

EXTENSIONS OF THE COEFFECTIVE COMPLEX

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ABSTRACT. The coeffective differential complex on a symplectic manifold is extended both in length and in scope, unifying the constructions of various other authors.

1. Introduction

This article is both an addendum to [4] and a precursor to [7]. In [4], we discussed the construction of differential complexes on manifolds equipped with various geometric structures. Mostly, these geometries were *parabolic* [5] but there were two exceptions, specifically *contact* geometry for which there is the *Rumin* complex [14] and *symplectic* for which there is a very similar complex [15], which we dubbed the *Rumin–Seshadri* complex (it was independently discovered by Tseng and Yau [17], [18]). This article extends the realm of these complexes, specifically covering *conformally symplectic* manifolds and *conformally calibrated* G_2 manifolds (see, for example, [1], [19] and [9], respectively).

In [2], T. Bouche introduced a differential complex naturally defined on any symplectic manifold M and coined the term *coeffective complex* for it (see also [8]). If M has dimension $2n$, then it is the subcomplex of the second half of the de Rham complex

$$\begin{array}{ccccccccccc} \Lambda^n & \xrightarrow{d} & \Lambda^{n+1} & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Lambda^{2n-2} & \xrightarrow{d} & \Lambda^{2n-1} & \xrightarrow{d} & \Lambda^{2n} \rightarrow 0 \\ \cup & & \cup & & & & \cup & & \parallel & & \parallel \\ \Lambda^n_{\perp} & \rightarrow & \Lambda^{n+1}_{\perp} & \rightarrow & \cdots & \rightarrow & \Lambda^{2n-2}_{\perp} & \rightarrow & \Lambda^{2n-1}_{\perp} & \rightarrow & \Lambda^{2n}_{\perp} \rightarrow 0, \end{array}$$

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where, if J denotes the symplectic form, then the bundle Λ_{\perp}^k is defined as the kernel of $\Lambda^k \xrightarrow{J \wedge} \Lambda^{k+2}$. Under the canonical isomorphisms

$$\underbrace{J \wedge J \wedge \cdots \wedge J}_{n-k} \wedge : \Lambda^k \xrightarrow{\simeq} \Lambda^{2n-k} \quad \text{for } k = 0, 1, 2, \dots, n$$

the bundle Λ_{\perp}^{2n-k} may equally well be regarded as a subbundle of Λ^k , which we shall write as Λ_{\perp}^k and, as such, provides a natural complement to the range of $\Lambda^{k-2} \xrightarrow{J \wedge} \Lambda^k$ for $k = 2, 3, \dots, n$. Using indices (more precisely, *abstract indices* in the sense of [13]), sections of the bundle Λ_{\perp}^k for $k = 2, 3, \dots, n$ are precisely the k -forms that are *trace-free* with respect to J_{ab} , that is,

$$J^{ab} \omega_{abc \dots d} = 0,$$

where J^{ab} is the inverse of J_{ab} (let us say $J_{ac} J^{bc} = \delta_a^b$, where δ_a^b is the Kronecker delta). Thus, we may rewrite the coeffective complex as

$$(1) \quad \Lambda_{\perp}^n \xrightarrow{d_{\perp}} \Lambda_{\perp}^{n-1} \xrightarrow{d_{\perp}} \cdots \xrightarrow{d_{\perp}} \Lambda_{\perp}^2 \xrightarrow{d_{\perp}} \Lambda_{\perp}^1 \xrightarrow{d_{\perp}} \Lambda^0 \rightarrow 0.$$

Bouche [2] showed that it is elliptic except at Λ_{\perp}^n . Since the diagrams with exact rows

$$\begin{array}{ccccccc} 0 & \rightarrow & \Lambda^{k-2} & \xrightarrow{J \wedge} & \Lambda^k & \rightarrow & \Lambda_{\perp}^k \rightarrow 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & \Lambda^{k-1} & \xrightarrow{J \wedge} & \Lambda^{k+1} & \rightarrow & \Lambda_{\perp}^{k+1} \rightarrow 0 \end{array}$$

commute, there is a canonically defined differential complex going the other way:

$$(2) \quad 0 \rightarrow \Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d_{\perp}} \Lambda_{\perp}^2 \xrightarrow{d_{\perp}} \cdots \xrightarrow{d_{\perp}} \Lambda_{\perp}^{n-1} \xrightarrow{d_{\perp}} \Lambda_{\perp}^n.$$

