d_i + d_j | i \neq j\}$. Moreover, Brenner showed for $q \gg 0$, if $m > q\nu_t + \frac{\deg\omega_Y}{\deg Y}$, then $H^1(Y,\mathcal{S}^q(m)) = 0$ (see [5] for more details).

Let g denotes the genus of Y. With the above set-up, we have the following theorem.

THEOREM 2.6. (1) t. s. d$(R/I^{[q]}) \leq \nu_t q + \frac{\deg\omega_Y}{\deg Y}$, *for $q \gg 0$.*

(2) *If we further assume* $\nu_t > \max_i\{d_i\}$, *then for* $q \gg 0$,

$$\text{t. s. d}\big(R/I^{[q]}\big) \geq \begin{cases} \lceil \nu_t q \rceil - 1, & \textit{if } g \geq 1, \\ \lceil \nu_t q \rceil - 2, & \textit{if } g = 0. \end{cases}$$

In particular, $c^I(R)$ *exists and equals* ν_t.

Proof. (1) Let $l = \text{t. s. d}(R/I^{[q]})$. Thus, the cokernel of

$$\bigoplus_{i=1}^{s} H^0\big(Y, \mathcal{O}(l - qd_i)\big) \overset{f_1^q,\dots,f_s^q}{\longrightarrow} H^0\big(Y, \mathcal{O}(l)\big)$$

must be nonzero, which implies $H^1(Y, \mathcal{S}^q(m)) \neq 0$. This shows $l \leq q\nu_t + \frac{\deg \omega_Y}{\deg Y}$ when $q \gg 0$.

(2) Here we only treat the case $g \geq 1$, the computations for the other case are almost identical. So we assume $g \geq 1$. To get the lower bound $\lceil \nu_t q \rceil - 1$ in this case, let $\ell = \lceil \nu_t q \rceil - 1$, it suffices to show the cokernel of

$$\bigoplus_{i=1}^{s} H^0\big(Y, \mathcal{O}(\ell - qd_i)\big) \overset{f_1^q,\dots,f_s^q}{\longrightarrow} H^0\big(Y, \mathcal{O}(\ell)\big)$$

is nonzero. To this end, we apply the following result from [5], p. 102:

For $q \gg 0$ and $q\nu_{t-1} + \frac{\deg \omega_Y}{\deg Y} < m < q\nu_t$,

$$\begin{aligned} h^0\big(Y, \mathcal{S}^q(m)\big) = {} & q(-r_1\nu_1 - \dots - r_{t-1}\nu_{t-1}) \deg Y \\ & + m(r_1 + \dots + r_{t-1}) \deg Y + \operatorname{rank}(\mathcal{S}_{t-1})(1 - g). \end{aligned} \tag{2.7}$$

Here, r_i is defined to be the rank of $\mathcal{S}_i/\mathcal{S}_{i-1}$ for $i = 1, \dots, t$. These numbers $r_1, \dots, r_t$ satisfy

$$r_1 + \dots + r_t = \operatorname{rank} \mathcal{S} = s - 1 \tag{2.8}$$

and

$$r_1\nu_1 + \dots + r_t\nu_t = \sum_{i=1}^{s} d_i. \tag{2.9}$$

Let $\varepsilon(q) = \lceil \nu_t q \rceil - \nu_t q$, then $\ell = \nu_t q - 1 + \varepsilon(q)$. Therefore, by (2.7), we have

$$\begin{aligned} h^0\big(Y, \mathcal{S}^q(\ell)\big) & \\ = {} & q(-r_1\nu_1 - \dots - r_{t-1}\nu_{t-1}) \deg Y \\ & + \big(\nu_t q - 1 + \varepsilon(q)\big)(r_1 + \dots + r_{t-1}) \deg Y + \operatorname{rank}(\mathcal{S}_{t-1})(1 - g) \\ = {} & q\big(\nu_t(s - 1 - r_t) - r_1\nu_1 - \dots - r_{t-1}\nu_{t-1}\big) \deg Y \\ & + \big(-1 + \varepsilon(q)\big)(s - 1 - r_t) \deg Y + \operatorname{rank}(\mathcal{S}_{t-1})(1 - g). \end{aligned} \tag{2.10}$$

On the other hand, since $\nu_t > \max\{d_i\}$, we have $-\ell + qd_i < 0$ for $q \gg 0$, whence

$$H^1\big(Y, \mathcal{O}(\ell - qd_i)\big) = H^0\big(Y, \mathcal{O}(-\ell + qd_i)\big) \otimes \omega_Y) = 0 \quad \text{for } q \gg 0.$$

So from Riemann–Roch theorem, we get

$$h^0\big(\mathcal{O}(\ell)\big) - \sum_{i=1}^{s} h^0\big(\mathcal{O}(\ell - qd_i)\big) = \ell \deg Y + (1-g) - \sum_{i=1}^{s} (\ell - qd_i)\deg Y - s(1-g) \tag{2.11}$$

which can be simplified to

$$\begin{aligned}(s-1)(g-1) - \left((s-1)\ell - \sum_{i=1}^{s} d_i q\right)\deg Y \\ = q\left(\sum_{i=1}^{s} d_i - (s-1)\nu_t\right)\deg Y + (s-1)\big(\big(1-\varepsilon(q)\big)\deg Y + g - 1\big).\end{aligned} \tag{2.12}$$

Thus, by (2.6), the length of the cokernel of

$$\bigoplus_{i=1}^{s} H^0\big(Y, \mathcal{O}(\ell - qd_i)\big) \xrightarrow{f_1^q,\dots,f_s^q} H^0\big(Y, \mathcal{O}(\ell)\big)$$

equals

$$h^0\big(Y, \mathcal{S}^q(\ell)\big) + h^0\big(\mathcal{O}(\ell)\big) - \sum_{i=1}^{s} h^0\big(\mathcal{O}(\ell - qd_i)\big) = c_1 q + c_0, \tag{2.13}$$

where, by adding the right-hand sides of (2.10) and (2.12) and using (2.9),

$$c_1 = \deg Y\left(\nu_t(s-1-r_t) - r_1\nu_1 - \cdots - r_{t-1}\nu_{t-1} + \sum_{i=1}^{s} d_i - (s-1)\nu_t\right) = 0$$

and

$$\begin{aligned} c_0 &= \big(-1+\varepsilon(q)\big)(s-1-r_t)\deg Y + \operatorname{rank}(\mathcal{S}_{t-1})(1-g) \\ &\quad + (s-1)\big(\big(1-\varepsilon(q)\big)\deg Y + g - 1\big) \\ &= \big(-1+\varepsilon(q)\big)(-r_t)\deg Y + \operatorname{rank}(\mathcal{S}_{t-1})(1-g) + (s-1)(g-1) \\ &= \big(1-\varepsilon(q)\big)r_t \deg Y + \big(s-1-\operatorname{rank}(\mathcal{S}_{t-1})\big)(g-1). \end{aligned}$$

