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ON THE LEFSCHETZ AND HODGE–RIEMANN THEOREMS

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ABSTRACT. We give an abstract version of the hard Lefschetz theorem, the Lefschetz decomposition and the Hodge–Riemann theorem for compact Kähler manifolds. Some examples are studied for compact symplectic Kähler manifolds.

1. Introduction

Let X be a compact Kähler manifold of dimension n and let ω be a Kähler form on X. Denote by $H^{p,q}(X,\mathbb{C})$ the Hodge cohomology group of bidegree (p,q) of X with the convention that $H^{p,q}(X,\mathbb{C})=0$ outside of the range $0\le p,q\le n$. When $p,q\ge 0$ and $p+q\le n$, put $\Omega:=\omega^{n-p-q}$ and define a Hermitian form $Q=Q_\Omega$ on $H^{p,q}(X,\mathbb{C})$ by

$$Q\big(\{\alpha\},\{\beta\}\big):=i^{p-q}(-1)^{\frac{(p+q)(p+q-1)}{2}}\int_X \alpha\wedge\overline{\beta}\wedge\Omega$$

for smooth closed (p,q)-forms α and β. The last integral depends only on the classes $\{\alpha\}$, $\{\beta\}$ of α, β in $H^{p,q}(X,\mathbb{C})$.

The classical Hodge–Riemann theorem asserts that Q is positive-definite on the primitive subspace $H^{p,q}(X,\mathbb{C})_{\mathrm{prim}}$ of $H^{p,q}(X,\mathbb{C})$ which depends on Ω and is given by

$$H^{p,q}(X,\mathbb{C})_{\mathrm{prim}}:=\big\{\{\alpha\}\in H^{p,q}(X,\mathbb{C}),\{\alpha\}\smile\{\Omega\}\smile\{\omega\}=0\big\},$$

where $\smile$ denotes the cup-product on the cohomology ring $\oplus H^*(X,\mathbb{C})$, see, for example, Demailly [5], Griffiths and Harris [15] and Voisin [25].

Still under the assumption that $\Omega:=\omega^{n-p-q}$, the hard Lefschetz theorem says that the linear map $\{\alpha\}\mapsto\{\alpha\}\smile\{\Omega\}$ defines an isomorphism between $H^{p,q}(X,\mathbb{C})$ and $H^{n-q,n-p}(X,\mathbb{C})$. Moreover, the following Lefschetz decomposition

$$H^{p,q}(X,\mathbb{C})=\{\omega\}\smile H^{p-1,q-1}(X,\mathbb{C})\oplus H^{p,q}(X,\mathbb{C})_{\mathrm{prim}}$$

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is orthogonal with respect to the Hermitian form Q. Consequently, we deduce easily from the above theorems the signature of Q in term of the Hodge numbers $h^{p,q} := \dim H^{p,q}(X,\mathbb{C})$. For example, when $p = q = 1$ the signature of Q is equal to $(h^{1,1} - 1, 1)$.

The above three theorems are not true if we replace $\{\Omega\}$ with an arbitrary class in $H^{n-p-q,n-p-q}(X,\mathbb{R})$, even when the class contains a strictly positive form, see, for example, Berndtsson and Sibony [4, §9] and Remark 2.9 below. Our aim here is to give sufficient conditions on $\{\Omega\}$ for which these theorems still hold. We will say that such a class $\{\Omega\}$ satisfies the Hodge–Riemann theorem, the hard Lefschetz theorem and the Lefschetz decomposition theorem for the bidegree (p,q).

If E is a complex vector space of dimension n and $\overline{E}$ its complex conjugate, we will introduce in the next section the notion of Hodge–Riemann cone in the exterior product $\bigwedge^k E \otimes \bigwedge^k \overline{E}$ with $0 \le k \le n$, see Definition 2.1 below. In practice, E is the complex cotangent space at a point x of X and we obtain a Hodge–Riemann cone associated with X. Here is our main result.

THEOREM 1.1. *Let (X,ω) be a compact Kähler manifold of dimension n. Let p,q be non-negative integers such that $p+q \le n$ and Ω a closed smooth form of bidegree $(n-p-q, n-p-q)$ on X. Assume that Ω takes values only in the Hodge–Riemann cone associated with X. Then $\{\Omega\}$ satisfies the Hodge–Riemann theorem, the hard Lefschetz theorem and the Lefschetz decomposition theorem for the bidegree (p,q).*

Roughly speaking, the hypothesis of Theorem 1.1 says that at every point x of X, we can deform Ω continuously to ω^{n-p-q} in a "nice way." However, we do not need that the deformation depends continuously on x and a priori the deformation does not preserve the closedness nor the smoothness of the form.

We deduce from Theorem 1.1 the following corollary using a result due to Timorin [24], see Proposition 2.2 below.

COROLLARY 1.2. *Let (X,ω) be a compact Kähler manifold of dimension n. Let p, q be non-negative integers such that $p+q \le n$ and $\omega_1, \dots, \omega_{n-p-q}$ be Kähler forms on X. Then the class $\{\omega_1 \wedge \cdots \wedge \omega_{n-p-q}\}$ satisfies the Hodge–Riemann theorem, the hard Lefschetz theorem and the Lefschetz decomposition theorem for the bidegree (p,q).*

The last result was obtained by the authors in [10], see also Cattani [6] for a proof using the theory of variations of Hodge structures. It solves a problem which has been considered in some important cases by Khovanskii [19], [20], Teissier [22], [23], Gromov [16] and Timorin [24]. The reader will find some related results and applications of the above corollary in Cattani [6], de Cataldo and Migliorini [7], Gromov [16], Dinh and Sibony [9], [11] and Keum, Oguiso and Zhang [18], [28].

This paper is organized as follows. We begin Section 2 by defining the notion of Hodge–Riemann forms. This notion plays a key role in this work. Next, we will establish some of its important properties. This preparatory material is necessary for us to prove Theorem 1.1 in Section 3. Section 4 is devoted to a thorough study of an explicit family of Hodge–Riemann forms in the context of compact symplectic Kähler manifolds.

2. Hodge–Riemann forms

In this section, we introduce the notion of Hodge–Riemann form in the linear setting and we will discuss some basic properties of these forms.