In fact, one can easily check that (1) and (2) are adjoint to each other under the pairing

$$\Lambda_{\perp}^k \otimes \Lambda_{\perp}^k \xrightarrow{\simeq} \Lambda_{\perp}^{2n-k} \otimes \Lambda_{\perp}^k \xrightarrow{-\wedge} \Lambda^{2n}.$$

The *Rumin–Seshadri* complex joins (1) and (2) with a symplectically invariant *second order* linear differential operator $d_{\perp}^{(2)} : \Lambda_{\perp}^n \rightarrow \Lambda_{\perp}^n$ to obtain an elliptic complex

$$(3) \quad \begin{array}{ccccccccccc} 0 & \rightarrow & \Lambda^0 & \xrightarrow{d} & \Lambda^1 & \xrightarrow{d_{\perp}} & \Lambda_{\perp}^2 & \xrightarrow{d_{\perp}} & \Lambda_{\perp}^3 & \xrightarrow{d_{\perp}} & \cdots & \xrightarrow{d_{\perp}} & \Lambda_{\perp}^n \\ & & & & & & & & & & & & \downarrow d_{\perp}^{(2)} \\ 0 & \leftarrow & \Lambda^0 & \xleftarrow{d_{\perp}} & \Lambda^1 & \xleftarrow{d_{\perp}} & \Lambda_{\perp}^2 & \xleftarrow{d_{\perp}} & \Lambda_{\perp}^3 & \xleftarrow{d_{\perp}} & \cdots & \xleftarrow{d_{\perp}} & \Lambda_{\perp}^n. \end{array}$$

In four dimensions this complex is due to Smith [16] and in higher dimensions it was also found by L.-S. Tseng and S.-T. Yau [17], [18] who go on to study its cohomology on compact manifolds. The construction of (3) given in [4] will be generalised in the following section.

2. Conformally symplectic manifolds

A *conformally symplectic structure* on an even dimensional manifold M of dimension at least 6 is defined by a non-degenerate 2-form J but, instead of requiring that J be closed, as one would for a symplectic structure, one requires only that

$$(4) \quad dJ = 2\alpha \wedge J$$

for some 1-form α (the factor of 2 being chosen only for convenience). Non-degeneracy of J implies that α is uniquely defined by (4). It is called the *Lee form* [12]. Differentiating (4) gives

$$0 = d^2 J = 2 d\alpha \wedge J + 2\alpha \wedge dJ = 2 d\alpha \wedge J + 4\alpha \wedge \alpha \wedge J = 2 d\alpha \wedge J$$

and, as $J \wedge _ : \Lambda^2 \rightarrow \Lambda^4$ is injective since $\dim M \geq 6$, we see that α is closed. In dimension 4, equation (4) defines a unique Lee form α and, for the definition of conformally symplectic, we require that α be closed. If we rescale J by a positive smooth function, say $\tilde{J} = \Omega^2 J$, then (4) remains valid with α replaced by $\hat{\alpha} = \alpha + \Upsilon$ for $\Upsilon \equiv d \log \Omega$. Hence, the notion of conformally symplectic is invariant under such rescalings (and also in dimension 4 since $d\Upsilon = 0$). Locally, we may use this freedom to eliminate α and obtain an ordinary symplectic structure. Globally, however, this need not be the case. For example, the rescaled symplectic form

$$J \equiv (1/\|x\|)^2 (dx^1 \wedge dx^2 + dx^3 \wedge dx^4 + \dots)$$

on \mathbb{R}^{2n} is invariant under dilation $x \mapsto \lambda x$ and, therefore, descends to a conformally symplectic structure on $S^1 \times S^{2n-1}$ whereas there is no global symplectic form on this manifold. If we continue to denote the inverse of J_{ab} by J^{ab} , and consider the vector field $X^a \equiv J^{ab} \alpha_b$, then the identities