Since $\operatorname{rank}(\mathcal{S}_{t-1}) + r_t = \operatorname{rank}\mathcal{S} = s-1$,

$$c_0 = r_t\big(\big(1-\varepsilon(q)\big)\deg Y + g - 1\big) > 0.$$

The last inequality here is due to our assumption $g \geq 1$. Therefore, the left-hand side of (2.13) is positive for $q \gg 0$. □

Corollary 2.7. *Assume R is of the form $k[x,y,z]/(f)$ such that the genus of $\operatorname{Proj} R$ is at least one. $I = (f_1, f_2, f_3)$ and $\deg f_i = d$. Then $c^I(R) \geq \frac{3d}{2}$, and $c^I(R) = \frac{3d}{2}$ if and only if the syzygy bundle of I is strongly semistable.*

Proof. If the syzygy bundle of I is strongly semistable, then $c^I(R) = \frac{3d}{2}$ follows from Theorem 2.5.

Assume the syzygy bundle of I is not strongly semistable. Here we have $t = 2$, $\nu_1 < \nu_2$ and $\nu_1 + \nu_2 = 3d$. Therefore, $c^I(R) = \nu_2 > \frac{3d}{2}$. □

REMARK 2.8. Assume R is of the form $k[x,y,z]/(f)$ where the degree of f is h. By Corollary 4.6 in [5], we have

$$\frac{3}{2} \leq c(R) \leq 2$$

and $c(R) = 3/2$ if and only if the syzygy bundle of (x,y,z) is strongly semistable. Moreover, the Hilbert–Kunz multiplicity and the diagonal F-threshold of R have the relation

$$e_{\mathrm{HK}}(R) = h\big(c(R)^2 - 3c(R) + 3\big).$$

EXAMPLE 2.9. Consider $R_p = \mathbb{Z}/p\mathbb{Z}[x,y,z]/(x^4 + y^4 + z^4)$. The rank two syzygy bundle $\mathcal{S}$ of (x,y,z) is strongly semistable if $p \equiv \pm 1 \mod 8$ and therefore, $c(R_p) = \frac{3}{2}$ in this case. On the other hand, $\mathcal{S}$ is not strongly semistable if $p \equiv \pm 3 \mod 8$ (See [13, Example 4.1.8], [6], [19] or Example 3.5). So

$$c(R_p) = \nu_t = \nu_2 = \frac{3}{2} + \frac{1}{2p}.$$

Our Macaulay 2 experiments give us the following formulae of the top socle degree functions for $p = 3, 5, 7$:

$$\text{t. s. d}\big(R_p/\mathfrak{m}^{[q]}\big) = \begin{cases} \frac{5q}{3} + 1, & \text{if } p = 3, \\ \frac{8q}{5} + 1, & \text{if } p = 5, \\ \lfloor \frac{3q}{2} \rfloor + 1, & \text{if } p = 7. \end{cases}$$

One might expect the following precise formula of t. s. d$(R/\mathfrak{m}^{[q]})$, depending only on $c(R)$ and $a(R)$,

$$\text{t. s. d}\big(R/\mathfrak{m}^{[q]}\big) = \lfloor c(R)q \rfloor + a(R).$$

However, the following example, suggested by Brenner to the author, indicates this is wrong.

EXAMPLE 2.10. Let $R = \mathbb{Z}/2\mathbb{Z}[x,y,z]/(x^4 + y^4 + z^4 + x^3y + y^3z + z^3x)$. Then

$$\text{t. s. d}\big(R/\mathfrak{m}^{[q]}\big) = \frac{3}{2}q.$$

Example 2.10 also shows the inequality in Theorem 2.1 could be strict for all q.

3. Some applications

Throughout this section, let R be the diagonal hypersurface $k[x,y,z]/(x^n+y^n+z^n)$ where $\operatorname{char} k = p$ and I be the ideal (x^d,y^d,z^d) of R. In a recent paper [15], Kustin, Rahamati and Vraciu completely determined how the property $\operatorname{pd} I < \infty$ depends on the parameters p, n and d. For every prime number p, they introduced the sets S_p and T_p, which form a partition for the set of all nonnegative integers. One of their main results in that paper is that I has finite projective dimension if and only if $n \mid d$ or $\lfloor \frac{d}{n} \rfloor \in T_p$ (see [15], Theorem 6.2). In addition to that, they also explicitly described the minimal free resolutions for such ideals. Our purpose here is to use the results in [15], together with the characterization of the strong semistability of the syzygy bundle of I obtained in Corollary 2.7, to study how the strong semistability of the syzygy bundle of I depends on parameters p, n and d. We would like to point out that the strong semistability of this particular syzygy bundle has also been studied in great detail in [13], Chapter 4. In the rest of this section, we adopt all the notation of [15] without explanation and refer to [15] for details. Since we will apply Corollary 2.7, we also assume $p \nmid n$ and $n \geq 3$ throughout this section. With this assumption, $\operatorname{Proj} R$ is a smooth curve. We first prove a sufficient condition for the strong semistability of syzygy bundle.

THEOREM 3.1. *If there are infinitely many q such that $\operatorname{pd} I^{[q]} = \infty$, then the syzygy bundle of I is strongly semistable.*

We remark here that this sufficient condition is equivalent to: (a) $n \nmid d$; and (b) there are infinitely many q such that $\lfloor \frac{qd}{n} \rfloor \in S_p$.

Proof of Theorem 3.1. We know by Theorem 3.5 of [15], when $\operatorname{pd} I^{[q]} = \infty$, the leading term of t. s. $\mathrm{d}(R/I^{[q]})$ is $(\frac{3d}{2})q$. Therefore, if there are infinitely many q such that $\operatorname{pd} I^{[q]} = \infty$, then $c^I(R) = \frac{3d}{2}$. It then follows from Corollary 2.7 that the syzygy bundle of I is strongly semistable. $\square$

We next discuss the converse of Theorem 3.1. Assume $\operatorname{pd} I^{[q_0]} < \infty$ for some q_0 (hence $\operatorname{pd} I^{[q]} < \infty$ for all $q \geq q_0$, see [22]). We know from Corollary 1.7 of [14], $c^{I^{[q_0]}}(R)$ equals the largest back twist in the minimal homogeneous resolution of $R/I^{[q_0]}$ by free R-modules, that is, the largest b_i in the following resolution of $R/I^{[q_0]}$:

$$0 \longrightarrow R(-b_1) \oplus R(-b_2) \longrightarrow \bigoplus_{i=1}^{3} R(-q_0 d) \longrightarrow R \longrightarrow 0.$$

Since $b_1 + b_2 - 2q_0 d = q_0 d$, we have:

$$b_1 \neq b_2 \quad \Leftrightarrow \quad \max\{b_1, b_2\} > \frac{3q_0 d}{2},$$

i.e.

$$c^{I^{[q_0]}}(R) > \frac{3q_0 d}{2},$$

which is equivalent to (observe that $c^I(R) = \frac{1}{q_0} c^{I^{[q_0]}}(R)$)

$$c^I(R) > \frac{3d}{2}.$$

Thus, by Corollary 2.7, we have the following theorem.

