

# OPERATOR ANALOGUES OF MAHLER'S MEASURE

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**ABSTRACT.** Motivated by a geometric meaning of Mahler's measure, we introduce two operator analogues of Mahler's measure. This leads to some interesting equalities and inequalities between the two operator-theoretic Mahler measures and the classical Mahler measure. In order to apply these results to the operator version of Lehmer's problem, we introduce and study an important class of operators, the so-called subharmonic operators. It is shown that the operator version of Lehmer's problem fails under some mild condition.

## 1. Introduction

Let  $\mathbb{Z}[z]$  and  $\mathbb{C}[z]$  denote the polynomial rings in  $z$  with integer and complex coefficients, respectively. Denote the open unit disk by  $\mathbb{D}$ , and the unit circle by  $\mathbb{T}$ . In this paper,  $H$  always denotes a Hilbert space, and  $B(H)$  denotes the set of all linear bounded operators acting on  $H$ .

In order to manufacture large primes, Lehmer paid his attention to monic integral polynomial

$$p(z) = z^d + a_{d-1}z^{d-1} + \cdots + a_1z + a_0 \in \mathbb{Z}[z].$$

Decompose  $p(z)$  on  $\mathbb{C}$  as

$$p(z) = \prod_{i=1}^d (z - \alpha_i),$$

and define

$$\Delta_n(p) = \prod_{i=1}^d (\alpha_i^n - 1), \quad n = 1, 2, \dots$$

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Since  $p(z)$  is a monic integral polynomial, it is easy to see that  $\Delta_n(p) \in \mathbb{Z}$ . The function  $\Delta_n(p)$  was introduced by Pierce [Pi]. In 1933, Lehmer [Le] proved that  $\Delta_n(p)$  is more likely to produce primes if it does not grow too quickly. Let  $\Omega(p)$  be the absolute value of the product of those roots of  $p$  which lie outside the unit circle. If  $p$  has no root on the unit circle, then  $\lim_{n \rightarrow \infty} |\frac{\Delta_{n+1}(p)}{\Delta_n(p)}| = \Omega(p)$ . Thus for any monic integral polynomial  $p$ , Lehmer used  $\Omega(p)$  to measure the rate of growth of the sequence  $\{\Delta_n(p)\}_{n=1}^\infty$ . It is clear that  $\Omega(p) \geq 1$ . Lehmer noticed the polynomial

$$L(z) = z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1$$

with  $\Omega(L) = 1.176280\dots$ . However, he failed to find a monic integral polynomial  $p$  such that  $1 < \Omega(p) < \Omega(L)$ . Then he asked if for every  $\varepsilon > 0$  there exists a monic polynomial  $p \in \mathbb{Z}[z]$  satisfying  $1 < \Omega(p) < 1 + \varepsilon$ . This is known as “Lehmer’s problem” or “Lehmer’s conjecture”, which remains to be an open problem.

Thirty years after Lehmer’s paper [Le], Mahler gave a generalized definition of  $\Omega(p)$ . For a nonzero polynomial

$$p(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_0 = a_d \prod_{i=1}^d (z - \alpha_i) \in \mathbb{C}[z],$$

he defined

$$M(p) = |a_d| \cdot \prod_{i=1}^d \max\{1, |\alpha_i|\}.$$

$M(p)$  is called the Mahler measure of  $p$ . Observe that  $M(p) \geq 1$  for each  $p \in \mathbb{Z}[z]$  and when  $a_d = 1$ ,  $M(p) = \Omega(p)$ . By a classical theorem of Kronecker [EW, p. 27, Theorem 1.31], [Kr], for any  $p \in \mathbb{Z}[z]$ ,  $M(p) = 1$  if and only if  $p(z) = z^n q(z)$  for some nonnegative integer  $n$  and a cyclotomic polynomial  $q$ . Recall that a *cyclotomic polynomial* is a monic integral polynomial all of whose zeros are roots of unity. Thus Lehmer’s problem is equivalent to the question: Is there a sequence of noncyclotomic integral polynomial  $\{p_n\}$  with  $p_n(0) \neq 0$  for all  $n$  such that

$$(1) \quad \lim_{n \rightarrow \infty} M(p_n) = 1?$$

Lehmer’s problem and Mahler’s measure arise in different areas of mathematics, for example, iteration of complex functions, transcendence and diophantine approximation theory, Fuglede–Kadison determinant in operator algebra [De], ergodic theory [Li], knot theory [Hi], and etc. See [Sm] for survey of Lehmer’s problem, and also refer to [Mo].

An important observation was also made in Mahler’s paper [Ma60]:

$$M(p) = \exp \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |p(e^{i\theta})| d\theta \right], \quad p \in \mathbb{C}[z].$$

This means that  $M(p)$  is multiplicative. Combining Szegő's theorem, one can give a geometric meaning of Mahler's measure in the context of Hilbert space, that is,

$$(2) \quad M(p) = \text{dist}(p, [zp]) = \inf_{q \in \mathbb{C}[z]} \|[1 - q(z)z]p(z)\|_{H^2}, \quad p \in \mathbb{C}[z].$$

Here

$$H^2 = \{f : f \in L^2(\mathbb{T}), \widehat{f}(n) = 0 \text{ for all } n < 0\}$$

is the Hardy space on the unit circle  $\mathbb{T}$  and  $[zp]$  denotes the closed invariant subspace of the Hardy shift generated by  $zp$ , that is,

$$[zp] = \text{cl}\{q(z)zp(z), q \in \mathbb{C}[z]\}.$$

In (2), the equality  $M(p) = \inf_{q \in \mathbb{C}[z]} \|[1 - q(z)z]p(z)\|_{H^2}$  was known in [Deg].

Let  $S$  denote the Hardy shift, that is,

$$(Sf)(z) = zf(z), \quad f \in H^2.$$

Then we can rewrite (2) as follows:

$$M(p) = \text{dist}(p(S)1, [Sp(S)1]) = \inf_{q \in \mathbb{C}[z]} \|[I - q(S)S]p(S)1\|.$$

By using the inner-outer factorization of functions in  $H^2$ , one will see that for any unit vector  $e$  in  $H^2$

$$\text{dist}(p(S)e, [Sp(S)e]) \leq M(p), \quad p \in \mathbb{C}[z] \text{ (see Section 2 for details).}$$

Hence

$$M(p) = \sup_{\|e\|=1} \text{dist}(p(S)e, [Sp(S)e]).$$

Inspired by these observations, we will introduce and study two operator analogues of Mahler's measure.