$$J^{ad} J^{be} J^{cf} (\nabla_{[d} J_{ef]} - 2\alpha_{[d} J_{ef]}) = J^{d[a} \nabla_d J^{bc]} - 2X^{[a} J^{bc]},$$

$$J^{ad} J^{be} (2\nabla_{[d} \alpha_{e]} + 3X^c (\nabla_{[d} J_{ec]} - 2\alpha_{[d} J_{ec]})) = -X^c \nabla_c J^{ab} - 2J^{c[a} \nabla_c X^{b]}$$

are readily established for any torsion-free connection ∇_a and show that a conformally symplectic structure is equivalent to a *Jacobi structure* (J^{ab}, X^a) if we insist that J^{ab} be non-degenerate (as discussed in [1]).

THEOREM 1. *On any conformally symplectic manifold (M, J) , there is a canonically defined elliptic complex*

$$(5) \quad \begin{array}{ccccccc} 0 & \rightarrow & \Lambda^0 & \rightarrow & \Lambda^1 & \rightarrow & \Lambda^2_{\perp} \rightarrow \Lambda^3_{\perp} \rightarrow \dots \rightarrow \Lambda^n_{\perp} \\ & & & & & & \downarrow \\ 0 & \leftarrow & \Lambda^0 & \leftarrow & \Lambda^1 & \leftarrow & \Lambda^2_{\perp} \leftarrow \Lambda^3_{\perp} \leftarrow \dots \leftarrow \Lambda^n_{\perp}, \end{array}$$

where Λ^k_{\perp} denotes the bundle of k -forms that are trace-free with respect to J . All operators are first order except for the middle operator, which is second order. In the symplectic case, the second half of the complex coincides with the

coeffective complex. This complex is locally exact except at Λ^0 and Λ^1 near the beginning.

Proof. Consider the diagram

$$(6) \quad \begin{array}{ccccccc} \longrightarrow & \Lambda^p & \xrightarrow{d-2\alpha\wedge} & \Lambda^{p+1} & \xrightarrow{d-2\alpha\wedge} & \Lambda^{p+2} & \longrightarrow \\ & \uparrow J\wedge_- & & \uparrow J\wedge_- & & \uparrow J\wedge_- & \\ \longrightarrow & \Lambda^{p-2} & \xrightarrow{d} & \Lambda^{p-1} & \xrightarrow{d} & \Lambda^p & \longrightarrow . \end{array}$$

The bottom row is the de Rham complex and, in particular, is locally exact except at Λ^0 . Since $d\alpha = 0$, the same is true of the top row. Since $dJ = 2\alpha \wedge J$, the diagram commutes. Now consider the columns. In the middle, non-degeneracy of J ensures that

$$J \wedge_- : \Lambda^{n-1} \rightarrow \Lambda^{n+1}$$

is an isomorphism. To the left of this, we have injections and, to the right, we have surjections. As discussed in Section 1, the trace-free forms Λ_{\perp}^k may be canonically identified with the cokernel of

$$J \wedge_- : \Lambda^{k-2} \rightarrow \Lambda^k \quad \text{for } k = 2, 3, \dots, n$$

but also with the kernel of

$$J \wedge_- : \Lambda^{2n-k} \rightarrow \Lambda^{2n-k+2} \quad \text{for } k = 2, 3, \dots, n.$$

The spectral sequence of a double complex completes the proof. \square

Explicit formulæ for the operators in this complex can be given by using indices and an arbitrarily chosen torsion-free connection but are quite complicated since they necessarily employ the decomposition of arbitrary k -forms into their trace-free parts

$$\Lambda^k = \Lambda_{\perp}^k \oplus \Lambda_{\perp}^{k-2} \oplus \Lambda_{\perp}^{k-4} \oplus \dots \quad \text{for } k = 2, 3, \dots, n$$

corresponding to the branching of $\Lambda^k \mathbb{R}^{2n}$ under $\mathrm{Sp}(2n, \mathbb{R}) \subset \mathrm{SL}(2n, \mathbb{R})$ (cf. the combinatorial formulæ in [18, §2.1]).