THEOREM 3.2. *Assume* $\operatorname{pd} I^{[q_0]} < \infty$ *for some* q_0. *Then the syzygy bundle of* I *is strongly semistable if and only if* $b_1 = b_2$.

The following example shows the converse of Theorem 3.1 does not hold in general.

EXAMPLE 3.3. Let $k = \mathbb{Z}/5\mathbb{Z}$, $n = 3$ and $d = 2$. Then $\operatorname{pd} I^{[p]} < \infty$ (hence $\operatorname{pd} I^{[q]} < \infty$ for all $q > p$). On the other hand, one can check the largest back twist in the minimal homogeneous resolution of $R/I^{[p]}$ is 15 (i.e., $b_1 = b_2 = 15$). So the syzygy bundle of I is strongly semistable.

Nonetheless, we could still expect the converse of Theorem 3.1 to hold when the parameters p, n, d satisfy special conditions. We first treat the case when $n \nmid d$.

THEOREM 3.4. *Assume* $n \nmid d$. *Then the converse of Theorem* 3.1 *holds for all the following cases*:

Case 1: $p = 3$;
Case 2: $p \equiv 1 \mod 3$;
Case 3: p *and* d *are both odd*;
Case 4: $3 \nmid n$.

In particular, if $p = 2$, $3 \nmid n$ *and* $n \nmid d$, *then the syzygy bundle of* I *is never strongly semistable.*

Proof. When $\operatorname{pd} I^{[q_0]} < \infty$, the minimal free resolution of $I^{[q_0]}$ has been given explicitly in [15], Section 5. It is then a bookkeeping job (but a little tedious) to check case by case that, in any of the above cases, $b_1 \neq b_2$ (we refer to particularly Observation 5.3, Lemmas 5.4, 5.5, 5.12, Observation 5.13, Theorem 5.14, Lemma 5.15 of [15]). Therefore by Theorem 3.2, the syzygy bundle of I is not strongly semistable. The $p = 2$ case of this theorem follows from the fact that T_2 is the set of all nonnegative integers, hence $\operatorname{pd} I^{[q]}$ is never finite in such a case. $\square$

EXAMPLE 3.5. We apply this theorem to recover the known result we mentioned earlier in Example 2.9: Suppose p is odd. The syzygy bundle of (x, y, z) over $\mathbb{Z}/p\mathbb{Z}[x, y, z]/(x^4 + y^4 + z^4)$ is strongly semistable if and only if $p \equiv \pm 1 \mod 8$. Here we have $n = 4$ which is not divisible by 3, so Theorem 3.4 is applicable. First, consider the case $p = 8c + 1$. We claim that $\lfloor \frac{p^e}{n} \rfloor \in S_p$ for every positive integer e. To see this, notice that $\lfloor \frac{p^e}{4} \rfloor = \frac{p^e-1}{4}$. We can express this number in the following base-p expansion (see [15], Notation 1.5)

$$\frac{p^e-1}{4} = 2cp^{e-1} + 2cp^{e-2} + \cdots + 2cp + 2c,$$

which involves only even digits. So the claim follows from [15], Remark 1.6. Therefore, the syzygy bundle is strongly semistable in this case. For the case $p = 8c+3$, we claim that $\lfloor \frac{p^2}{4} \rfloor \in T_p$. This is because $\lfloor \frac{p^2}{4} \rfloor = 16c^2 + 12c + 2 = (2c+1)p - (2c+1)$, which is a base-$p$ expansion involves at least one odd digit. Thus the syzygy bundle is not strongly semistable. We leave the remaining cases for the interested readers to verify.

One can also apply the above base-p expansion method to reinvestigate the strong semistability of the syzygy bundle $\mathcal{S}$ of (x^2, y^2, z^2) on the Fermat quintic $x^5 + y^5 + z^5 = 0$ (here $n = 5$ and $d = 2$ so Theorem 3.4 is applicable), which has been studied in [3], Section 2 and in [13], Example 4.1.9. In particular, one can recover Corollary 2.1 of [3], which says $\mathcal{S}$ is not strongly semistable when $p \equiv \pm 2 \mod 5$, by looking at the base-p expansions of $\lfloor \frac{p^2}{5} \rfloor$ of such primes. Not only that, one also obtains that $\mathcal{S}$ is strongly semistable when $p \equiv \pm 1 \mod 5$ via the same method. Again, we leave the detail for the interested readers.

For the case $n \mid d$, we know there exists a $q_0 \gg 0$ such that $\operatorname{pd} I^{[q_0]} < \infty$. In such a case, the strong semistability is determined by the syzygy gap. We refer to [20] or [7] for the definition of syzygy gap.

Theorem 3.6. *Assume $n \mid d$. Let $a = \frac{d}{n}$. The syzygy bundle of I is strongly semistable if and only if the syzygy gap of $x^a, y^a, (x+y)^a$ in $k[x,y]$ is zero. In particular, if a is odd, then the syzygy bundle of I is never strongly semistable.*

Proof. Let δ be the syzygy gap of $x^a, y^a, (x+y)^a$ in $k[x,y]$. From [15], Observation 5.3, we know $b_1 = b_2$ if and only if $\delta = 0$. So our conclusion follows from Theorem 3.2. There is a formula $\delta^2 = 4\ell(k[x,y]/(x^a, y^a, (x+y)^a)) - 3a^2$ ([20], Lemma 1). Hence if a is odd, $\delta \neq 0$. □

In the case $n = 3$ and $p \equiv 1 \mod 3$, we have the following characterization of strong semistability.

Theorem 3.7. *Assume $n = 3$, $p \equiv 1 \mod 3$ and $3 \nmid d$. Then the syzygy bundle of I is strongly semistable if and only of* $\operatorname{pd} I = \infty$.

Proof. This follows from Proposition 8.5 of [15], Theorem 3.1 and Theorem 3.4 immediately. □

We finally use Theorem 3.4 to recover a result of Brenner (Proposition 1 in [4], see also Lemma 4.2.8 in [13] for an extended version).