Let  $T \in B(H)$ . For  $h \in H$ , let  $[h]$  denote the closed invariant subspace of  $T$  generated by  $h$ , that is,

$$[h] = \overline{\text{span}}\{h, Th, T^2h, \dots\}.$$

Then for each polynomial  $p$  and  $e \in H$ , define

$$M_T^e(p) = \text{dist}(p(T)e, [Tp(T)e]) = \inf_{q \in \mathbb{C}[z]} \|[I - q(T)T]p(T)e\|,$$

called the  $T$ -Mahler measure of  $p$  on  $e$ ; and set

$$M_T(p) = \sup_{\|e\|=1} M_T^e(p),$$

called the  $T$ -Mahler measure of  $p$ . We will establish some connection between the  $T$ -Mahler measure and the classical Mahler measure. This makes it possible to study Lehmer's problem in the context of operator theory.

This paper is organized as follows.

In Section 2, we will pay attention to the properties of the  $T$ -Mahler measure. In particular, when  $T$  is a contraction,  $M_T(p) \leq M(p)$  for all  $p \in \mathbb{C}[z]$ . The multiplicativity properties of  $M_T$  and  $M_T^e$  are also studied.

In Section 3, in order to generalize Lehmer's problem in the context of operator theory, we will introduce and study an important class of operators, the so-called subharmonic operators which is closely related to the operator-theoretic Mahler measure.

Section 4 is devoted to applications of our results in previous sections. It is shown that the operator version of Lehmer's problem fails under some mild condition. As an application, one gives new proofs of some known results in [Pr08a] and [Hu], see Example 4.5 and Remark 4.7.

## 2. Operator analogues of Mahler's measure

In this section, we present a geometric meaning of Mahler's measure.

Motivated by this, two operator analogues of Mahler's measure are introduced. Some connection between the  $T$ -Mahler measure and the classical Mahler measure are realized. Finally, we will pay attention to the multiplicativity properties of the  $T$ -Mahler measure.

Let us recall the classical Szegő theorem [Sz], [Ho, p. 49]. Let  $\mu$  be a finite positive Borel measure on the unit circle  $\mathbb{T}$  and  $h$  be the derivative of  $\mu$  with respect to the normalized Lebesgue measure. That is,

$$d\mu = h(e^{i\theta}) \frac{d\theta}{2\pi} + d\mu_s,$$

where  $d\mu_s$  and  $\frac{d\theta}{2\pi}$  are mutually singular and  $h \in L^1(\frac{d\theta}{2\pi})$ . Szegő's theorem states that

$$\inf_{f \in A_0} \int |1 - f|^2 d\mu = \exp \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log h(e^{i\theta}) d\theta \right],$$

where  $A_0$  is the disk algebra defined by

$$\left\{ f : f \in C(\mathbb{T}), \hat{f}(n) \triangleq \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta} \frac{d\theta}{2\pi} = 0, n \leq 0 \right\}.$$

In particular, if  $p \in \mathbb{C}[z]$  and  $d\mu = |p|^2 \frac{d\theta}{2\pi}$ , then by Szegő's theorem we have

$$\begin{aligned} M^2(p) &= \exp \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |p|^2 d\theta \right] \\ &= \inf_{f \in A_0} \int_T |1 - f|^2 |p|^2 \frac{d\theta}{2\pi} \\ &= \inf_{q \in \mathbb{C}[z]} \int_T |(1 - qz)p|^2 \frac{d\theta}{2\pi} \end{aligned}$$

$$\begin{aligned} (3) \quad &= \inf_{q \in \mathbb{C}[z]} \|(1 - qz)p\|^2 \\ (4) \quad &= \text{dist}^2(p, [zp]). \end{aligned}$$

As mentioned in the [Introduction](#), let  $S$  be the Hardy shift. Then we have the following operator-theoretic form of Mahler's measure

$$\begin{aligned} M(p) &= \text{dist}(p, [zp]) \\ &= \inf_{q \in \mathbb{C}[z]} \|[1 - q(z)z]p(z)\| \\ &= \inf_{q \in \mathbb{C}[z]} \|[I - q(S)S]p(S)1\| \\ (5) \quad &= \text{dist}(p(S)1, [Sp(S)1]). \end{aligned}$$

Inspired by this observation, we have the following definition.

**DEFINITION 2.1.** Let  $T \in B(H)$  and  $[h]$  denote the closed invariant subspace of  $T$  generated by  $h \in H$ . Then for each vector  $e \in H$ , define

$$M_T^e(p) = \text{dist}(p(T)e, [Tp(T)e]) = \inf_{q \in \mathbb{C}[z]} \|[I - q(T)T]p(T)e\|, \quad p \in \mathbb{C}[z],$$

called the  $T$ -Mahler measure on  $e$ .

For example, let  $T = S$ . For each unit vector  $e \in H^2$  and any polynomial  $p$ , we have

$$\begin{aligned} M_S^e(p) &= \text{dist}(p(S)e, [Sp(S)e]) \\ &= \left( \inf_{f \in \mathbb{C}[z], f(0)=0} \int_{\mathbb{T}} |pe - fpe|^2 \frac{d\theta}{2\pi} \right)^{\frac{1}{2}} \\ &= \exp \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |pe| d\theta \right] \\ &= \exp \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |p| d\theta \right] \exp \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |e| d\theta \right] \\ &\leq \exp \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |p| d\theta \right] \\ &= M(p). \end{aligned}$$

Thus

$$\sup_{\|e\|=1} M_S^e(p) = M(p).$$

This leads to the following definition.

**DEFINITION 2.2.** Let  $T \in B(H)$ , set

$$M_T(p) = \sup_{\|e\|=1} M_T^e(p), \quad p \in \mathbb{C}[z].$$

Then  $M_T$  is called the  $T$ -Mahler measure.

It is easy to verify that  $M_T$  is unitary invariant for  $T$ . This means that if  $T_1$  is unitarily equivalent to  $T_2$ , then  $M_{T_1} = M_{T_2}$ . Also, observe that if  $M_T \neq 0$  then  $T$  has a nontrivial invariant subspace.

From the discussion before Definition 2.2, one sees that both  $M_S^1$  and  $M_S$  are equal to Mahler's measure. In the following, we will give more results related to Mahler's measure.

**2.1. Some connection between the  $T$ -Mahler measure and the classical Mahler measure.** We state our main result in this section as follows, and its proof is placed at the end of Section 2.1.

**THEOREM 2.3.** *For any contraction  $T \in B(H)$ , that is,  $\|T\| \leq 1$ , we have*

$$M_T(p) \leq M(p), p \in \mathbb{C}[z].$$

First, we establish the following lemma.