Let E be a complex vector space of dimension n and $\overline{E}$ its conjugate space. Denote by $V^{p,q}$ the space $\bigwedge^p E \otimes \bigwedge^q \overline{E}$ of (p,q)-forms with the convention that $V^{p,q} := 0$ unless $0 \le p, q \le n$. Recall that a form ω in $V^{1,1}$ is a *Kähler form* if it can be written as

$$\omega = i dz_1 \wedge d\overline{z}_1 + \cdots + i dz_n \wedge d\overline{z}_n$$

for some coordinate system $(z_1, \ldots, z_n)$ of E, where $z_i \otimes \overline{z}_j$ is identified with $dz_i \wedge d\overline{z}_j$.

Recall also that a form Ω in $V^{k,k}$ with $0 \le k \le n$, is *real* if $\Omega = \overline{\Omega}$. Let $V_{\mathbb{R}}^{k,k}$ denote the space of real (k,k)-forms. A form Ω in $V^{k,k}$ is *positive*[1] if it is a combination with positive coefficients of forms of type $i^{k^2} \alpha \wedge \overline{\alpha}$ with $\alpha \in V^{k,0}$. So, positive forms are real. If Ω is positive its restriction to any subspace of E is positive. A positive (k,k)-form Ω is *strictly positive*, if its restriction to any subspace of dimension k of E does not vanish. The powers of a Kähler form are strictly positive forms. Fix a Kähler form ω as above.

Definition 2.1. A (k,k)-form Ω in $V^{k,k}$ is said to be *a Lefschetz form for the bidegree* (p,q) if $k = n - p - q$ and the map $\alpha \mapsto \alpha \wedge \Omega$ is an isomorphism between $V^{p,q}$ and $V^{n-q,n-p}$. A real (k,k)-form Ω in $V_{\mathbb{R}}^{k,k}$ is said to be *a Hodge–Riemann form for the bidegree* (p,q) if there is a continuous deformation $\Omega_t \in V_{\mathbb{R}}^{k,k}$ with $0 \le t \le 1$, $\Omega_0 = \Omega$ and $\Omega_1 = \omega^k$ such that

$(*)$ $\qquad \Omega_t \wedge \omega^{2r}$ is a Lefschetz form for the bidegree $(p-r, q-r)$

for every $0 \le r \le \min\{p,q\}$ and $0 \le t \le 1$. The cone of such forms Ω is called *the Hodge–Riemann cone for the bidegree* (p,q). We say that Ω is *Hodge–Riemann* if it is a Hodge–Riemann form for any bidegree (p,q) with $p + q = n - k$.

Note that the property $(*)$ for $t = 1$ is a consequence of the linear version of the classical hard Lefschetz theorem. The Hodge–Riemann cone is open in $V_{\mathbb{R}}^{k,k}$ and a priori depends on the choice of ω. In practice, to check that a

[1] There are two other notions of positivity but we will not use them here.

form is Hodge–Riemann is usually not a simple matter. We have the following result due to Timorin in [24].

PROPOSITION 2.2. *Let k be an integer such that $0 \le k \le n$. Let $\omega_1, \dots, \omega_k$ be Kähler forms. Then $\Omega := \omega_1 \wedge \cdots \wedge \omega_k$ is a Hodge–Riemann form.*

Consider a square matrix $M = (\alpha_{ij})_{1 \le i,j \le k}$ with entries in $V^{1,1}$. Assume that M is Hermitian, that is, $\alpha_{ij} = \overline{\alpha}_{ji}$ for all i, j. We say that M is *Griffiths positive* if for any row vector $\theta = (\theta_1, \dots, \theta_k)$ in $\mathbb{C}^k \setminus \{0\}$ and its transpose ${}^t\theta$, $\theta M {}^t\overline{\theta}$ is a Kähler form. We call *Griffiths cone* the set of (k,k)-forms in $V^{k,k}$ which can be obtained as the determinant of a Griffiths positive matrix M as above. We are still unable to answer the following question.

PROBLEM 2.3. *Is the Griffiths cone contained in the Hodge–Riemann cone?*

The affirmative answer to the question would allow us to obtain a transcendental version of the hyperplane Lefschetz theorem which is known for the last Chern class associated with a Griffiths positive vector bundle, see Voisin [25, p. 312]. Another fact which allows us to believe in the affirmative answer is that the Griffiths cone contains the wedge-products of Kähler forms (case where M is diagonal) which are Hodge–Riemann according to Proposition 2.2.

Note also that for the above problem it is enough to check the condition $(*)$ for $t = 0$ and $r = 0$. Indeed, we can consider Ω_t, the determinant of the Griffiths positive matrix $M_t := (1-t)M + tI\omega$, where I is the identity matrix. It is enough to observe that $\Omega_t \wedge \omega^{2r}$ is the determinant of the Griffiths positive $(k+2r) \times (k+2r)$ matrix which is obtained by adding to M_t a square block equal to ω times the identity $2r \times 2r$ matrix.

The following question is also open.

PROBLEM 2.4. *Let Ω_t, $0 \le t \le 1$, be a continuous family of strictly positive (k,k)-forms in $V^{k,k}_{\mathbb{R}}$ with $\Omega_0 = \Omega$ and $\Omega_1 = \omega^k$. Assume the property $(*)$ in Definition 2.1 for $r = 0$ and for this family Ω_t. Is Ω always a Hodge–Riemann form for the bidegree (p,q)?*

Note that the strict positivity of Ω_t implies the property $(*)$ for $r = \min\{p,q\}$. This is perhaps a reason to believe that the answer to the above problem is affirmative. An interesting point here is that the cone of all forms Ω as in Problem 2.4 does not depend on ω. The following result gives a partial answer to the question.

PROPOSITION 2.5. *Let Ω_t be as in Problem 2.4. Assume moreover that $\min\{p,q\} \le 2$. Then Ω is a Hodge–Riemann form for the bidegree (p,q).*

Fix a coordinate system $(z_1, \dots, z_n)$ of E such that $\omega = idz_1 \wedge d\overline{z}_1 + \cdots + idz_n \wedge d\overline{z}_n$. So, this Kähler form is invariant under the natural action of the unitary group $\mathrm{U}(n)$. We will need the following lemma.