THEOREM 3.8. *Fix $d > 0$ and a prime number p. For every positive integer n_0, there exists $n > n_0$ such that the syzygy bundle of $I = (x^d, y^d, z^d)$ is not strongly semistable on the smooth projective curve defined by $x^n + y^n + z^n = 0$.*

Proof. By Remark 1.6 in [15], the set T_p is not empty. Assume $c \in T_p$. For every n_0, we choose q large enough such that $\frac{qd}{c} > \max\{n_0 + 4, 4c + 4, d + 4\}$. Let a be an integer such that $a \leq \frac{qd}{c} < a + 1$. If $3 \mid a - 1$, let n be one of $a - 2, a - 3$ which is not divisible by p. If $3 \nmid a - 1$, let n be one of $a - 2, a - 1, a$ which is neither divisible by 3 nor divisible by p. Then such an n satisfies conditions $3 \nmid n$, $n \nmid qd$ and $\lfloor \frac{qd}{n} \rfloor = c \in T_p$. Thus, the conclusion follows from Theorem 3.4. $\square$

4. Asymptotic behavior of length of socle of Frobenius powers

Our main question in this section is:

How does $\lambda(\mathrm{soc}(R/I^{[q]}))$ vary as q varies?

In the case that I has finite projective dimension, it is well known that $\lambda(\mathrm{soc}(R/I^{[q]}))$ equals the constant $\lambda(\mathrm{soc}(R/I))$. In the general situation, we recall a result of Yackel [23], which asserts such a length function cannot grow faster than $O(q^{\max\{\dim R - 2, 0\}})$.

THEOREM 4.1 (Yackel). *There exists a constant c_R, s.t.*

$$\lambda\big(\mathrm{soc}\big(R/I^{[q]}\big)\big) \leq c_R q^{\max\{0, n-2\}},$$

where $n = \dim R$.

Unfortunately, other than the above result of Yackel, very little is known regarding the asymptotic behavior of this length function when I has infinite projective dimension, even in the hypersurface situation. Therefore, any explicit computation on such a length function in special cases would be valuable, and even could potentially initiate some new study. The main purpose of this section is to explicitly calculate such length functions for $I = \mathfrak{m}$ over a special class of hypersurface rings, which will be specified below.

In [8], Buchweitz and Chen investigated a lower bound $m(q)$ of the top socle degree of $R/\mathfrak{m}^{[q]}$ among all dimension n hypersurfaces $(R, \mathfrak{m})$, and its relation with minimal Hilbert–Kunz function. Among other things, they showed the following theorem.

THEOREM 4.2 (Buchweitz–Chen). *Fix $n, d > 0$. Let q be a power of p. Let*

$$m(q) = \left\lfloor \frac{(n+1)(q-1) + (d-1)}{2} \right\rfloor$$

and

$$L(q) = \text{the coefficient of } t^{m(q)} \text{ in } \big(1 - t^d\big)\big(1 - t^q\big)^{n+1}(1 - t)^{-n-2}.$$

Assume f is a homogeneous polynomial of degree d in $S=k[x_0,x_1,\dots,x_n]$. Let R be the hypersurface ring S/fS and $\mathfrak{m}$ the ideal of R generated by the images of $x_0,x_1,\dots,x_n$ in R. Then the top socle degree t. s. d$(R/\mathfrak{m}^{[q]})$ *is at least $m(q)$ and the Hilbert–Kunz function of R with respect to $\mathfrak{m}$ is at least $L(q)$. Moreover, the following are equivalent:*

(1) t. s. d$(R/\mathfrak{m}^{[q]})=m(q)$;

(2) *the Hilbert–Kunz function of R with respect to $\mathfrak{m}$ equals $L(q)$.*

DEFINITION 4.3. We say a hypersurface ring *attains the minimal Hilbert–Kunz function* if it satisfies any of the equivalent conditions in the above theorem of Buchweitz and Chen.

EXAMPLE 4.4 (Buchweitz–Chen). The hypersurfaces $k[x,y,z,w]/(xy-zw)$ and the Cayley's cubic surface $k[x,y,z,w]/(xyz+xyw+xzw+yzw)$ both attain the minimal Hilbert–Kunz function, see [8] for details.

The following theorem is the main result of this section.

THEOREM 4.5. *Adopt all of the notation of Theorem 4.2. Assume $R=S/fS$ attains the minimal Hilbert–Kunz function. Then the entire socle of $R/\mathfrak{m}^{[q]}$ must live in the top degree spot. In other words, no socle element of $R/\mathfrak{m}^{[q]}$ is in degree $<m(q)$.*

Proof. We use $\mathbf{x}^{[q]}$ to denote the ideal $(x_0,\dots,x_n)^{[q]}$ of S. Let

$$\mathbf{\Theta}=\frac{S}{\mathbf{x}^{[q]}}=\bigoplus_{i\ge 0}\mathbf{\Theta}_i$$

and

$$\theta=\frac{S}{fS+\mathbf{x}^{[q]}}=\bigoplus_{i\ge 0}\theta_i.$$

Since we assume R attains the minimal Hilbert–Kunz function, by the argument contained in the proof of Theorem 4.2 in [8], we have short exact sequences

$$0\longrightarrow \mathbf{\Theta}_{i-d}\xrightarrow{f}\mathbf{\Theta}_i\longrightarrow\theta_i\longrightarrow 0 \tag{4.1}$$

for every $i\le m(q)$.

Assume $i\le m(q)$. Consider the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{\Theta}_{i-d-1} & \xrightarrow{f} & \mathbf{\Theta}_{i-1} & \longrightarrow & \theta_{i-1} & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \phi_{i-d}} & & \Big\downarrow{\scriptstyle \phi_i} & & \Big\downarrow{\scriptstyle \psi_i} & & \\ 0 & \longrightarrow & \mathbf{\Theta}_{i-d}^{n+1} & \xrightarrow{\oplus f} & \mathbf{\Theta}_i^{n+1} & \longrightarrow & \theta_i^{n+1} & \longrightarrow & 0, \end{array} \tag{4.2}$$

where ϕ_i sends every $r\in\mathbf{\Theta}_{i-1}$ to $r(x_0,x_1,\dots,x_n)\in\mathbf{\Theta}_i^{n+1}$ and ψ_i is induced by ϕ_i in the natural way. Then the degree $(i-1)$ component of the socle of θ is just $\operatorname{Ker}(\psi_i)$. The map ϕ_i is injective since the socle of $\boldsymbol{\Theta}$ is the one-dimensional vector space over k generated by $x_0^{q-1}\cdots x_n^{q-1}$, an element of degree $> m(q)$. Therefore, we have the injection

$$\operatorname{Ker}\psi_i \hookrightarrow (0:\mathfrak{m})_{\operatorname{Coker}(\phi_{i-d})}.$$

While the following lemma guarantees $(0:\mathfrak{m})_{\operatorname{Coker}(\phi_{i-d})} = 0$, our theorem follows. □

We fix some notation which are valid for this lemma only. Let k be an arbitrary field in any characteristic. Let R be the standard graded Artinian ring $k[x_0,\dots,x_n]/(x_0^t,\dots,x_n^t)$, where t is a fixed positive integer. Let $\mathfrak{m}_R$ denote the maximal ideal $(x_0,\dots,x_n)$ of R.