To discuss the global cohomology of the complex (5), let us relabel its terms as B^r for $r = 0, 1, 2, \dots, 2n, 2n+1$ and define

$$(7) \quad H_J^r(M) \equiv \frac{\ker : \Gamma(M, B^r) \rightarrow \Gamma(M, B^{r+1})}{\mathrm{im} : \Gamma(M, B^{r-1}) \rightarrow \Gamma(M, B^r)}.$$

In comparison with [8] in the symplectic case, we have

$$H_J^r(M) = H^{r-1}(\mathcal{A}(M)) \quad \text{for } r = n+2, n+3, \dots, 2n+1$$

for their *coeffective cohomology* but now, for compact M , we have finite-dimensional vector spaces for all $r = 0, 1, 2, \dots, 2n, 2n+1$. Also in the symplectic case, these cohomologies were introduced and studied by Tseng and

Yau [17], [18] and our notation compares as follows.

$$\begin{aligned} H_J^r(M) &= PH_{\partial_+}^r(M) \quad \text{for } 0 \leq r < n, \\ H_J^n(M) &= PH_{dd^A}^n(M), \quad H_J^{n+1}(M) = PH_{d+d^A}^n(M), \\ H_J^r(M) &= PH_{\partial_-}^{2n+1-r}(M) \quad \text{for } n+1 < r \leq 2n+1. \end{aligned}$$

(Tseng and Yau refer to these and similar cohomologies as ‘primitive.’) According to Theorem 1, the cohomology \mathcal{H}^\bullet of (5) on the level of sheaves of germs of smooth functions occurs only at B^0 and B^1 and, from its proof, we see that $\mathcal{H}^1 = \mathbb{R}$. Also \mathcal{H}^0 is a locally constant sheaf. Specifically,

$$\mathcal{H}^0 = \{f \text{ s.t. } df - 2f\alpha = 0\},$$

and may equivalently be viewed as parallel sections of the trivial bundle equipped with the flat connection defined by -2α as connection form. In the symplectic case, we have $\mathcal{H}^0 = \mathbb{R}$. Evidently, the top row of (6) provides a fine resolution of \mathcal{H}^0 and so the sheaf cohomology $H^r(M, \mathcal{H}^0)$ may be identified as the cohomology of the complex $\Gamma(M, \Lambda^\bullet)$ with $\omega \mapsto d\omega - 2\alpha \wedge \omega$ as differential. The following theorem extends the long exact sequence [8, (5)].

THEOREM 2. *On a conformally symplectic manifold (M, J) , we have*

$$H_J^0(M) = H^0(M, \mathcal{H}^0), \quad H_J^{2n+1}(M) = H^{2n}(M, \mathbb{R}),$$

and a long exact sequence

$$\begin{aligned} 0 \rightarrow H^1(M, \mathcal{H}^0) \rightarrow H_J^1(M) \rightarrow H^0(M, \mathbb{R}) \xrightarrow{\delta} H^2(M, \mathcal{H}^0) \rightarrow \cdots \\ \rightarrow H_J^r(M) \rightarrow H^{r-1}(M, \mathbb{R}) \xrightarrow{\delta} H^{r+1}(M, \mathcal{H}^0) \rightarrow \cdots \\ \rightarrow H_J^{2n}(M) \rightarrow H^{2n-1}(M, \mathbb{R}) \rightarrow 0, \end{aligned}$$

where $\delta : H^{r-1}(M, \mathbb{R}) \rightarrow H^{r+1}(M, \mathcal{H}^0)$ is given by cup product with the cohomology class $[J] \in H^2(M, \mathcal{H}^0)$.

Proof. The hypercohomology spectral sequence for the complex B^\bullet as a complex of sheaves reads, at the E_2 -level

$$\begin{array}{ccccccc} \uparrow & H^0(M, \mathbb{R}) & \xrightarrow{\quad} & H^1(M, \mathbb{R}) & \xrightarrow{\quad} & H^2(M, \mathbb{R}) & \xrightarrow{\quad} & H^3(M, \mathbb{R}) & \xrightarrow{\quad} & \cdots \\ & \searrow & & \searrow & & \searrow & & \searrow & & \\ & H^0(M, \mathcal{H}^0) & \xrightarrow{\quad} & H^1(M, \mathcal{H}^0) & \xrightarrow{\quad} & H^2(M, \mathcal{H}^0) & \xrightarrow{\quad} & H^3(M, \mathcal{H}^0) & \xrightarrow{\quad} & \cdots \end{array}$$

and the desired conclusions follow. \square

(Spectral sequence reasoning can always be replaced by an appropriate diagram chase, in this case on the double complex (6).)