LEMMA 2.6. *Let α be a form in $V^{p,q-1}$ with $q \geq 2$ and $p+q \leq n$. Assume that for every $\varphi \in V^{0,1}$ we can write $\alpha \wedge \varphi = \omega \wedge \beta$ for some $\beta \in V^{p-1,q-1}$. Then we can write $\alpha = \omega \wedge \gamma$ for some $\gamma \in V^{p-1,q-2}$.*

Proof. Let M denote the set of all forms $\alpha \in V^{p,q-1}$ satisfying the hypothesis of the lemma. Observe that M is invariant under the action of $\mathrm{U}(n)$. So, it is a linear representation of this group. Let P_j denote the primitive subspace of $V^{p-j,q-1-j}$, that is, the set of $\phi \in V^{p-j,q-1-j}$ such that $\phi \wedge \omega^{n-p-q+2+2j} = 0$. It is well-known that the P_j are irreducible representations of $\mathrm{U}(n)$ and they are not isomorphic one to another, see, for example, Fujiki [13, Proposition 2.2]. Moreover, we have the Lefschetz decomposition

$$V^{p,q-1} = \bigoplus_{0 \leq j \leq \min\{p,q-1\}} \omega^j \wedge P_j.$$

The space $\omega^j \wedge P_j$ is also a representation of $\mathrm{U}(n)$ which is isomorphic to P_j. Therefore, it is enough to show that M does not contain P_0.

Consider the form

$$\alpha := d\overline{z}_2 \wedge \cdots \wedge d\overline{z}_q \wedge dz_{q+1} \wedge \cdots \wedge dz_{p+q}.$$

A direct computation shows that α is a form in P_0. Observe that $\alpha \wedge d\overline{z}_1$ does not contain any factor $dz_j \wedge d\overline{z}_j$. Therefore, $\alpha \notin M$ because $\alpha \wedge d\overline{z}_1$ does not belong to $\omega \wedge V^{p-1,q-1}$. The lemma follows. □

Given nonnegative integers p, q such that $p+q \leq n$ and a real form Ω of bidegree $(n-p-q, n-p-q)$, define the Hermitian form Q by

$$Q(\alpha,\beta) := i^{p-q}(-1)^{\frac{(p+q)(p+q-1)}{2}} * (\alpha \wedge \overline{\beta} \wedge \Omega) \quad \text{for } \alpha,\beta \in V^{p,q},$$

where $*$ is the Hodge star operator. Define also *the primitive subspace*

$$P^{p,q} := \big\{\alpha \in V^{p,q} : \alpha \wedge \Omega \wedge \omega = 0\big\}.$$

The classical Lefschetz theorem asserts that the wedge-product with ω defines a surjective map from $V^{n-q,n-p}$ to $V^{n-q+1,n-p+1}$. Its kernel is of dimension $\dim V^{p,q} - \dim V^{p-1,q-1}$. Therefore, if the map $\alpha \mapsto \Omega \wedge \alpha$ is injective on $V^{p,q}$, the above primitive space has dimension $\dim V^{p,q} - \dim V^{p-1,q-1}$ which does not depend on Ω.

We also need the following lemma.

LEMMA 2.7. *Let Ω_t be a continuous family of real (k,k)-forms in $V^{k,k}_{\mathbb{R}}$ with $\Omega_0 = \Omega$, $\Omega_1 = \omega^k$ and $0 \leq t \leq 1$. Assume that Ω_t is Lefschetz for the bidegree (p,q) for every $0 \leq t \leq 1$ and $\Omega_t \wedge \omega^2 \wedge \alpha$ is Lefschetz for the bidegree $(p-1,q-1)$ for every $0 < t \leq 1$. Then, for every form α in $V^{p,q-1}$ (resp. $V^{p-1,q}$) satisfying $\alpha \wedge \Omega \wedge \omega = 0$, α belongs to $\omega \wedge V^{p-1,q-2}$ (resp. $\omega \wedge V^{p-2,q-1}$).*

It is worthy to note here that since $\alpha \mapsto \Omega_t \wedge \omega^2 \wedge \alpha$ is isomorphic from $V^{p-1,q-1}$ to $V^{n-q+1,n-p+1}$ for only $0 < t \leq 1$ (and not for every $0 \leq t \leq 1$!), the intersection $\omega \wedge V^{p-1,q-1} \cap P^{p,q}$ is, in general, non-zero.

Proof. Let V denote the space of forms $\beta \in V^{p,q}$ such that $Q(\beta,\phi)=0$ for every ϕ in $\omega\wedge V^{p-1,q-1}+P^{p,q}$. The hypothesis implies that Q is non-degenerate. Therefore, we obtain

$$\begin{aligned}\dim\omega\wedge V^{p-1,q-1}+\dim P^{p,q} &= \dim V^{p-1,q-1}+\dim V^{p,q}-\dim V^{p-1,q-1}\\ &= \dim V^{p,q},\end{aligned}$$

and hence

$$\dim V=\dim V^{p,q}-\dim\big(\omega\wedge V^{p-1,q-1}+P^{p,q}\big)=\dim\big(\omega\wedge V^{p-1,q-1}\cap P^{p,q}\big).$$

On the other hand, by definition of $P^{p,q}$, the space $\omega\wedge V^{p-1,q-1}\cap P^{p,q}$ is contained in V. We deduce that these two spaces coincide.

Let $\alpha\in V^{p,q-1}$ such that $\alpha\wedge\Omega\wedge\omega=0$ (the case $\alpha\in V^{p-1,q}$ can be treated in the same way). Fix a form φ in $V^{0,1}$. By Lemma 2.6 and the above discussion, we only need to show that $\alpha\wedge\varphi$ belongs to V. It is clear that $Q(\alpha\wedge\varphi,\phi)=0$ for $\phi\in\omega\wedge V^{p-1,q-1}$. It remains to show that $Q(\alpha\wedge\varphi,\phi)=0$ for $\phi\in P^{p,q}$. For this purpose, it is enough to consider the case where $\varphi=d\overline{z}_j$ since $\{d\overline{z}_1,\dots,d\overline{z}_n\}$ is a basis of $V^{0,1}$.