Lemma 4.6. *Let ϕ be the map from R to R^{n+1} defined by $\phi(r) = r(x_0,x_1,\dots,x_n)$. Let M be the cokernel of ϕ. Then the socle degree of M is at least $n(t-1)$, that is, if a is a nonzero homogeneous element in $(0:\mathfrak{m}_R)_M$, then the degree of $a \geq n(t-1)$.*

Proof. We first point out the following fact, which is easy to verify.

Fact 1. *In R, we have $(0:x_i) = (x_i^{t-1})$ $\forall i$.*

To prove the lemma, we induct on n. The case $n=0$ is trivial. Assume the theorem holds for all Artinian rings of the form $k[x_0,x_1,\dots,x_l]/(x_0^t,x_1^t,\dots,x_l^t)$ for all $l<n$. Consider the case $R = k[x_0,x_1,\dots,x_n]/(x_0^t,x_1^t,\dots,x_n^t)$. We use R_i to denote the degree i component of R.

Suppose the lemma fails, then there exists an nonzero homogeneous element $\overline{a} \in (0:\mathfrak{m}_R)_M$, whose degree is $d < n(t-1)$. Since M is a quotient module of R^{n+1}, we can assume $\overline{a}$ is the image of $a = (a_0,a_1,\dots,a_n)$ where $a_0,a_1,\dots,a_n \in R_d$. The condition $\overline{a} \in (0:\mathfrak{m}_R)_M$ implies that there exist $u_0,u_1,\dots,u_n \in R_d$ such that

$$ax_i = u_i(x_0,x_1,\dots,x_n), \quad 0 \leq i \leq n. \tag{4.3}$$

It follows that for every $j = 0,1,\dots,n$,

$$a_j x_i = u_i x_j, \quad 0 \leq i \leq n. \tag{4.4}$$

This implies $a_j \mathfrak{m}_R \subseteq (x_j)$, that is,

$$a_j \in (x_j : \mathfrak{m}_R). \tag{4.5}$$

Let $\widetilde{R}$ denote R/x_jR, $\widetilde{a}_j$ be the image of a_j in $\widetilde{R}$ and $\mathfrak{m}_{\widetilde{R}}$ the maximal ideal of $\widetilde{R}$. Then (4.5) becomes

$$\widetilde{a}_j \in (0:\mathfrak{m}_{\widetilde{R}}).$$

Applying Fact 1 to $\widetilde{R}$, since degree of $a_j < n(t-1)$, we see that $\widetilde{a}_j$ must be the zero in $\widetilde{R}$. This means

$$a_j \in (x_j).$$

Hence, we can assume there exists $a'_j \in R_{d-1}$ such that $a_j = a'_j x_j$, for $j = 0, 1, \ldots, n$. Let $a' = a'_0(x_0, \ldots, x_n)$, which is an element in $\operatorname{Im}\phi$. Then $\overline{a}$ is also the image of $a - a'$. So, we can replace a by $a - a'$ to assume a is of the form $(0, a_1, \ldots, a_n)$ in the first place. Therefore, by (4.3), we get

$$u_i x_0 = 0, \quad 0 \le i \le n. \tag{4.6}$$

Multiplying both sides of (4.4) by x_0, we then obtain

$$a_j x_i x_0 = 0, \quad 0 \le i, j \le n. \tag{4.7}$$

Therefore, for every $j = 0, 1, \ldots, n$, $a_j x_0 \in (0 : \mathfrak{m}_R)$. Thus, by Fact 1 again, we get

$$a_j x_0 = \lambda_j x_0^{t-1} \cdots x_n^{t-1} \tag{4.8}$$

for some $\lambda_j \in k$. This contradicts the fact that $\deg a_j < n(t-1)$. □

Corollary 4.7. *Adopt all of the notation of Theorem 4.2. Assume $R = S/fS$ attains the minimal Hilbert–Kunz function. Then there exists a constant c, such that*

$$\lambda\big(\operatorname{soc}\big(R/\mathfrak{m}^{[q]}\big)\big) = cq^{n-2} + O\big(q^{n-3}\big).$$

Moreover, the constant c has the following expressions:

$$c = \begin{cases} \frac{d((-1)^{n-d}-3)}{2n!}\binom{n}{2}\sum_{i=0}^{\nu-1}(-1)^i\binom{n+1}{i}(\nu-i)^{n-2}, & \text{if } n = 2\nu - 1 \text{ is odd,} \\ \frac{d((-1)^{n-d}-3)}{2n!}\binom{n}{2}\sum_{i=0}^{\nu-1}(-1)^i\binom{n+1}{i}(\nu+\frac{1}{2}-i)^{n-2}, & \text{if } n = 2\nu \text{ is even.} \end{cases}$$

Proof. By Theorem 4.5, it remains to calculate $\dim_k \theta_{m(q)}$, which is equal to $\dim_k \mathbf{\Theta}_{m(q)} - \dim_k \mathbf{\Theta}_{m(q-d)}$ by (4.1). Note that $\dim_k \mathbf{\Theta}_i$ is the coefficient of t^i in the polynomial $(1 + t + t^2 + \cdots + t^{q-1})^{n+1}$. The rest of the calculation is completely elementary, which will be carried out in detail in the next section. □

Remark 4.8. Applying Corollary 4.7 to the 3-dimensional hypersurface $k[x,y,z,w]/(xy - zw)$, we see that $\lambda(\operatorname{soc}(R/\mathfrak{m}^{[q]})) = 4q - 3$; to the Cayley's cubic surface $k[x,y,z,w]/(xyz + xyw + xzw + yzw)$, we get $\lambda(\operatorname{soc}(R/\mathfrak{m}^{[q]})) = 3q - 3$. One might expect that in general, the limit of $\lambda(\operatorname{soc}(R/\mathfrak{m}^{[q]}))/q^{\max\{0,n-2\}}$ (as $q \to \infty$) exists. However, the following example in dimension two provides a negative answer for this question. Let R be the coordinate ring of the rational quintic curve in $\mathbb{P}^3$ parametrized by $(s,t) \to (t^5, st^4, s^4t, s^5)$ in characteristic 2. Our Macaulay 2 experiment gives the following eventually periodic sequence for the lengths of $\operatorname{soc}(R/\mathfrak{m}^{[q]})$:

q	1	2	3	4	5	6	7	8	9	10	11	12	...
Socle length	5	9	13	21	19	17	17	21	19	17	17	21	...