As an application of Theorem 2, if we consider complex projective space \mathbb{CP}_n with J its usual Kähler form, then

$$[J] \cup _ : H^{r-1}(\mathbb{CP}_n, \mathbb{R}) \rightarrow H^{r+1}(\mathbb{CP}_n, \mathbb{R})$$

is an isomorphism for $1 \leq r \leq 2n - 1$. Therefore,

$$H_J^0(\mathbb{CP}) = \mathbb{R}, \quad H_J^r(\mathbb{CP}_n) = 0 \quad \text{for } 1 \leq r \leq 2n, \quad H_J^{2n+1}(\mathbb{CP}_n) = \mathbb{R}.$$

More generally, Theorem 2 shows that the cohomology $H_J^r(M)$ of a symplectic manifold is determined by its de Rham cohomology and the action of the symplectic class $[J] \in H^2(M, \mathbb{R})$. In particular, there are evident inequalities concerning $\dim H_J^r$ and the Betti numbers of a compact symplectic manifold (including those of [8, Theorem 3.1]).

3. Conformally calibrated G_2 -manifolds

Following [9], a *conformally calibrated G_2 -manifold* is defined as a G_2 -manifold (M, ϕ) such that

$$(8) \quad d\phi = 2\alpha \wedge \phi$$

for some 1-form α . Recall [3], [6], [9] that ϕ is the *fundamental 3-form* defining a reduction of structure group on the 7-dimensional smooth manifold M from $\mathrm{GL}(7, \mathbb{R})$ to $G_2 \subset \mathrm{SO}(7) \subset \mathrm{GL}(7, \mathbb{R})$. In parallel with the symplectic case, the form ϕ may be locally rescaled so that it is closed (and a G_2 -manifold with closed fundamental form is said to be calibrated). As in the symplectic case and as detailed in [9], the form ϕ , pointwise sometimes known as the *Cayley form* [6], is sufficiently non-degenerate that

$$(9) \quad \begin{aligned} \phi \wedge _ : \Lambda^k &\longrightarrow \Lambda^{k+3} \quad \text{is injective for } k = 0, 1, \\ \phi \wedge _ : \Lambda^2 &\xrightarrow{\cong} \Lambda^5, \\ \phi \wedge _ : \Lambda^k &\longrightarrow \Lambda^{k+3} \quad \text{is surjective for } k = 3, 4. \end{aligned}$$

One way to see this is to decompose the forms on M into G_2 -irreducibles

$$(10) \quad \begin{aligned} \Lambda^0 &= \begin{smallmatrix} 0 & 0 \\ \hline \hline \end{smallmatrix} & \Lambda^1 &= \begin{smallmatrix} 1 & 0 \\ \hline \hline \end{smallmatrix} & \Lambda^2 &= \begin{smallmatrix} 0 & 1 \\ \hline \hline \end{smallmatrix} \oplus \begin{smallmatrix} 1 & 0 \\ \hline \hline \end{smallmatrix} \\ \Lambda^3 &= \begin{smallmatrix} 2 & 0 \\ \hline \hline \end{smallmatrix} \oplus \begin{smallmatrix} 1 & 0 \\ \hline \hline \end{smallmatrix} \oplus \begin{smallmatrix} 0 & 0 \\ \hline \hline \end{smallmatrix} & \Lambda^4 &= \begin{smallmatrix} 2 & 0 \\ \hline \hline \end{smallmatrix} \oplus \begin{smallmatrix} 1 & 0 \\ \hline \hline \end{smallmatrix} \oplus \begin{smallmatrix} 0 & 0 \\ \hline \hline \end{smallmatrix} \\ \Lambda^5 &= \begin{smallmatrix} 0 & 1 \\ \hline \hline \end{smallmatrix} \oplus \begin{smallmatrix} 1 & 0 \\ \hline \hline \end{smallmatrix} & \Lambda^6 &= \begin{smallmatrix} 1 & 0 \\ \hline \hline \end{smallmatrix} & \Lambda^7 &= \begin{smallmatrix} 0 & 0 \\ \hline \hline \end{smallmatrix} \end{aligned}$$