**LEMMA 2.4.** *Suppose  $V$  is an isometry on  $H$ . Then*

$$M_V(p) = M_V(1) \cdot M(p), \quad p \in \mathbb{C}[z].$$

*Proof.* Suppose  $V$  is an isometry on  $H$ . It is well known that  $V$  has a unitary extension  $U$ , where  $U$  can be decomposed as

$$U = \begin{bmatrix} V & I_H - VV^* \\ 0 & V^* \end{bmatrix},$$

with respect to  $K \triangleq H \oplus H$ , see [Pa, p. 6].

For each unit vector  $e \in H$ , denote by  $[e]$  the closed reducing subspace of  $U$  generated by  $e$ , and by  $U_{[e]}$  the restriction of  $U$  on  $[e]$ . Then  $U$  can be decomposed as

$$U = \begin{bmatrix} U_{[e]} & 0 \\ 0 & U' \end{bmatrix},$$

with respect to  $[e] \oplus [e]^\perp$ . It is easy to verify that

$$M_V^e(p) = M_U^e(p) = M_{U_{[e]}}^e(p), \quad p \in \mathbb{C}[z].$$

Recall that an operator  $T \in B(H)$  is called *star-cyclic* if there is a vector  $h \in H$  such that

$$H = \text{cl}\{p(T, T^*)h : p \text{ is noncommutative polynomial in two variables}\}.$$

Observe that  $U_{[e]}$  is a star-cyclic unitary operator. By the classical theory of normal operator [Co90, p. 269], [Co00, p. 51], there exists a unitary operator  $U_0 : [e] \rightarrow L^2(\mu)$  satisfying

$$N_\mu = U_0 U_{[e]} U_0^{-1} \quad \text{and} \quad U_0 e = 1.$$

Here,  $\mu$  is a probability Borel measure on the unit circle  $\mathbb{T}$ , and  $N_\mu$  is the multiplication operator on  $L^2(\mu)$  defined by

$$(N_\mu f)(z) = zf(z), \quad f \in L^2(\mu).$$

Observe that if there is a unitary operator  $U_1$  such that  $T_1 = U_1 T_2 U_1^*$  and  $U_1 e_2 = e_1$ , then  $M_{T_1}^{e_1} = M_{T_2}^{e_2}$ . Then for each  $p \in \mathbb{C}[z]$ , we have

$$\begin{aligned} M_{U_{[e]}}^e(p) &= M_{N_\mu}^1(p) \\ &= \left( \inf_{f \in A_0} \int |p - fp|^2 d\mu \right)^{\frac{1}{2}} \\ &= \left( \inf_{f \in A_0} \int |1 - f|^2 |p|^2 d\mu \right)^{\frac{1}{2}} \left( d\mu = h(e^{i\theta}) \frac{d\theta}{2\pi} + d\mu_s \right) \\ &= \left( \exp \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |p(\theta)|^2 |h(e^{i\theta})| d\theta \right] \right)^{\frac{1}{2}} \quad (\text{by Szegő's theorem}) \\ &= \exp \left[ \frac{1}{4\pi} \int_{-\pi}^{\pi} \log |h(\theta)| d\theta \right] \cdot M(p), \end{aligned}$$

where  $d\mu = h(e^{i\theta}) \frac{d\theta}{2\pi} + d\mu_s$  is the Lebesgue decomposition of  $d\mu$  relative to  $\frac{d\theta}{2\pi}$ . Thus

$$M_V^e(p) = \exp \left[ \frac{1}{4\pi} \int_{-\pi}^{\pi} \log |h(\theta)| d\theta \right] \cdot M(p), \quad p \in \mathbb{C}[z].$$

In particular, we have

$$M_V^e(1) = \exp \left[ \frac{1}{4\pi} \int_{-\pi}^{\pi} \log |h(\theta)| d\theta \right],$$

which implies that

$$(6) \quad M_V^e(p) = M_V^e(1) \cdot M(p), \quad p \in \mathbb{C}[z].$$

Therefore, for each  $p \in \mathbb{C}[z]$  we have

$$M_V(p) = \sup_{\|e\|=1} M_V^e(p) = \left( \sup_{\|e\|=1} M_V^e(1) \right) \cdot M(p) = M_V(1) \cdot M(p).$$

This completes the proof. □

The following lemma is of independent interest.

LEMMA 2.5. *If  $V$  is an isometry on  $H$ , then*

$$M_V(1) = 0 \text{ or } 1.$$

*Proof.* Since  $V$  is an isometry. By the von Neumann–Wold Decomposition theorem [Co00, p. 112, Theorem 23.7]

$$V = S' \oplus U,$$

where  $S'$  is a unilateral shift and  $U$  is a unitary operator.

If  $S' \neq 0$ , then for any unit vector  $e \in H \ominus VH$ ,

$$M_V^e(1) = M_{S'}^e(1) = 1,$$

and hence  $M_V(1) = 1$ .

If  $V$  is a unitary operator. Then there are two cases under consideration:

- I.  $M_V^e(1) = 0$  for all unit vectors  $e \in H$ .
- II. There is a unit vector  $e$  such that  $M_V^e(1) \neq 0$ .

*Case I.*  $M_V^e(1) = 0$  for all unit vectors  $e \in H$ . In this case,  $M_V(1) = 0$ .

*Case II.* There is a unit vector  $e$  such that  $M_V^e(1) \neq 0$ . By the same reasoning as in the proof of Lemma 2.4, there is a probability Borel measure  $\mu$  on  $\mathbb{T}$  such that there exists a unitary operator  $U_0 : [e] \rightarrow L^2(\mu)$  satisfying

$$N_\mu = U_0 V_{[e]} U_0^{-1} \quad \text{and} \quad U_0 e = 1.$$

Then

$$\begin{aligned} M_V^e(1) &= M_{V_{[e]}}^e(1) = M_{N_\mu}^1(1) \\ &= \left( \inf_{f \in A_0} \int |1 - f|^2 d\mu \right)^{\frac{1}{2}} \left( d\mu = h \frac{d\theta}{2\pi} + d\mu_s \right) \\ &= \exp \left[ \frac{1}{2} \int_{\mathbb{T}} \log |h| \left| \frac{d\theta}{2\pi} \right| \right] \neq 0. \end{aligned}$$

This shows that  $\log |h| \in L^1(\frac{d\theta}{2\pi})$ .