Denote by Q_t and $P_t^{p,q}$ the Hermitian form and the primitive space associated with Ω_t which are defined as above. Moreover, since $\Omega_t\wedge\omega^2\wedge\alpha$ is Lefschetz for the bidegree $(p-1,q-1)$, the intersection $\omega\wedge V^{p-1,q-1}\cap P_t^{p,q}$ is zero for every $0<t\leq 1$. Using the continuous deformation Ω_t of Ω, we obtain as in Proposition 2.8 below that Q_t is positive-definite on $P_t^{p,q}$ for every $0<t\leq 1$. Since the dimension of $P_t^{p,q}$ is constant, this space depends continuously on t. Hence, the restriction of Q to $P^{p,q}$ is semi-positive. Observe that $\alpha\wedge d\overline{z}_j$ is in $P^{p,q}$. Hence,

$$Q(\alpha\wedge d\overline{z}_j,\alpha\wedge d\overline{z}_j)\geq 0.$$

The sum over j of $Q(\alpha\wedge d\overline{z}_j,\alpha\wedge d\overline{z}_j)$ vanishes since $\alpha\wedge\Omega\wedge\omega=0$. We deduce that all the above inequalities are in fact equalities. Now, since Q is semi-positive on $P^{p,q}$, by Cauchy–Schwarz's inequality, $Q(\alpha\wedge d\overline{z}_j,\phi)=0$ for $\phi\in P^{p,q}$. This completes the proof. □

Proof of Proposition 2.5. Assume without loss of generality that $q\leq p$. Observe that for every α non-zero in $V^{n-k-s,0}$ we have $i^{(n-k-s)^2}\alpha\wedge\overline{\alpha}\wedge\Omega_t\wedge\omega^s>0$. So, we only have to consider the case $q=2$ and to check the property $(*)$ for $r=1$. We will show that the map $\alpha\mapsto\Omega_t\wedge\omega\wedge\alpha$ is injective on $V^{p,1}$ and the map $\alpha\mapsto\Omega_t\wedge\omega^2\wedge\alpha$ is injective on $V^{p-1,1}$. The result will follow easily.

Let Σ denote the set of t satisfying the above property. By continuity, Σ is open in $[0,1]$. Moreover, by the Lefschetz theorem, it contains the point 1. Assume that Σ is not equal to $[0,1]$. Let $t_0<1$ be the minimal number such that $]t_0,1]\subset\Sigma$. We will show that $t_0\in\Sigma$ which is a contradiction. Up to a re-parametrization of the family Ω_t, we can assume for simplicity that $t_0=0$.

Consider a form $\alpha \in V^{p,1}$ such that $\Omega \wedge \omega \wedge \alpha = 0$. We deduce from Lemma 2.7 that $\alpha = \omega \wedge \gamma$ with $\gamma \in V^{p-1,0}$. We have $\gamma \wedge \overline{\gamma} \wedge \Omega \wedge \omega^2 = 0$. The positivity of Ω implies that $\gamma = 0$ and then $\alpha = 0$. So, the map $\alpha \mapsto \Omega \wedge \omega \wedge \alpha$ is injective on $V^{p,1}$. By dimension reason, this map is bijective from $V^{p,1}$ to $V^{n-1,n-p}$. So $\Omega_t \wedge \omega$ is Lefschetz for the bidegree $(p,1)$ for every $0 \leq t \leq 1$. By the positivity of Ω, the form $\Omega_t \wedge \omega^3$ is Lefschetz for the bidegree $(p-1,0)$ for every $0 < t \leq 1$. Consequently, we are in the position to apply again Lemma 2.7 but to $\Omega_t \wedge \omega$ instead of Ω_t and $(p,1)$ instead of (p,q). We obtain as above that the map $\alpha \mapsto \Omega \wedge \omega^2 \wedge \alpha$ is injective on $V^{p-1,1}$. Therefore, 0 is a point in Σ. This completes the proof. $\square$

We give now fundamental properties of Hodge–Riemann forms that we will use in the next section. We fix a norm on each space $V^{*,*}$.

Proposition 2.8. *Let Ω be a form satisfying the condition $(*)$ in Definition 2.1 for $r = 0, 1$. Then the space $V^{p,q}$ splits into the Q-orthogonal direct sum*

$$V^{p,q} = P^{p,q} \oplus \omega \wedge V^{p-1,q-1}$$

and the Hermitian form Q is positive-definite on $P^{p,q}$. Moreover, for any constant $c_1 > 0$ large enough, there is a constant $c_2 > 0$ such that

$$\|\alpha\|^2 \leq c_1 Q(\alpha,\alpha) + c_2 \|\alpha \wedge \Omega \wedge \omega\|^2 \quad \text{for } \alpha \in V^{p,q}.$$

Proof. The Q-orthogonality is obvious. By the classical Lefschetz theorem, the wedge-product with ω defines an injective map from $V^{p-1,q-1}$ to $V^{p,q}$. Therefore, we have

$$\dim V^{p,q} = \dim P^{p,q} + \dim V^{p-1,q-1} = \dim P^{p,q} + \dim \omega \wedge V^{p-1,q-1}.$$

On the other hand, the property $(*)$ for $r = 1$ implies that the intersection of $P^{p,q}$ and $\omega \wedge V^{p-1,q-1}$ is reduced to 0. We then deduce the above decomposition of $V^{p,q}$. Of course, this property still holds if we replace Ω with Ω_t.

Denote by Q_t and $P_t^{p,q}$ the Hermitian form and the primitive space associated with Ω_t which are defined as above. Since the dimension of $P_t^{p,q}$ is constant, this space depends continuously on t. By the classical Hodge–Riemann theorem, Q_1 is positive-definite on $P_1^{p,q}$. If Q is not positive-definite on $P^{p,q}$, there is a maximal number t such that Q_t is not positive-definite. The maximality of t implies that Q_s is positive-definite on $P_s^{p,q}$ when $s > t$. It follows by continuity that there is an element $\alpha \in P_t^{p,q}$, $\alpha \neq 0$, such that $Q_t(\alpha,\beta) = 0$ for $\beta \in P_t^{p,q}$. By definition of $P_t^{p,q}$, this identity holds also for $\beta \in \omega \wedge V^{p-1,q-1}$. We then deduce that the identity holds for all $\beta \in V^{p,q}$. It follows that $\alpha \wedge \Omega_t = 0$. This is a contradiction. So, Q is positive-definite on $P^{p,q}$.