The author thanks Jason McCullough for providing some help on Macaulay 2 programming.

REMARK 4.9. This remark is due to Florian Enescu and Yongwei Yao. Let $(R,\mathfrak{m})$ be a local ring and $S=R[x]_{(\mathfrak{m},x)}$ with maximal ideal $\mathfrak{n}=(\mathfrak{m},x)$. Then $\lambda(\operatorname{soc}(R/\mathfrak{m}^{[q]}))=\lambda(\operatorname{soc}(S/\mathfrak{n}^{[q]}))$ for all q. We leave the verification of this to the interested readers.

5. Two combinatorial identities

This entire section is elementary. We prove a combinatorial result Theorem 5.1, which contains the calculation we mentioned in the proof of Corollary 4.7 as a special case.

Fix an integer $n>0$, consider the following function

$$\sum_{i\geq 0}\Gamma(i)t^i=\left(1+t+t^2+\cdots+t^{q-1}\right)^{n+1}. \tag{5.1}$$

Assume $m(q)$ is an integer-valued function of the following form

$$m(q)=\left(\frac{n+1}{2}\right)q+\xi+\varepsilon,$$

where ξ is a constant and ε is defined according to

$$\varepsilon=\begin{cases}0, & \text{if } n \text{ is odd},\\ (1/2)(q-2\lfloor q/2\rfloor), & \text{if } n \text{ is even}.\end{cases}$$

For any fixed integer $d>0$ define

$$h(q)=\Gamma\big(m(q)\big)-\Gamma\big(m(q)-d\big). \tag{5.2}$$

Then, we have the following estimate about $h(q)$.

THEOREM 5.1. *There exists a constant c, such that*

$$h(q)=cq^{n-2}+O\big(q^{n-3}\big).$$

Moreover, the constant c has the following expressions

$$c=\begin{cases}\frac{d}{n!}\binom{n}{2}(2\xi+2\varepsilon+n-d+1)\sum_{i=0}^{\nu-1}(-1)^i\binom{n+1}{i}(\nu-i)^{n-2},\\ \quad \textit{if } n=2\nu-1,\\ \frac{d}{n!}\binom{n}{2}(2\xi+2\varepsilon+n-d+1)\sum_{i=0}^{\nu-1}(-1)^i\binom{n+1}{i}(\nu+\frac{1}{2}-i)^{n-2},\\ \quad \textit{if } n=2\nu.\end{cases}$$

Proof. By binomial theorem, we have

$$\begin{aligned}&\left(1+t+t^2+\cdots+t^{q-1}\right)^{n+1}\\ &\quad=\frac{(1-t^q)^{n+1}}{(1-t)^{n+1}}\\ &\quad=\left(\sum_{i=0}^{n+1}(-1)^i\binom{n+1}{i}t^{qi}\right)\left(\sum_{i=0}^{\infty}\binom{n+i}{n}t^i\right).\end{aligned} \tag{5.3}$$

Case 1. $n = 2\nu - 1$. Then (we leave ε here for the purpose of Case 2, even though it is 0 here)

$$m(q) = \nu q + \xi + \varepsilon.$$

By comparing the coefficients of $t^{m(q)}$ in (5.1) and (5.3), we get

$$\Gamma\big(m(q)\big) = \sum_{i=0}^{\nu-1} (-1)^i \binom{n+1}{i} \binom{(\nu-i)q + \xi + \varepsilon + n}{n} \tag{5.4}$$

and

$$\Gamma\big(m(q) - d\big) = \sum_{i=0}^{\nu-1} (-1)^i \binom{n+1}{i} \binom{(\nu-i)q + \xi + \varepsilon + n - d}{n}. \tag{5.5}$$

Let

$$g(q) = \binom{(\nu-i)q + \xi + \varepsilon + n}{n} - \binom{(\nu-i)q + \xi + \varepsilon + n - d}{n}, \tag{5.6}$$

then, one could rewrite $h(q)$ as

$$h(q) = \sum_{i=0}^{\nu-1} (-1)^i \binom{n+1}{i} g(g). \tag{5.7}$$

To estimate $g(q)$, we use *Stirling numbers of the first kind* $s(n,k)$ to expand $g(q)$ as a polynomial of $(\nu - i)q$. Recall by definition, $s(n,k)$ is the coefficient of x^k in the polynomial $x(x-1)\cdots(x-n+1)$, i.e.,

$$x(x-1)\cdots(x-n+1) = \sum_{k=0}^{n} s(n,k) x^k. \tag{5.8}$$

Therefore, for any integer Z, we have

$$\begin{aligned}
n!&\binom{(\nu-i)q + Z}{n} \\
&= \sum_{k=0}^{n} s(n,k)\big((\nu-i)q + Z\big)^k \\
&= \sum_{k=0}^{n} s(n,k)\left(\sum_{j=0}^{k}\binom{k}{j}\big((\nu-i)^j q^j Z^{k-j}\big)\right) \\
&= (\nu-i)^n q^n + \left(\binom{n}{1}\big((v-i)q\big)^{n-1} Z - \binom{n}{2}\big((v-i)q\big)^{n-1}\right) \\
&\quad + \left(\binom{n}{2}\big((v-i)q\big)^{n-2} Z^2 - \binom{n}{2}\binom{n-1}{1}\big((v-i)q\big)^{n-2} Z \right.\\
&\quad \left. + s(n,n-2)\big((v-i)q\big)^{n-2}\right) \\
&\quad + o\big(q^{n-3}\big).
\end{aligned}$$

Here in the last equality, we use the fact $s(n,n)=1$ and $s(n,n-1)=-\binom{n}{2}$. It follows that

$$
\begin{aligned}
n!g(q) &= d\big((v-i)q\big)^{n-1}\binom{n}{1} \\
&\quad + d\big((v-i)q\big)^{n-2}\left(\binom{n}{2}(2\xi+2\varepsilon+2n-d)-\binom{n}{2}\binom{n-1}{1}\right) \\
&\quad + o\big(q^{n-3}\big) \\
&= d\big((v-i)q\big)^{n-1}\binom{n}{1} + d\big((v-i)q\big)^{n-2}\binom{n}{2}(2\xi+2\varepsilon+n-d+1) \\
&\quad + o\big(q^{n-3}\big).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
h(q) = q^{n-1}&\left(\frac{d}{n!}\binom{n}{1}\sum_{i=0}^{\nu-1}(-1)^i\binom{n+1}{i}(\nu-i)^{n-1}\right) \\
&+ q^{n-2}\left(\frac{d}{n!}\binom{n}{2}(2\xi+2\varepsilon+n-d+1)\sum_{i=0}^{\nu-1}(-1)^i\binom{n+1}{i}(\nu-i)^{n-2}\right) \\
&+ o\big(q^{n-3}\big).
\end{aligned}
$$