Since  $M_T$  is unitarily invariant for  $T$ , we have

$$\begin{aligned} M_{V_{[e]}}(1) &= M_{N_\mu}(1) \\ &= \sup_{\|f\|_{L^2(d\mu)}=1} M_{N_\mu}^f(1) \\ &= \sup_{\|f\|_{L^2(d\mu)}=1} \left( \inf_{p \in A_0} \int |1 - p|^2 \cdot |f|^2 d\mu \right)^{\frac{1}{2}} \\ &= \sup_{\|f\|_{L^2(d\mu)}=1} \exp \left[ \frac{1}{4\pi} \int_{-\pi}^{\pi} \log |h(\theta)| |f(\theta)|^2 d\theta \right]. \end{aligned}$$

That is,

$$(7) \quad M_{V_{[e]}}(1) = M_{N_\mu}(1) = \sup_{\|f\|_{L^2(d\mu)}=1} \exp \left[ \frac{1}{4\pi} \int_{-\pi}^{\pi} \log |h(\theta)| |f(\theta)|^2 d\theta \right].$$

Put

$$E = \text{supp } \mu_s \cup h^{-1}\{0, +\infty\},$$

and then  $E$  has Lebesgue measure zero. Set

$$f(x) = \begin{cases} 0, & x \in E, \\ \frac{1}{\sqrt{h(x)}}, & x \notin E. \end{cases}$$

Clearly,

$$\int_{\mathbb{T}} |f|^2 d\mu = 1.$$



It is easy to see that

$$M_{N_\mu}^f(1) = \exp \left[ \frac{1}{4\pi} \int_{-\pi}^{\pi} \log |h(\theta)| |f(\theta)|^2 d\theta \right] = 1.$$

Then

$$1 = M_{N_\mu}(1) = M_{V_{[e]}}(1) \leq M_V(1) \leq 1,$$

forcing  $M_V(1) = 1$ . The proof is complete.  $\square$

Combining Lemma 2.4 with Lemma 2.5 yields the following.

PROPOSITION 2.6. *Suppose  $V$  is an isometry on  $H$ . Then*

$$M_V \equiv 0 \quad \text{or} \quad M_V(p) = M(p), \quad p \in \mathbb{C}[z].$$

*In particular, if  $V$  is a nonunitary isometry, then*

$$M_V(p) = M(p), \quad p \in \mathbb{C}[z].$$

Now we are ready to give the proof of Theorem 2.3.

*Proof of Theorem 2.3.* From the classical Sz.-Nagy's dilation theorem [Pa, p. 7, Theorem 1.1],  $T$  has a unitary dilation. That is, there is a Hilbert space  $K$  with  $H \subseteq K$  and a unitary operator  $U \in B(K)$  such that

$$P_H U^n|_H = T^n, \quad n \geq 1,$$

where  $P_H$  is the projection from  $K$  to  $H$ . Since for each  $p \in \mathbb{C}[z]$ ,

$$p(T)e = P_H p(U)e,$$

we have  $\|p(T)e\| \leq \|p(U)e\|$ . This, combined with (6), shows that for any unit vector  $e \in H$ ,

$$M_T^e(p) \leq M_U^e(p) = M_U^e(1) \cdot M(p), \quad p \in \mathbb{C}[z].$$

Since  $M_U^e(1) \leq \|e\| = 1$ , it follows that

$$(8) \quad M_T^e(p) \leq M(p), \quad p \in \mathbb{C}[z].$$

Thus

$$M_T(p) = \sup_{\|e\|=1} M_T^e(p) \leq M(p), \quad p \in \mathbb{C}[z].$$

The proof is complete.  $\square$

REMARK 2.7. One may compare the inequality

$$M_T(p) \leq M(p), \quad p \in \mathbb{C}[z]$$

with the well-known von Neumann's inequality [Pa, p. 7]:

$$\|p(T)\| \leq \|p\|_\infty \triangleq \sup\{|p(z)| : |z| = 1\}, \quad p \in \mathbb{C}[z],$$

where  $T$  is a contraction in both cases.

It is not difficult to verify the followings:

$$(9) \quad \begin{aligned} M_T^{ce}(p) &= |c| M_T^e(p), \quad c \in \mathbb{C}; \\ M_{cT}^e(p) &= M_T^e(p(cz)), \quad c \in \mathbb{C} \setminus \{0\}. \end{aligned}$$

Combining (8) with (9) shows that for any  $T \in B(H)$  and  $e \in H$ ,

$$(10) \quad M_T^e(p) \leq \|e\| \cdot M(p(\|T\|z)).$$

**COROLLARY 2.8.** *Let  $T \in B(H)$  be a contraction. Then for any unit vector  $e \in H$ ,*

$$M_T^e(p) \leq M(p), \quad p \in \mathbb{C}[z].$$

*Moreover, the equality holds for all  $p \in \mathbb{C}[z]$  if and only if  $\{T^n e\}_{n=0}^\infty$  is an orthonormal sequence.*

*Proof.* The inequality follows from (8).

Suppose that  $M_T^e(p) = M(p)$  holds for all  $p \in \mathbb{C}[z]$ . We will show that  $\{T^n e\}_{n=0}^\infty$  is an orthonormal sequence. For this, notice that

$$M_T^e(z^n) = M(z^n) = 1 = \text{dist}(T^n e, \overline{\text{span}}\{T^{n+1}e, T^{n+2}e, \dots\}), \quad n \geq 0.$$

Since

$$\|T^n e\| \leq 1 \quad \text{and} \quad \text{dist}(T^n e, \overline{\text{span}}\{T^{n+1}e, T^{n+2}e, \dots\}) = 1,$$

we have

$$\|T^n e\| = 1 \quad \text{and} \quad T^n e \perp \overline{\text{span}}\{T^{n+1}e, T^{n+2}e, \dots\}.$$

On the other hand, if  $\{T^n e\}_{n=0}^\infty$  is an orthonormal sequence, and write

$$[e] = \overline{\text{span}}\{e, Te, \dots, T^n e, \dots\}.$$

Then the restriction  $T|_{[e]}$  of  $T$  on  $[e]$  is an isometric operator, and

$$M_{T|_{[e]}}^e(1) = \text{dist}(e, [Te]) = 1.$$

Then by (6), we have

$$M_T^e(p) = M_{T|_{[e]}}^e(p) = M_{T|_{[e]}}^e(1) \cdot M(p) = M(p), \quad p \in \mathbb{C}[z].$$

This completes the proof.  $\square$

**2.2. Multiplicativity.** Let  $\Phi$  be a map from  $\mathbb{C}[z]$  to  $\mathbb{R}$ . If  $\Phi(pq) = \Phi(p)\Phi(q)$  for all  $p, q \in \mathbb{C}[z]$ , then  $\Phi$  is called *multiplicative*. Clearly, Mahler's measure is multiplicative. The remaining part of this section focuses on the multiplicativity properties of the  $T$ -Mahler measure.