We prove now the last assertion in the proposition for a fixed constant c_1 large enough. Consider a form $\alpha \in V^{p,q}$. The first assertion implies that we can write

$$\alpha = \beta + \omega \wedge \gamma \quad \text{with } \beta \in P^{p,q} \text{ and } \gamma \in V^{p-1,q-1}$$

and we have

$$Q(\alpha,\alpha) = Q(\beta,\beta) + Q(\omega\wedge\gamma,\omega\wedge\gamma).$$

Since the wedge-product with $\Omega\wedge\omega^2$ defines an isomorphism between $V^{p-1,q-1}$ and $V^{n-q+1,n-p+1}$, there is a constant $c>0$ such that

$$c^{-1}\big\|\gamma\wedge\Omega\wedge\omega^2\big\| \le \|\gamma\| \le c\big\|\gamma\wedge\Omega\wedge\omega^2\big\| = c\|\alpha\wedge\Omega\wedge\omega\|.$$

Therefore, there is a constant $c'>0$ such that

$$\|\alpha\|^2 \le c'\big(\|\beta\|^2 + \|\gamma\|^2\big) \le c'\|\beta\|^2 + c'c^2\|\alpha\wedge\Omega\wedge\omega\|^2.$$

Finally, since Q is positive-definite on $P^{p,q}$ and since $c_1>0$ is large enough, we obtain

$$\begin{aligned} c'\|\beta\|^2 \le c_1 Q(\beta,\beta) &= c_1\big(Q(\alpha,\alpha) - Q(\omega\wedge\gamma,\omega\wedge\gamma)\big) \\ &\le c_1 Q(\alpha,\alpha) + c_1 c\|\gamma\|^2 \\ &\le c_1 Q(\alpha,\alpha) + c_1 c^3\big\|\gamma\wedge\Omega\wedge\omega^2\big\|^2 \\ &= c_1 Q(\alpha,\alpha) + c_1 c^3\|\alpha\wedge\Omega\wedge\omega\|^2. \end{aligned}$$

We then deduce the estimate in the proposition by taking $c_2 := c'c^2 + c_1c^3$. $\square$

REMARK 2.9. Consider the following strictly positive forms, exhibited by Berndtsson and Sibony [4, §9],

$$\Omega_\varepsilon := (idz_1\wedge d\overline{z}_1)\wedge(idz_2\wedge d\overline{z}_2) + (idz_3\wedge d\overline{z}_3)\wedge(idz_4\wedge d\overline{z}_4) + \varepsilon\omega^2 \in V^{2,2},$$

where $\varepsilon>0$ and $\dim E = 4$. Ω_ε is not a Lefschetz form for the bidegree $(1,1)$ if and only if the determinant of the linear map $V^{1,1}\ni\alpha\mapsto\Omega_\varepsilon\wedge\alpha$ with respect to any fixed bases of $V^{1,1}$ and $V^{3,3}$ vanishes. So by expanding this determinant it is not difficult to see that Ω_ε is not a Lefschetz form if and only if ε is a root of a suitable finite family of polynomials. Moreover, for ε large enough, the determinant of the linear map associated to $\varepsilon^{-1}\Omega_\varepsilon$ tends to that of ω^2 which is non-zero since ω^2 is a Lefschetz form. So the above family contains a non-zero polynomial. Consequently, for all but a finite number of values of $\varepsilon>0$, Ω_ε is a Lefschetz form for the bidegree $(1,1)$. In particular, Ω_ε is a Lefschetz form for all $\varepsilon>0$ small enough. By the positivity of Ω, $\Omega_\varepsilon\wedge\omega^2$ is clearly a Lefschetz form for the bidegree $(0,0)$. Recall from [4, §9] that for every $\varepsilon>0$ small enough, there is $\gamma^\pm = \gamma^\pm_\varepsilon \in V^{1,1}\setminus\{0\}$ such that $\gamma^\pm\wedge\Omega_\varepsilon\wedge\omega = 0$ and that

$$\gamma^+\wedge\overline{\gamma^+}\wedge\Omega_\varepsilon > 0 > \gamma^-\wedge\overline{\gamma^-}\wedge\Omega_\varepsilon.$$

So by Proposition 2.8, Ω_ε is not Hodge–Riemann for the bidegree $(1,1)$. This example shows that the condition on the existence of a continuous deformation in Definition 2.1 is necessary.

3. Lefschetz and Hodge–Riemann theorems

In this section, we prove Theorem 1.1. Corollary 1.2 is then deduced from that theorem and Proposition 2.2. We will use the results of the last section for E the complex cotangent space of X at a point and ω the Kähler form on X. So, we can define at every point of X a Hodge–Riemann cone for bidegree (p,q). We now use the notation in Theorem 1.1. Let $\mathscr{E}^{p,q}(X)$ (resp. $L^2_{p,q}(X)$) denote the spaces of smooth (resp. L^2) forms on X of bidegree (p,q). Recall that $\Omega\in\mathscr{E}^{n-p-q,n-p-q}(X)$ is a closed form that takes values only in the Hodge–Riemann cone.

Proposition 3.1. *Assume that $p,q\geq 1$. Then, for every closed form $f\in\mathscr{E}^{p,q}(X)$ such that $\{f\}\in H^{p,q}(X,\mathbb{C})_{\mathrm{prim}}$, there is a form $u\in L^2_{p-1,q-1}(X)$ such that*

$$dd^cu\wedge\Omega\wedge\omega=f\wedge\Omega\wedge\omega.$$

Proof. Consider the subspace H of $L^2_{n-p+1,n-q+1}(X)$ defined by

$$H:=\big\{dd^c\alpha\wedge\Omega\wedge\omega:\alpha\in\mathscr{E}^{q-1,p-1}(X)\big\}$$

and the linear form h on H given by

$$h\big(dd^c\alpha\wedge\Omega\wedge\omega\big):=(-1)^{p+q+1}\int_X\alpha\wedge f\wedge\Omega\wedge\omega.$$

We prove that h is a well-defined bounded linear form with respect to the L^2-norm restricted to H.