Hence, by (5.9), the coefficient of q^{n-1} in $h(q)$ is 0. Moreover,

$$
c = \frac{d}{n!}\binom{n}{2}(2\xi+2\varepsilon+n-d+1)\sum_{i=0}^{\nu-1}(-1)^i\binom{n+1}{i}(\nu-i)^{n-2}.
$$

Case 2. $n=2\nu$. In this case,

$$
m(q) = \left(\nu+\frac{1}{2}\right)q+\xi+\varepsilon.
$$

Exactly the same computations yield

$$
\begin{aligned}
h(q) = q^{n-1}&\left(\frac{d}{n!}\binom{n}{1}\sum_{i=0}^{\nu-1}(-1)^i\binom{n+1}{i}\left(\nu+\frac{1}{2}-i\right)^{n-1}\right) \\
&+ q^{n-2}\left(\frac{d}{n!}\binom{n}{2}(2\xi+2\varepsilon+n-d+1)\right. \\
&\left.\times\sum_{i=0}^{\nu-1}(-1)^i\binom{n+1}{i}\left(\nu+\frac{1}{2}-i\right)^{n-2}\right) \\
&+ o\big(q^{n-3}\big).
\end{aligned}
$$

So by (5.10), the coefficient of q^{n-1} in $h(q)$ is 0 and

$$c = \frac{d}{n!}\binom{n}{2}(2\xi + 2\varepsilon + n - d + 1)\sum_{i=0}^{\nu-1}(-1)^i\binom{n+1}{i}\left(\nu + \frac{1}{2} - i\right)^{n-2}. \qquad \square$$

LEMMA 5.2. *For any positive integer n, the following identities hold:*

$$\sum_{i=0}^{n}(-1)^i\binom{2n}{i}(n-i)^{2n-2} = 0 \tag{5.9}$$

and

$$\sum_{i=0}^{n}(-1)^i\binom{2n+1}{i}\left(n - i + \frac{1}{2}\right)^{2n-1} = 0. \tag{5.10}$$

Proof. We only prove (5.9). The proof of (5.10) is similar. The following elementary proof of (5.9) is suggested by Daniel Smith-Tone to the author. First, we notice that

$$\begin{aligned}
&2\sum_{i=0}^{n}(-1)^i\binom{2n}{i}(n-i)^{2n-2}\\
&\quad= \sum_{i=0}^{n}(-1)^i\binom{2n}{i}(n-i)^{2n-2} + \sum_{j=n+1}^{2n}(-1)^j\binom{2n}{j}(n-j)^{2n-2}\\
&\quad= \sum_{i=0}^{2n}(-1)^i\binom{2n}{i}(n-i)^{2n-2}.
\end{aligned}$$

For any function $f(x)$, one defines $\triangle f(x)$, the *forward difference* of $f(x)$, to be the function $f(x+1) - f(x)$. The higher order forward difference is defined recursively by $\triangle^n f(x) = \triangle^{n-1}(\triangle f(x))$. It is then easy to check that for any positive integer k,

$$\triangle^k f(x) = \sum_{i=0}^{k}(-1)^i\binom{k}{i}f(x+k-i).$$

Apply the above to $f(x) = x^{2n-2}$. Since $f(x)$ is a polynomial of degree $2n-2$, $\triangle^{2n} f(x)$ must be zero, that is,

$$\sum_{i=0}^{2n}(-1)^i\binom{2n}{i}(x+2n-i)^{2n-2} = 0.$$

In particular, we can take $x = -n$ and the identity is proved. $\square$

6. Socle length and Betti number

In this section, we provide some connections between the socle length function we considered in Section 4 and the asymptotic growth of some other invariants in characteristic p, such as Betti numbers. The main result of this section is the following theorem.

THEOREM 6.1. *Let $(R, \mathfrak{m}, k)$ be a local ring in characteristic p. Let I and $\mathfrak{a}$ be $\mathfrak{m}$-primary ideals. Suppose $I = J + uR$ for some $u \in R$, and J is an ideal of finite projective dimension which satisfies the condition that $R/J^{[q]}$ is Artinian Gorenstein for $q \gg 0$. Then the differences between any of the following two numerical functions (as functions on q) are bounded as $q \to \infty$:*

(1) $\lambda(\mathrm{Hom}(R/\mathfrak{a}, R/I^{[q]}))$;
(2) $\lambda(\mathrm{Tor}_1(R/I, {}^{f^n}\!R) \otimes R/\mathfrak{a})$;
(3) $\lambda(\mathrm{Tor}_2(R/I^{[q]}, R/\mathfrak{a}))$.

Proof. We first show for a given R-module M,

$$\lambda\big(\mathrm{Hom}(R/\mathfrak{a}, M)\big) = \lambda\big(\mathrm{Hom}(M, E) \otimes R/\mathfrak{a}\big), \tag{6.1}$$

where $E = E(k)$ is the injective hull of k. To see this, suppose $\mathfrak{a} = (a_1, \ldots, a_c)$ and consider the following exact sequence

$$0 \longrightarrow \mathrm{Hom}(R/\mathfrak{a}, M) \longrightarrow M \longrightarrow \bigoplus_1^c M, \tag{6.2}$$

where the rightmost map sends every $\tau \in M$ to $(a_1\tau, \ldots, a_c\tau) \in \bigoplus_1^c M$. Then the equality (6.1) follows easily from taking the Matlis dual of (6.2).