Recall that the Bergman space  $L_a^2(\mathbb{D})$  is defined by

$$L_a^2(\mathbb{D}) = \left\{ f : f \text{ is holomorphic on } \mathbb{D} \text{ such that } \int_{\mathbb{D}} |f(z)|^2 dA(z) < +\infty \right\},$$

where  $dA(z) = \frac{dx dy}{\pi}$  is the normalized area measure on  $\mathbb{D}$ . Denote by  $B$  the Bergman shift, defined by

$$(Bf)(z) = zf(z), \quad f \in L_a^2(\mathbb{D}).$$

Then we have  $M_B^1(z^n) = \|z^n\| = \frac{1}{\sqrt{n+1}}$ , which implies that  $M_B^1$  is not multiplicative. In general, we have the following lemma.

**LEMMA 2.9.** *Let  $\{e_n\}_{n \geq 1}$  be an orthonormal basis of  $H$  and suppose that  $T \in B(H)$  is a weighted shift such that*

$$Te_n = a_n e_{n+1}, \quad a_n \neq 0, n \geq 1.$$

*Then  $M_T^{e_1}$  is multiplicative if and only if  $|a_n| = |a_1|$  for all  $n \geq 1$ , and in this case  $M_T^{e_1}(p) = M(p(|a_1|z))$ .*

*Proof.* First assume that  $M_T^{e_1}$  is multiplicative. Then it is easy to see that  $M_T^{e_1}(z^n) = \prod_{i=1}^n |a_i|$ . Thus

$$\prod_{i=1}^n |a_i| = |a_1|^n, \quad n \in \mathbb{N}.$$

Therefore  $|a_n| = |a_1|$ ,  $n \geq 1$ .

Now assume that  $|a_n| = |a_1|$ ,  $n \geq 1$ . Then we can write

$$T = |a_1|US,$$

where  $U$  is a unitary operator satisfying  $Ue_n = e^{i\theta_n}e_n$ ,  $\theta_n \in \mathbb{R}$ ,  $n \geq 1$  and  $S$  is the Hardy shift.

Observe that

$$\|p(cUS)e_1\| = \|p(cS)e_1\|$$

holds for all  $c \in \mathbb{R}$  and  $p \in \mathbb{C}[z]$ . In fact, this identity is trivial if  $p$  is monomial; and in general, it follows from the orthogonality of  $\{e_n : n \geq 1\}$ .

Therefore, for each  $p \in \mathbb{C}[z]$  we have

$$\begin{aligned} M_T^{e_1}(p) &= \inf \left\{ \| [I - q(T)T]p(T)e_1 \| : q \in \mathbb{C}[z] \right\} \\ &= \inf \left\{ \| [I - q(|a_1|S)|a_1|S]p(|a_1|S)e_1 \| : q \in \mathbb{C}[z] \right\} \\ &= \inf \left\{ \| [I - q(S)S]p(|a_1|S)e_1 \| : q \in \mathbb{C}[z] \right\} \quad (\text{since } |a_1| \neq 0) \\ &= M_S^{e_1}(p(|a_1|z)) \\ &= M(p(|a_1|z)). \end{aligned}$$

The proof is complete. □

If one replaces  $M_T^{e_1}$  with  $M_T$  in Lemma 2.9, then we get a similar result.

**PROPOSITION 2.10.** *With the same assumption as in Lemma 2.9 and assume that  $\{|a_n|\}_{n=1}^\infty$  is a decreasing sequence. Then  $M_T$  is multiplicative if and only if  $|a_1| = |a_n|$ ,  $n \geq 1$ . In this case,  $M_T(p) = M(p(|a_1|z))$ ,  $p \in \mathbb{C}[z]$ .*

*Proof.* If  $|a_1| = |a_n|$  for all  $n \geq 1$ , then by (10)

$$M_T(p) \leq M(p(|a_1|z)).$$

By Lemma 2.9, we have

$$M_T^{e_1}(p) = M(p(|a_1|z)).$$

Thus

$$M_T(p) = M(p(|a_1|z)).$$

This implies that  $M_T$  is multiplicative.

On the other hand, assume that  $M_T$  is multiplicative. Since

$$M_T^{e_1}(z^n) = \text{dist}(T^n e_1, [T^{n+1} e_1]) = \|T^n e_1\| = \prod_{i=1}^n |a_i|,$$

and for any unit vector  $e \in H$ ,

$$M_T^e(z^n) = \text{dist}(T^n e, [T^{n+1} e]) \leq \|T^n e\| \leq \sup_{m \geq 1} \prod_{i=m}^{n+m-1} |a_i| = \prod_{i=1}^n |a_i|,$$

we get

$$M_T(z^n) = \prod_{i=1}^n |a_i|.$$

Since  $M_T$  is multiplicative, we have

$$\prod_{i=1}^n |a_i| = |a_1|^n, \quad n \geq 1.$$

By induction, we get

$$|a_1| = |a_n|, \quad n \geq 1,$$

as desired. This completes the proof.  $\square$

**COROLLARY 2.11.** *Let  $T \in B(H)$ . If there is a unit vector  $e \in H$  such that  $M_T^e(z) = \|T\| \neq 0$ , then  $M_T^e$  is multiplicative if and only if*

$$M_T^e(p) = M(p(\|T\|z)), \quad p \in \mathbb{C}[z].$$

*Proof.* The sufficiency is trivial.

Suppose that  $M_T^e$  is multiplicative. Then it is not difficult to verify that  $M_{\frac{T}{\|T\|}}^e$  is multiplicative and

$$M_{\frac{T}{\|T\|}}^e(z) = M_T^e\left(\frac{z}{\|T\|}\right) = \frac{M_T^e(z)}{\|T\|} = 1.$$

Thus

$$M_{\frac{T}{\|T\|}}^e(z^n) = 1, \quad n \geq 0.$$

By the proof of Corollary 2.8, one sees that  $\{(\frac{T}{\|T\|})^n e\}_{n=0}^\infty$  is an orthonormal sequence. Thus

$$M_{\frac{T}{\|T\|}}^e(p) = M(p), \quad p \in \mathbb{C}[z].$$

Then by (9), we have

$$M_T^e(p) = M_{\|T\| \cdot \frac{T}{\|T\|}}^e(p) = M_{\frac{T}{\|T\|}}^e(p(\|T\|z)) = M(p(\|T\|z)).$$

The proof is complete.  $\square$

### 3. Subharmonic operators

In order to generalize Lehmer's problem in the context of operator theory, we will introduce and study an important class of operators, the so-called subharmonic operators which is closely related to the operator-theoretic Mahler measure.