We claim that there is a constant $c>0$ such that

$$\big\|dd^c\alpha\big\|_{L^2}\leq c\big\|dd^c\alpha\wedge\Omega\wedge\omega\big\|_{L^2}.$$

Indeed, recall that $\Omega(x)$ is a Hodge–Riemann form for the bidegree (p,q) for all $x\in X$. Therefore, we use the inequality in Proposition 2.8 applied to $dd^c\alpha$ instead of α and the complex cotangent spaces of X instead of E. Since X is compact, we can find common constants c_1 and c_2 for all cotangent spaces. We then integrate over X and obtain

$$\big\|dd^c\alpha\big\|^2_{L^2}\leq c_1Q\big(dd^c\alpha,dd^c\alpha\big)+c_2\big\|dd^c\alpha\wedge\Omega\wedge\omega\big\|^2_{L^2},$$

where Q is defined in Section 1. Using Stokes' formula, we obtain

$$Q\big(dd^c\alpha,dd^c\alpha\big)=i^{p-q}(-1)^{\frac{(p+q)(p+q-1)}{2}}\int_X dd^c\alpha\wedge dd^c\overline{\alpha}\wedge\Omega=0.$$

We then deduce easily the claim.

Now, by hypothesis the smooth form $f\wedge\Omega\wedge\omega$ is exact. Therefore, there is a form $g\in\mathscr{E}^{n-q,n-p}(X)$ such that

$$dd^cg=f\wedge\Omega\wedge\omega,$$

see, for example, [5, p. 41]. Using again Stokes' formula and the above claim, we obtain

$$\left|\int_X \alpha\wedge f\wedge\Omega\wedge\omega\right| = \left|\int_X \alpha\wedge dd^c g\right| = \left|\int_X dd^c\alpha\wedge g\right|$$
$$\leq \|g\|_{L^2}\big\|dd^c\alpha\big\|_{L^2} \leq c\|g\|_{L^2}\big\|dd^c\alpha\wedge\Omega\wedge\omega\big\|_{L^2}.$$

It follows that h is a well-defined form whose norm in L^2 is bounded by $c\|g\|_{L^2}$.

By the Hahn–Banach theorem, we can extend h to a bounded linear form on $L^2_{n-p+1,n-q+1}(X)$. Let u be a form in $L^2_{p-1,q-1}(X)$ that represents h. It follows from the definition of h that

$$\int_X u\wedge dd^c\alpha\wedge\Omega\wedge\omega = (-1)^{p+q+1}\int_X \alpha\wedge f\wedge\Omega\wedge\omega = -\int_X f\wedge\alpha\wedge\Omega\wedge\omega$$

for all test forms $\alpha\in\mathscr{E}^{q-1,p-1}(X)$. The form u satisfies the proposition. □

We have the following result.

PROPOSITION 3.2. *Let u be as in Proposition 3.1. Then there is a form $v\in\mathscr{E}^{p-1,q-1}(X)$ such that $dd^c v = dd^c u$.*

Proof. We can assume without loss of generality that $p\leq q$. The idea is to use the ellipticity of the Laplacian operator associated with $\overline{\partial}$ and a special inner product on $\mathscr{E}^{p,q}(X)$. We first construct this inner product. Fix an arbitrary Hermitian metric on the vector bundle $\bigwedge^{r,s}(X)$ of differential (r,s)-forms on X with $(r,s)\neq(p,q)$ and denote by $\langle\cdot,\cdot\rangle$ the associated inner product on $\mathscr{E}^{r,s}(X)$.

Using the first assertion in Proposition 2.8, for any $\alpha,\alpha'\in\mathscr{E}^{p,q}(X)$, we can write in a unique way

$$\alpha = \beta+\omega\wedge\gamma \quad\text{and}\quad \alpha' = \beta'+\omega\wedge\gamma'$$

with $\beta,\beta'\in\mathscr{E}^{p,q}(X)$ and $\gamma,\gamma'\in\mathscr{E}^{p-1,q-1}(X)$ such that $\beta\wedge\Omega\wedge\omega=0$ and $\beta'\wedge\Omega\wedge\omega=0$. Define an inner product $\langle\cdot,\cdot\rangle$ on $\mathscr{E}^{p,q}(X)$ by setting

$$\big\langle\alpha,\alpha'\big\rangle := Q\big(\beta,\beta'\big)+\big\langle\gamma,\gamma'\big\rangle = Q\big(\alpha,\beta'\big)+\big\langle\gamma,\gamma'\big\rangle,$$

where $\langle\gamma,\gamma'\rangle$ is calculated using the previously fixed Hermitian metric on the vector bundle $\bigwedge^{p-1,q-1}(X)$. This inner product is associated with a Hermitian metric on $\bigwedge^{p,q}(X)$.

Using the positivity of Q given in Proposition 2.8, we see that $\langle\cdot,\cdot\rangle$ defines a Hermitian metric on $\mathscr{E}^{p,q}(X)$. Consider now the norm $\|\alpha\| := \sqrt{\langle\alpha,\alpha\rangle}$. Then there is a constant $c>0$ such that

$$c^{-1}\big(\|\beta\|_{L^2}+\|\gamma\|_{L^2}\big)\leq\|\alpha\|\leq c\big(\|\beta\|_{L^2}+\|\gamma\|_{L^2}\big).$$

Consider the (p,q)-current $h := dd^c u - f$ which belongs to a Sobolev space. We have

$$\overline{\partial}h = 0,\qquad \partial h = 0\quad\text{and}\quad h\wedge\Omega\wedge\omega = 0.$$

The last identity says that if we decompose h as we did above for α, α', the second component in the decomposition vanishes. Therefore, $\langle\overline{\partial}\alpha, h\rangle = Q(\overline{\partial}\alpha, h)$ for any form $\alpha \in \mathscr{E}^{p,q-1}(X)$. Using Stokes' formula, we obtain

$$\langle\overline{\partial}\alpha, h\rangle = Q(\overline{\partial}\alpha, h) = i^{p-q}(-1)^{p+q-1+\frac{(p+q)(p+q-1)}{2}} \int_X \alpha \wedge \overline{\partial h} \wedge \Omega = 0.$$