and check (9) on the level of highest weights. The canonical Hodge isomorphism $\Lambda^k \cong \Lambda^{7-k}$ is evident in this decomposition. Parallel to the symplectic case, let us write

$$\Lambda_\perp^4 \equiv \ker \phi \wedge _ : \Lambda^3 \rightarrow \Lambda^6, \quad \Lambda_\perp^3 \equiv \ker \phi \wedge _ : \Lambda^4 \rightarrow \Lambda^7$$

and, by inspecting (10), note that

$$\Lambda_\perp^3 = \begin{smallmatrix} 2 & 0 \\ \hline \hline \end{smallmatrix} \oplus \begin{smallmatrix} 1 & 0 \\ \hline \hline \end{smallmatrix} \quad \Lambda_\perp^4 = \begin{smallmatrix} 2 & 0 \\ \hline \hline \end{smallmatrix} \oplus \begin{smallmatrix} 0 & 0 \\ \hline \hline \end{smallmatrix}$$

also provide canonical complements to the ranges of $\phi \wedge _ : \Lambda^0 \rightarrow \Lambda^3$ and $\phi \wedge _ : \Lambda^1 \rightarrow \Lambda^4$, respectively. From (9) we see that, as in the conformally symplectic case, α is uniquely defined by (8) and is closed.

THEOREM 3. *On any conformally calibrated G_2 -manifold (M, ϕ) , there is a canonically defined elliptic complex*

$$(11) \quad \begin{array}{ccccccc} 0 & \rightarrow & \Lambda^0 & \rightarrow & \Lambda^1 & \rightarrow & \Lambda^2 \rightarrow \Lambda^3_{\perp} \rightarrow \Lambda^4_{\perp} \\ & & & & & & \downarrow \\ 0 & \leftarrow & \Lambda^0 & \leftarrow & \Lambda^1 & \leftarrow & \Lambda^2 \leftarrow \Lambda^3_{\perp} \leftarrow \Lambda^4_{\perp}. \end{array}$$

All differential operators are first order except for the middle operator, which is second order. The second half of this complex coincides with the coefficient complex defined in [8]. It is locally exact except at Λ^0 and Λ^2 near the beginning.

Proof. Consider the diagram

$$\begin{array}{ccccccc} \rightarrow & \Lambda^p & \xrightarrow{d-2\alpha\wedge} & \Lambda^{p+1} & \xrightarrow{d-2\alpha\wedge} & \Lambda^{p+2} & \rightarrow \\ & \uparrow _ \wedge \phi & & \uparrow _ \wedge \phi & & \uparrow _ \wedge \phi & \\ \rightarrow & \Lambda^{p-3} & \xrightarrow{d} & \Lambda^{p-2} & \xrightarrow{d} & \Lambda^{p-1} & \rightarrow. \end{array}$$

The bottom row is the de Rham complex and, in particular, is locally exact except at Λ^0 . Since $d\alpha = 0$, the same is true of the top row. Since $d\phi = 2\alpha \wedge \phi$, the diagram commutes. The columns behave according to (9). Hence, the first spectral sequence of this double complex reads, at the E_1 -level

$$\begin{array}{ccccccccccc} \uparrow & \Lambda^0 & \rightarrow & \Lambda^1 & \rightarrow & \Lambda^2 & \rightarrow & \Lambda^3_{\perp} & & 0 & & 0 & & 0 & & 0 & & 0 \\ & 0 & & 0 & & 0 & & 0 & & 0 & & \Lambda^3_{\perp} & \rightarrow & \Lambda^4 & \rightarrow & \Lambda^5 & \rightarrow & \Lambda^7 \\ \rightarrow & & & & & & & & & & & & & & & & & \end{array}$$