First, the definition of subharmonic operators is given as follows:

DEFINITION 3.1. For an operator  $T \in B(H)$ , if there is a unit vector  $e \in H$  such that

$$\|p(T)e\| \geq |p(0)|, \quad p \in \mathbb{C}[z],$$

then  $T$  is called to be subharmonic on  $e$ .

$T$  is called subharmonic if for any  $\varepsilon > 0$  there is a unit vector  $e \in H$  such that

$$\|p(T)e\| \geq |p(0)|(1 - \varepsilon), \quad p \in \mathbb{C}[z].$$

Let us see a simple example of subharmonic operators.

EXAMPLE 3.2. As mentioned before,  $S$  denotes the Hardy shift. For any  $p \in \mathbb{C}[z]$ ,

$$\|p(S)1\| = \|p\|_{H^2} = \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |p^2(e^{i\theta})| d\theta \right)^{\frac{1}{2}}.$$

Since  $|p^2(z)|$  is a subharmonic function on  $\mathbb{C}$ , we have

$$\left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |p^2(e^{i\theta})| d\theta \right)^{\frac{1}{2}} \geq |p(0)|.$$

Then

$$\|p(S)1\| \geq |p(0)|, \quad p \in \mathbb{C}[z].$$

Hence,  $S$  is subharmonic on 1. Similarly, the Bergman shift  $B$  is also a subharmonic operator.

By definition, it is easy to see that if  $T$  is subharmonic, then

$$(11) \quad M_T(p) \geq |p(0)|, \quad p \in \mathbb{C}[z].$$

Similarly, if  $T$  is subharmonic on some unit vector  $e$ , then

$$(12) \quad M_T^e(p) \geq |p(0)|, \quad p \in \mathbb{C}[z].$$

By (1), the original Lehmer's problem is equivalent to the question:

*Is there a sequence of noncyclotomic integral polynomial  $p_n$  satisfying  $p_n(0) \neq 0$  such that*

$$\lim_{n \rightarrow \infty} M_S(p_n) = 1 \quad \left( \text{or } \lim_{n \rightarrow \infty} M_S^1(p_n) = 1 \right)?$$

Inspired by this, if  $T$  is subharmonic (or  $T$  is subharmonic on some unit vector  $e$ ). Then we raise the following question for  $M_T$  (or  $M_T^e$ ):

*Is there a sequence of noncyclotomic integral polynomial  $p_n$  satisfying  $p_n(0) \neq 0$  such that*

$$(13) \quad \lim_{n \rightarrow \infty} M_T(p_n) = 1 \quad \left( \text{or } \lim_{n \rightarrow \infty} M_T^e(p_n) = 1 \right)?$$

This is the operator version of Lehmer's problem and it will be answered under some mild condition in Section 4.

**3.1. Properties of subharmonic operators.** Furthermore, the following theorem describes subharmonic operators, and its proof is placed at the end of Section 3.2.

**THEOREM 3.3.** *Suppose  $T \in B(H)$ , then the following statements are equivalent.*

- (1)  $M_T(1) = 1$ .
- (2)  $T$  is subharmonic.
- (3)  $T$  is subharmonic on some unit vector  $e$ .
- (4) There is a unit vector  $e \in H$  such that  $e \perp \overline{\text{span}}\{Te, T^2e, \dots\}$ .

The following are some other examples of subharmonic operators.

**LEMMA 3.4.** *Suppose  $T \in B(H)$ . Then the followings hold:*

- (1) *If  $\ker T \neq \{0\}$ , or  $\overline{\text{ran } T} \neq H$ , then  $T$  is subharmonic.*
- (2) *All weighted shift operators are subharmonic.*
- (3) *All semi-Fredholm operators with nonzero index are subharmonic.*

*Proof.* (1) Assume that either  $\ker T \neq \{0\}$  or  $\overline{\text{ran } T} \neq H$ . Then pick a unit vector  $e$  such that  $e \in \ker T$  or  $e \in (\overline{\text{ran } T})^\perp$ , and in either case we have

$$e \perp \overline{\text{span}}\{Te, T^2e, \dots\}.$$

Then it follows that  $T$  is subharmonic on  $e$ .

Both (2) and (3) follow directly from (1). The proof is complete. □

By Lemma 3.4, one sees that many analytic multiplication operators on function spaces are subharmonic. For example, the Hardy shift  $S$ , the Dirichlet shift [ARSW] and the Bergman shift  $B$  [DS].

Applying Theorem 3.3 and Lemma 3.4, and using a matrix decomposition technique we get the following:

**COROLLARY 3.5.** *If  $T \in M_n(\mathbb{C})$ , then  $T$  is subharmonic if and only if  $\ker T \neq \{0\}$ .*

*Proof.* If  $\ker T \neq \{0\}$ , then by Lemma 3.4(1)  $T$  is subharmonic.

Suppose  $T$  is subharmonic. By Theorem 3.3(3), there is a unit vector  $e$  such that  $T$  is subharmonic on  $e$ . Write

$$[e] = \text{span}\{e, Te, T^2e, \dots\} \quad \text{and} \quad T_1 = T|_{[e]}.$$

Then decompose  $T$  as

$$T = \begin{bmatrix} T_1 & T_2 \\ 0 & T_3 \end{bmatrix}$$

with respect to  $[e] \oplus [e]^\perp$ .

Assume conversely that  $\ker T = \{0\}$ , then  $T$  is invertible, and so is  $T_1$ . However, by Theorem 3.3(4), one can decompose  $T_1$  as follows:

$$T_1 = \begin{bmatrix} 0 & 0 \\ T_{1,1} & T_{1,2} \end{bmatrix}$$

corresponding to  $[e] = \text{span}\{e\} \oplus \text{span}\{Te, T^2e, \dots\}$ . This is a contradiction. The proof is complete.  $\square$

By the proof of Corollary 3.5, one can see that: if  $T \in B(H)$  and  $T^* = T$ , then  $T$  is subharmonic if and only if  $\ker T \neq \{0\}$ .

**3.2. The quantity  $E(T)$ .** Recall that for  $T \in B(H)$ ,

$$M_T(1) = \sup_{\|e\|=1} \inf_{q \in \mathbb{C}[z]} \| [I - q(T)T]e \| = \sup_{\|e\|=1} \text{dist}(e, [Te]).$$

We write  $E(T) = M_T(1)$ . The quantity  $E(T)$  carries key information of  $T$  and it will play an important role in the proof of Theorem 3.3.

The following inequality establish an interesting connection between the  $T$ -Mahler measure and  $E(T)$ .