If $\overline{\partial}^*$ is the adjoint of $\overline{\partial}$ with respect to the considered inner products, we deduce that $\overline{\partial}^* h = 0$. On the other hand, $\overline{\partial} h = 0$. Therefore, h is a harmonic current with respect to the Laplacian operator $\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$, see Section 5 in [26, Chapter IV]. Consequently, by elliptic regularity, h is smooth, see, for example, Theorem 4.9 in [26, Chapter IV]). Hence, $dd^c u$ is smooth. We deduce the existence of $v \in \mathscr{E}^{p-1,q-1}(X)$ such that $dd^c v = dd^c u$, see, for example, [5, p. 41]. □

End of the proof of Theorem 1.1. Let f be a closed form in $\mathscr{E}^{p,q}(X)$ such that $\{f\} \in H^{p,q}(X, \mathbb{C})_{\mathrm{prim}}$. We first show that $Q(\{f\}, \{f\}) \geq 0$. Let v be the smooth $(p-1, q-1)$-form given by Proposition 3.2. Then we have

$$\big(f - dd^c v\big) \wedge \Omega \wedge \omega = 0.$$

Here, we should replace $dd^c v$ with 0 when either $p = 0$ or $q = 0$. Using Proposition 2.8 at each point of X, after an integration on X, we obtain

$$i^{p-q}(-1)^{\frac{(p+q)(p+q-1)}{2}} \int_X \big(f - dd^c v\big) \wedge \big(\overline{f} - dd^c \overline{v}\big) \wedge \Omega \geq 0.$$

Using Stokes' formula and that f is closed, we obtain

$$\int_X f \wedge \overline{f} \wedge \Omega = \int_X \big(f - dd^c v\big) \wedge \big(\overline{f} - dd^c \overline{v}\big) \wedge \Omega.$$

Therefore, $Q(\{f\}, \{f\}) \geq 0$. The equality occurs if and only if $f = dd^c v$, that is, $\{f\} = 0$. Hence, $\{\Omega\}$ satisfies the Hodge–Riemann theorem for the bidegree (p, q).

We deduce that the map $\{\alpha\} \mapsto \{\alpha\} \smile \{\Omega\}$ is injective on $H^{p,q}(X, \mathbb{C})_{\mathrm{prim}}$. If $\{\alpha\}$ is a class in $H^{p,q}(X, \mathbb{C})$ such that $\{\alpha\} \smile \{\Omega\} = 0$, $\{\alpha\}$ is a primitive class and hence $\{\alpha\} = 0$. Therefore, $\{\Omega\}$ satisfies the hard Lefschetz theorem for the bidegree (p, q).

The classical hard Lefschetz theorem implies that $\{\alpha\} \mapsto \{\alpha\} \smile \{\omega\}$ is an injective map from $H^{p-1,p-1}(X, \mathbb{C})$ to $H^{p,q}(X, \mathbb{C})$. Therefore,

$$\dim\{\omega\} \smile H^{p-1,q-1}(X, \mathbb{C}) = \dim H^{p-1,q-1}(X, \mathbb{C}).$$

This Lefschetz theorem also implies that $\{\alpha\} \mapsto \{\alpha\} \smile \{\omega\}$ is a surjective map from $H^{n-q,n-p}(X, \mathbb{C})$ to $H^{n-q+1,n-p+1}(X, \mathbb{C})$. This together with the hard

Lefschetz theorem for $\{\Omega\}$ yield

$$\begin{aligned}\dim H^{p,q}(X,\mathbb{C})_{\rm prim} &= \dim H^{p,q}(X,\mathbb{C}) - \dim H^{n-q+1,n-p+1}(X,\mathbb{C})\\ &= \dim H^{p,q}(X,\mathbb{C}) - \dim H^{p-1,q-1}(X,\mathbb{C})\\ &= \dim H^{p,q}(X,\mathbb{C}) - \dim\{\omega\}\smile H^{p-1,q-1}(X,\mathbb{C}).\end{aligned}$$

The hard Lefschetz theorem can also be applied to $\{\Omega\wedge\omega^2\}$ and to the bidegree $(p-1,q-1)$. We deduce that the intersection of $\{\omega\}\smile H^{p-1,q-1}(X,\mathbb{C})$ and $H^{p,q}(X,\mathbb{C})_{\rm prim}$ is reduced to 0. This together with the above dimension computation gives us the following decomposition into a direct sum

$$H^{p,q}(X,\mathbb{C}) = \{\omega\}\smile H^{p-1,q-1}(X,\mathbb{C}) \oplus H^{p,q}(X,\mathbb{C})_{\rm prim}.$$

Finally, the previous decomposition is orthogonal with respect to Q by definition of primitive space. So, $\{\Omega\}$ satisfies the Lefschetz decomposition theorem. □

REMARK 3.3. In order to obtain the Hodge–Riemann theorem and the hard Lefschetz theorem (resp. the Lefschetz decomposition), it is enough to assume the property $(*)$ in Definition 2.1 for $r=0,1$ (resp. $r=0,1,2$). When $(*)$ is satisfied for all r, we can apply inductively these theorems to $\Omega\wedge\omega^{2r}$ and then obtain the signature of Q on $H^{p,q}(X,\mathbb{C})$.

4. A family of Hodge–Riemann forms

This section contains an experimental study of Hodge–Riemann forms in the holomorphic symplectic setting. From now on, assume that $n=2m$ and we consider on $E=\mathbb{C}^{2m}$ the coordinate system $(x_1,\ldots,x_m,y_1,\ldots,y_m)$, the standard Kähler form

$$\omega := idx_1\wedge d\overline{x}_1+\cdots+idx_m\wedge d\overline{x}_m+idy_1\wedge d\overline{y}_1+\cdots+idy_m\wedge d\overline{y}_m$$

and the standard symplectic form

$$\sigma := dx_1\wedge dy_1+\cdots+dx_m\wedge dy_m.$$

The main purpose of this section is to establish the following result.