Passing to the E_2 -level constructs the complex and the second spectral sequence identifies its local cohomology \mathcal{H}^\bullet as

$$\mathcal{H}^0 = \{f \text{ s.t. } df - 2f\alpha = 0\}, \quad \mathcal{H}^2 = \mathbb{R},$$

with all others vanishing. Finally, ellipticity of this complex is inherited from that of the de Rham complex. Specifically, for $\Lambda^1 \ni \xi \neq 0$, the symbol complex of (11) is constructed from the double complex

$$\begin{array}{ccccccc} \rightarrow & \Lambda^p & \xrightarrow{\xi\wedge} & \Lambda^{p+1} & \xrightarrow{\xi\wedge} & \Lambda^{p+2} & \rightarrow \\ & \uparrow _ \wedge \phi & & \uparrow _ \wedge \phi & & \uparrow _ \wedge \phi & \\ \rightarrow & \Lambda^{p-3} & \xrightarrow{\xi\wedge} & \Lambda^{p-2} & \xrightarrow{\xi\wedge} & \Lambda^{p-1} & \rightarrow, \end{array}$$

the rows of which are exact (they are Koszul complexes). □

As in the (conformally) symplectic case, this construction (and this proof of ellipticity) avoids explicit formulæ for the operators. If such formulæ are needed, then one simply needs explicitly to write out the branching (9) (as is done in [9, p. 365]).

As in the conformally symplectic case (7), we may consider the global cohomology on M of the complex (11), which we shall denote by $H_\phi^r(M)$ for $0 \leq r \leq 9$.

THEOREM 4. *On a conformally calibrated G_2 -manifold (M, ϕ) , we have*

$$\begin{aligned} H_\phi^0(M) &= H^0(M, \mathcal{H}^0), & H_\phi^1(M) &= H^1(M, \mathcal{H}^0), \\ H_\phi^8(M) &= H^6(M, \mathbb{R}), & H_\phi^9(M) &= H^7(M, \mathbb{R}) \end{aligned}$$

and a long exact sequence

$$\begin{aligned} 0 \rightarrow H^2(M, \mathcal{H}^0) &\rightarrow H_\phi^2(M) \rightarrow H^0(M, \mathbb{R}) \xrightarrow{\delta} H^3(M, \mathcal{H}^0) \\ &\rightarrow H_\phi^3(M) \rightarrow H^1(M, \mathbb{R}) \xrightarrow{\delta} H^4(M, \mathcal{H}^0) \rightarrow \cdots \\ \cdots \rightarrow H_\phi^6(M) &\rightarrow H^4(M, \mathbb{R}) \xrightarrow{\delta} H^7(M, \mathcal{H}^0) \\ &\rightarrow H_\phi^7(M) \rightarrow H^5(M, \mathbb{R}) \rightarrow 0, \end{aligned}$$

where $\delta : H^r(M, \mathbb{R}) \rightarrow H^{r+3}(M, \mathcal{H}^0)$ is given by cup product with the cohomology class $[\phi] \in H^3(M, \mathcal{H}^0)$.

Proof. Immediate from the hypercohomology spectral sequence as for the proof of Theorem 2 except that the connecting homomorphism δ does not appear until the E_3 -level. \square

In the calibrated case (when $\alpha = 0$), $H^r(M, \mathcal{H}^0) = H^r(M, \mathbb{R})$ and we see that $H_\phi^r(M)$ is determined by the de Rham cohomology of M and the action of $[\phi] \in H^3(M, \mathbb{R})$ by cup product.

4. Other geometries

There are several other geometries defined by special k -forms for which one can apply similar reasoning. Certainly, there are $\text{Spin}(7)$ -geometries in dimension 8 defined [3] by a fundamental 4-form Φ . The construction given in this article extends to this case and, by the work of Joyce [11], there are non-trivial compact examples with $d\Phi = 0$.

Also, there are $\text{SO}(3) \times \text{SO}(3)$ -geometries in dimension 9 defined [10] by a fundamental 5-form and $\text{SU}(4) \times \text{U}(1)$ -geometries in dimension 10 defined [6] by a fundamental 6-form or 4-form but, for the moment, it is unclear whether there are any examples of such geometries that are not locally homogeneous.

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