**PROPOSITION 3.6.** *For each  $T \in B(H)$ , we have*

$$M_T(p) \leq \|p(T)\| \cdot E(T), \quad p \in \mathbb{C}[z].$$

*In particular, if  $T$  is a contraction, then*

$$M_T(p) \leq \|p\|_\infty \cdot E(T).$$

*Proof.* By the definition of  $M_T(p)$ , we have

$$\begin{aligned} M_T(p) &= \sup_{\|e\|=1} \inf_{q \in \mathbb{C}[z]} \| [I - q(T)T]p(T)e \| \\ &= \sup_{\|e\|=1} \left\{ \|p(T)e\| \cdot \inf_{q \in \mathbb{C}[z]} \left\| [I - q(T)T] \frac{p(T)e}{\|p(T)e\|} \right\| \right\} \\ &\leq \sup_{\|e\|=1} \|p(T)e\| \cdot M_T(1) \\ &= \|p(T)\| \cdot E(T). \end{aligned}$$

If  $T$  is a contraction, then by von Neumann's inequality  $\|p(T)\| \leq \|p\|_\infty$ , and hence

$$M_T(p) \leq \|p\|_\infty \cdot E(T).$$

The proof is complete.  $\square$

As a consequence of Proposition 3.6, if  $E(T) = 0$ , then  $M_T \equiv 0$ . Also,  $E(T)$  is a unitary invariant for  $T$ .

Lemma 2.5 says that if  $T$  is an isometry, then  $E(T) = 0$  or  $1$ . The following gives a generalization of this result, with a shorter proof.

PROPOSITION 3.7. *For each  $T \in B(H)$ ,  $E(T) \in \{0, 1\}$ .*

*Proof.* Suppose  $M_T(1) \neq 0$ , and we will show that  $M_T(1) = 1$  to finish the proof.

Since  $M_T(1) \neq 0$ , there is a unit vector  $e \in H$  such that

$$\text{dist}(e, [Te]) \neq 0.$$

Write  $H_0 = \overline{\text{span}}\{e, [Te]\}$ , and put  $T_0 = T|_{H_0}$ .

Since

$$\overline{\text{ran } T_0} = [Te] \subsetneq H_0,$$

there is a unit vector  $e_0 \in H_0 \ominus [Te]$ .

Since  $[T_0 e_0] \subseteq [Te]$ , we get

$$e_0 \perp [T_0 e_0].$$

Hence  $\text{dist}(e_0, [T_0 e_0]) = 1$ , forcing  $M_{T_0}(1) = 1$ . On the other hand, it is easy to see that

$$M_{T_0}(1) \leq M_T(1) \leq 1.$$

Therefore,  $M_T(1) = 1$ , completing the proof.  $\square$

We now will give the proof of Theorem 3.3.

*Proof of Theorem 3.3.* (2)  $\Rightarrow$  (1). If  $T$  is subharmonic, then for any  $\varepsilon > 0$  there is a unit vector  $e$  such that

$$\|p(T)e\| \geq |p(0)| \cdot (1 - \varepsilon), \quad p \in \mathbb{C}[z].$$

In particular, put  $p(z) = 1 - zq(z)$ , and then

$$\|[I - q(T)T]e\| \geq 1 - \varepsilon, \quad q \in \mathbb{C}[z].$$

This means that

$$1 - \varepsilon \leq M_T^e(1) \leq M_T(1) \leq 1.$$

By arbitrariness of  $\varepsilon$ , we have  $M_T(1) = 1$ .

(1)  $\Rightarrow$  (2). Suppose that  $M_T(1) = 1$ . Then for any  $\varepsilon > 0$  there is a unit vector  $e$  such that

$$1 - \varepsilon \leq M_T^e(1) \leq 1,$$



which implies that

$$\| [I - q(T)T]e \| \geq 1 - \varepsilon, \quad q \in \mathbb{C}[z].$$

Therefore

$$\| p(T)e \| \geq |p(0)| \cdot (1 - \varepsilon), \quad p \in \mathbb{C}[z].$$

(4)  $\Rightarrow$  (3) is trivial.

(3)  $\Rightarrow$  (4). If  $T$  is subharmonic on unit vector  $e$ , then

$$\| p(T)e \| \geq |p(0)|, \quad p \in \mathbb{C}[z].$$

In particular, for any  $q \in \mathbb{C}[z]$ , we have

$$\| [I - q(T)T]e \| \geq 1,$$

which gives  $\text{dist}(e, [Te]) = 1$ . Thus,  $e \perp \overline{\text{span}}\{Te, T^2e, \dots\}$ .

(3)  $\Rightarrow$  (2) is trivial.

(2)  $\Rightarrow$  (3). If  $T \in B(H)$  is subharmonic. Then  $M_T(1) = 1$ . By the proof of Proposition 3.7, there is a unit vector  $e \in H$  such that

$$e \perp \overline{\text{span}}\{Te, T^2e, \dots\}.$$

Thus  $T$  is subharmonic on  $e$ .

The proof of the theorem is complete. □

## 4. Applications

In this section, we will apply the results of previous sections to the operator version of Lehmer's problem (13).

The following is our main result in this section, which answers (13) under some mild condition. As applications, we give new proofs of some known results.

**THEOREM 4.1.** *Suppose  $T \in B(H)$  is subharmonic on some unit vector  $e$ , contractive and  $T^n \xrightarrow{SQT} 0$ . Then there is a sequence of noncyclotomic integral polynomials  $\{p_n\}$  with  $p_n(0) \neq 0$ , such that*

$$\lim_{n \rightarrow \infty} M_T^e(p_n) = 1.$$

The proof of Theorem 4.1 is given after Remark 4.4.

**COROLLARY 4.2.** *Suppose  $T \in B(H)$  is subharmonic and  $\|T\| < 1$ . Then there is a sequence of noncyclotomic polynomials  $\{p_n\} \in \mathbb{Z}[z]$  with  $p_n(0) \neq 0$ , such that*

$$\lim_{n \rightarrow \infty} M_T(p_n) = 1.$$

*Proof.* Combine (9), (11) and Theorem 4.1.  $\square$

To prove Theorem 4.1, we need the following proposition.

PROPOSITION 4.3. *Let  $T \in B(H)$  be a subharmonic contraction. Then for any cyclotomic polynomial  $p$ ,*

$$M_T(p) = 1.$$

*Proof.* Assume that  $T \in B(H)$  is a subharmonic operator with  $\|T\| \leq 1$ . Then by Theorem 2.3 we have

$$M_T(q) \leq M(q), \quad q \in \mathbb{C}[z].$$

For each cyclotomic polynomial  $p$ , we have

$$M_T(p) \leq M(p) = 1.$$

Since  $p$  is a cyclotomic polynomial, then  $|p(0)| = 1$ . By (11), we have

$$M_T(p) \geq |p(0)| = 1.$$

Therefore  $M_T(p) = 1$ , as desired. The proof is complete.  $\square$

REMARK 4.4. Similarly, one can show that if  $T$  is a contraction and  $T$  is subharmonic on some vector  $e$ , then  $M_T^e(p) = 1$  for any cyclotomic polynomial  $p$ . Indeed, it is a nontrivial result, because on Bergman space  $L_a^2(\mathbb{D})$  it is hard to compute  $M_B^1(z+1)$  via definition.