Applying (6.1) to the case $M = R/I^{[q]}$ (take q large enough such that $J^{[q]} \subseteq \mathfrak{a}$), since $R/J^{[q]}$ is Gorenstein, $E = R/J^{[q]}$, we obtain

$$\lambda\big(\mathrm{Hom}\big(R/\mathfrak{a}, R/I^{[q]}\big)\big) = \lambda\big(\mathrm{Hom}\big(R/I^{[q]}, R/J^{[q]}\big) \otimes R/\mathfrak{a}\big).$$

Observe that

$$\mathrm{Hom}\big(R/I^{[q]}, R/J^{[q]}\big) \cong \frac{J^{[q]} : I^{[q]}}{J^{[q]}} = \frac{J^{[q]} : u^q}{J^{[q]}}.$$

The last equality here is due to our assumption $I = J + uR$. We therefore have

$$\lambda\big(\mathrm{Hom}\big(R/\mathfrak{a}, R/I^{[q]}\big)\big) = \lambda\left(\frac{J^{[q]} : u^q}{J^{[q]}} \otimes R/\mathfrak{a}\right). \tag{6.3}$$

Again, since $I = J + uR$, we have a short exact sequence

$$0 \longrightarrow \frac{R}{J : u} \longrightarrow \frac{R}{J} \longrightarrow \frac{R}{I} \longrightarrow 0. \tag{6.4}$$

Tensoring (6.4) with ${}^{f^n}R$. Notice that $\mathrm{Tor}_1(R/J, {}^{f^n}R) = 0$ since J has finite projective dimension, we see there exists an exact sequence

$$0 \longrightarrow \mathrm{Tor}_1\big(R/I, {}^{f^n}R\big) \longrightarrow \frac{R}{(J:u)^{[q]}} \longrightarrow \frac{R}{J^{[q]}} \longrightarrow \frac{R}{I^{[q]}} \longrightarrow 0. \tag{6.5}$$

Comparing the exact sequence (6.5) with the short exact sequence

$$0 \longrightarrow \frac{R}{J^{[q]} : u^q} \longrightarrow \frac{R}{J^{[q]}} \longrightarrow \frac{R}{I^{[q]}} \longrightarrow 0,$$

which is obtained in a way similar to (6.4), we have

$$\mathrm{Tor}_1\big(R/I, {}^{f^n}R\big) \cong \frac{J^{[q]} : u^q}{(J:u)^{[q]}}.$$

Hence, we obtain a short exact sequence

$$0 \longrightarrow \frac{(J:u)^{[q]}}{J^{[q]}} \longrightarrow \frac{J^{[q]} : u^q}{J^{[q]}} \longrightarrow \mathrm{Tor}_1\big(R/I, {}^{f^n}R\big) \longrightarrow 0. \tag{6.6}$$

Tensoring (6.6) with $R/\mathfrak{a}$ gives rise to an exact sequence

$$\longrightarrow \frac{(J:u)^{[q]}}{J^{[q]}} \otimes R/\mathfrak{a} \longrightarrow \frac{J^{[q]} : u^q}{J^{[q]}} \otimes R/\mathfrak{a} \longrightarrow \mathrm{Tor}_1\big(R/I, {}^{f^n}R\big) \otimes R/\mathfrak{a} \longrightarrow 0.$$

It follows that

$$\begin{aligned} 0 &\le \lambda\left(\frac{J^{[q]} : u^q}{J^{[q]}} \otimes R/\mathfrak{a}\right) - \lambda\big(\mathrm{Tor}_1\big(R/I, {}^{f^n}R\big) \otimes R/\mathfrak{a}\big) \\ &\le \lambda\left(\frac{(J:u)^{[q]}}{J^{[q]}} \otimes R/\mathfrak{a}\right). \end{aligned}$$

Thus from (6.3), we conclude that

$$\begin{aligned} 0 &\le \lambda\big(\mathrm{Hom}\big(R/\mathfrak{a}, F^n(R/I)\big)\big) - \lambda\big(\mathrm{Tor}_1\big(R/I, {}^{f^n}R\big) \otimes R/\mathfrak{a}\big) \\ &\le \lambda\left(\frac{(J:u)^{[q]}}{J^{[q]}} \otimes R/\mathfrak{a}\right). \end{aligned}$$

But the right-hand side is

$$\le \lambda\left(\frac{(J:u)^{[q]}}{J^{[q]}} \otimes k\right)\lambda(R/\mathfrak{a}) \le \lambda\left(\frac{(J:u)}{J} \otimes k\right)\lambda(R/\mathfrak{a}) = O(1).$$

Therefore, the difference between (1) and (2) is bounded as $q \to \infty$.

To establish the boundedness of the difference between (2) and (3), we use some spectral sequence arguments. Let $F_\bullet$ be the minimal free resolution of R/I and $G_\bullet$ the minimal free resolution of $R/\mathfrak{a}$. The double complex $F^n(F_\bullet) \otimes G_\bullet$ yields the following spectral sequence

$$\mathrm{Tor}_i\big(\mathrm{Tor}_j\big(R/I, {}^{f^n}R\big), R/\mathfrak{a}\big) \Rightarrow H_{i+j}\big(F^n(F_\bullet) \otimes R/\mathfrak{a}\big).$$

From the exact sequence of low degree terms, we get the following exact sequence

$$\begin{aligned} H_2\big(F^n(F_\bullet)\otimes R/\mathfrak{a}\big) &\longrightarrow \mathrm{Tor}_2\big(F^n(R/I),R/\mathfrak{a}\big)\longrightarrow \mathrm{Tor}_1\big(R/I,{}^{f^n}\!R\big)\otimes R/\mathfrak{a}\\ &\xrightarrow{0} H_1\big(F^n(F_\bullet)\otimes R/\mathfrak{a}\big)\\ &\xrightarrow{\varsigma} \mathrm{Tor}_1\big(F^n(R/I),R/\mathfrak{a}\big)\longrightarrow 0.\end{aligned}$$

In this exact sequence, we choose $n \gg 0$ so that $\mathfrak{m}^{[q]} \subseteq \mathfrak{a}$, which forces the map ς to be an isomorphism. It then follows that

$$\begin{aligned} 0 &\le \lambda\big(\mathrm{Tor}_2\big(F^n(R/I),R/\mathfrak{a}\big)\big) - \lambda\big(\mathrm{Tor}_1\big(R/I,{}^{f^n}\!R\big)\otimes R/\mathfrak{a}\big)\\ &\le \lambda\big(H_2\big(F^n(F_\bullet)\otimes R/\mathfrak{a}\big)\big).\end{aligned} \tag{6.7}$$

Since $\mathfrak{m}^{[q]} \subseteq \mathfrak{a}$, we also have

$$H_i\big(F^n(F_\bullet)\otimes R/\mathfrak{a}\big) = \bigoplus_{1}^{\operatorname{rank} F_i} R/\mathfrak{a}.$$

So the right-hand side of (6.7) equals $(\operatorname{rank} F_2)\lambda(R/\mathfrak{a})$, which is independent of q. $\square$

We point out here that in Theorem 6.1, if we take $\mathfrak{a}$ to be the maximal ideal $\mathfrak{m}$, then (1) gives us the socle length function we considered in Section 4 and (3) gives the second Betti numbers of $R/I^{[q]}$.

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