Now we come to the proof of Theorem 4.1.

*Proof of Theorem 4.1.* Write  $p_n = z^n + z + 1$  ( $n > 2$ ), which is a sequence of noncyclotomic polynomials [EW, p. 78, Exercise 3.12]. Since  $T$  is subharmonic on  $e$  and  $\|T\| \leq 1$ , by Remark 4.4 we have

$$M_T^e(z+1) = 1.$$

Then by definition of  $M_T^e$ , for any  $\varepsilon > 0$ , there is a polynomial  $q$  such that

$$\| [I - q(T)T](T+1)e \| < 1 + \varepsilon.$$

Since  $\lim_{n \rightarrow \infty} \|T^n e\| = 0$ , we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \| [I - q(T)T](T^n + T + 1)e \| &= \| [I - q(T)T](T+1)e \| \\ &< 1 + \varepsilon. \end{aligned}$$

Then there is a natural number  $N$  such that

$$\| [I - q(T)T](T^n + T + 1)e \| < 1 + \varepsilon, \quad n \geq N.$$

By (12), we have  $M_T^e(p) \geq |p(0)|$  for all  $p \in \mathbb{C}[z]$ , and hence

$$1 \leq M_T^e(z^n + z + 1) < 1 + \varepsilon, \quad n \geq N.$$

By arbitrariness of  $\varepsilon$ , we have

$$\lim_{n \rightarrow \infty} M_T^e(z^n + z + 1) = 1,$$

completing the proof.  $\square$

As an application of Theorem 4.1, we have the following example.

EXAMPLE 4.5. Recently, I. Pritsker [Pr08a] defined an areal analog of Mahler's measure as the following: For each polynomial  $p \in \mathbb{C}[z]$

$$\|p\|_0 \triangleq \exp \left\{ \int_{\mathbb{D}} \log |p(z)| \, dA(z) \right\}.$$

This can be regarded as a Bergman-space version of Mahler's measure.

Notice that for each  $p \in \mathbb{C}[z]$  with  $p(0) \neq 0$ ,  $\log |p(z)|$  is subharmonic on  $\mathbb{D}$  and  $\log |p(0)| \geq 0$ , and then  $\|p\|_0 \geq 1$ . Later, below Proposition 4.6 we will apply Theorem 4.1 to show that

$$\lim_{n \rightarrow \infty} \|p_n\|_0 = 1.$$

We have the following relations on  $\|p\|_0$ ,  $M_B^1(p)$  and  $M(p)$ .

PROPOSITION 4.6. *On Bergman space  $L_a^2(\mathbb{D})$ ,*

$$\|p\|_0 \leq M_B^1(p) \leq M(p), \quad p \in \mathbb{C}[z],$$

where  $B$  is the Bergman shift.

*Proof.* Observe that  $B$  is subharmonic on the unit vector 1 and  $\|B\| \leq 1$ . Then by Corollary 2.8,

$$M_B^1(p) \leq M(p), \quad p \in \mathbb{C}[z].$$

It remains to show that

$$\|p\|_0 \leq M_B^1(p), \quad p \in \mathbb{C}[z].$$

For this, notice that

$$M_B^1(p) = \left( \inf_{q \in \mathbb{C}[z], q(0)=0} \int_{\mathbb{D}} |[1 - q(z)]p(z)|^2 \, dA(z) \right)^{\frac{1}{2}}.$$

Since  $dA(z)$  is a probability measure, then by Jensen's Inequality [Ru, p. 62, Theorem 3.3], for each  $q \in \mathbb{C}[z]$  with  $q(0) = 0$  we have

$$\begin{aligned} & \int_{\mathbb{D}} |[1 - q(z)]p(z)|^2 \, dA(z) \\ & \geq \exp \left\{ \int_{\mathbb{D}} \log |[1 - q(z)]p(z)|^2 \, dA(z) \right\} \\ & = \exp \left\{ \int_{\mathbb{D}} \log |1 - q(z)|^2 \, dA(z) + \int_{\mathbb{D}} \log |p(z)|^2 \, dA(z) \right\}. \end{aligned}$$

Since the function  $\log |1 - q(z)|^2$  is subharmonic on  $\mathbb{D}$  and  $\log |1 - q(0)|^2 = 0$ , we have

$$\begin{aligned} & \exp \left\{ \int_{\mathbb{D}} \log |1 - q(z)|^2 dA(z) + \int_{\mathbb{D}} \log |p(z)|^2 dA(z) \right\} \\ & \geq \exp \left\{ \int_{\mathbb{D}} \log |p(z)|^2 dA(z) \right\} \\ & = \|p\|_0^2. \end{aligned}$$

Thus

$$\|p\|_0 \leq M_B^1(p), \quad p \in \mathbb{C}[z]$$

as desired. This completes the proof.  $\square$

Observe that the Bergman shift  $B$  is subharmonic on the unit vector 1,  $\|B\| \leq 1$  and  $B^n \xrightarrow{SOT} 0$ . Write  $p_n(z) = z^n + z + 1$ . Then by Proposition 4.6,

$$1 \leq \|p_n\|_0 \leq M_B^1(p_n).$$

Applying Theorem 4.1 gives that

$$\lim_{n \rightarrow \infty} M_B^1(p_n) = 1.$$

Thus we obtain

$$\lim_{n \rightarrow \infty} \|p_n\|_0 = 1.$$

REMARK 4.7. Consider the weighted Bergman space

$$L_a^2(\mathbb{D}, \rho(|z|^2) dA(z)) = \left\{ f : f \text{ is analytic on } \mathbb{D}, \int_{\mathbb{D}} |f(z)|^2 \rho(|z|^2) dA(z) < \infty \right\}.$$

In [Hu, Example 1], the quantity  $\|p\|_\rho$  is defined by

$$\|p\|_\rho \triangleq \exp \left( \int_{\mathbb{D}} \log |p(z)| \rho(|z|^2) dA(z) \right).$$

Then by the same reasoning as above, one gets

$$\lim_{n \rightarrow \infty} \|p_n\|_\rho = 1.$$

This result was first obtained in [Hu, Corollary 4]. However, the proof presented here is an operator-theoretic approach, which is quite different from that of [Hu, Corollary 4].

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