PROPOSITION 4.1. *The form*

$$\Omega := \big(\sigma\overline{\sigma}+t\omega^2\big)\wedge(\sigma\overline{\sigma})^{m-p-v-1}\wedge\omega^{p-q+2v}$$

is a Hodge–Riemann form for the bidegree (p,q) for $q=0$ or 1, $q\le p\le m/2$, $v_q<v\le m-p-1$ and $t\in\mathbb{R}_+$, where $v_0:=-1$ and

$$v_1 := \begin{cases}\frac{p(m-p)}{p+1} & \text{when } p<\sqrt{2(m+1)}-1,\\ \frac{2m-p+3}{2}-\sqrt{2(m+1)} & \text{when } p\ge\sqrt{2(m+1)}-1.\end{cases}$$

Note that when $t=0$, Proposition 4.1 holds also for $v=m-p$. As a direct consequence of Theorem 1.1 and Proposition 4.1 applied to $t=0$, we obtain the following result.

THEOREM 4.2. *Let (X,ω,σ) be a compact symplectic Kähler manifold of dimension $n=2m$, where ω is a Kähler form and σ is a holomorphic symplectic $(2,0)$-form on X. Let p,q,v be non-negative integers such that $q\leq p\leq m/2$, $v_q<v\leq m-p$ and $q=0$ or 1, where v_q is defined as above. Then the class of $(\sigma\wedge\overline{\sigma})^{m-p-v}\wedge\omega^{p-q+2v}$ satisfies the Hodge–Riemann theorem, the hard Lefschetz theorem and the Lefschetz decomposition theorem for the bidegree (p,q).*

Theorem 4.2 may be useful in the study of the automorphism group of X, see, for example, [9], [11], [18], [21], [28]. Note that by Proposition 2.5 the results still hold if we use the primitive space associated to $\Omega\wedge\omega'$ for another Kähler form ω'. It is worthy to note also that the lower bound on v is necessary even when $p=q=1$, see Remark 4.6 below. However, when X is an irreducible compact symplectic Kähler manifold and $p=q=1$, Theorem 4.2 for $v=0$ can be deduced from results by Beauville [1] and Bogomolov [2], [3], see also Fujiki [13] and Enoki [12], Huybrechts [17]. In this case, we can show that Theorem 4.2 holds without lower bound for v.

The remaining part is devoted to the proof of Proposition 4.1. In order to simplify the notation, we often drop the letter d and the sign $\wedge$, for example, we will write

$$\omega=ix_1\overline{x}_1+\cdots+ix_m\overline{x}_m+iy_1\overline{y}_1+\cdots+iy_m\overline{y}_m$$

and

$$\sigma=x_1y_1+\cdots+x_my_m.$$

The most inconvenience due to this simplification is the identities like $x_1y_1=-y_1x_1$ involving in the next computation.

For a Lie group G we use the terminology: a G-module and a representation of G interchangeably. The unitary symplectic group $\mathrm{Sp}(m)$ is identified to the group of matrices in $\mathrm{GL}(2m,\mathbb{C})$ which preserve $\sigma,\overline{\sigma}$ and ω. Its action on E extends naturally to the vector spaces $V^{p,q}:=\bigwedge^p E\otimes\bigwedge^q\overline{E}$ and $V^k:=\bigoplus_{p+q=k}V^{p,q}$. In the sequel, we give some properties of $V^{p,q}$ and V^k which are seen as such $\mathrm{Sp}(m)$-modules. We refer to Fujiki [13] for details. Let $V^{p,q}_\varepsilon$ be the set of forms α in $V^{p,q}$ such that $\alpha\{\sigma,\overline{\sigma},\omega\}^{2m-p-q}=0$, where $\{\sigma,\overline{\sigma},\omega\}^{2m-p-q}$ is the family of monomials of degree $2m-p-q$ on $\sigma,\overline{\sigma},\omega$. This is the *universally effective* subspaces of $V^{p,q}$ which is also a representation of $\mathrm{Sp}(m)$. We will also consider the set $V^{p,q}_0$ of forms in $V^{p,q}$ which can be written as polynomials in $\sigma,\overline{\sigma}$ and ω. This is a representation of $\mathrm{Sp}(m)$ which is isomorphic to a direct sum of copies of the trivial representation since $\sigma,\overline{\sigma}$ and ω are invariant. Define also $V^k_\varepsilon:=\bigoplus_{p+q=k}V^{p,q}_\varepsilon$ and $V^k_0:=\bigoplus_{p+q=k}V^{p,q}_0$.

A representation is said to be *isotropic* or *W-isotropic* if it is isomorphic to a direct sum $W\oplus\cdots\oplus W$ of an irreducible representation W. If $\widehat{V}$ is a representation, there is a unique maximal representation $\widehat{V}_W \subset \widehat{V}$ which is W-isotropic and we call it the *W-isotropic component* of $\widehat{V}$. Any representation is isomorphic to the direct sum of its isotropic components.

Define for all non-negative integers k, ν, s such that $\nu \leq s$ and $\nu + s \leq k \leq m$

$$Z_{k,\nu,s} := x_1\overline{y}_1 \cdots x_\nu \overline{y}_\nu \sum \operatorname{sign}(I,J) x_I \overline{y}_J,$$

where the sum is taken over $\{I, J\}$ such that $I \subset \{\nu+1, \ldots, k-\nu\}$, $|I| = k-\nu-s$, J is the complement of I in $\{\nu+1, \ldots, k-\nu\}$ and $\operatorname{sign}(I,J)$ is the signature of the permutation $\{\nu+1, \ldots, k-\nu\} \mapsto \{I, J\}$. The form $Z_{k,\nu,s}$ is of bidegree $(k-s, s)$ in V^k_ε.

Consider the diagonal subgroup of $\operatorname{Sp}(m)$

$$D(m) := \big\{\operatorname{diag}\big(\varepsilon_1, \ldots, \varepsilon_m, \varepsilon_1^{-1}, \ldots, \varepsilon_m^{-1}\big), \varepsilon_i \in \mathbb{C}, |\varepsilon_i| = 1\big\},$$

and the set

$$\Psi := \big\{(k,r) \in \mathbb{N}^2, k + r \leq 2m, k \geq r \text{ and } k \equiv r \text{ modulo } 2\big\}.$$

Fix an arbitrary pair $(k,r) \in \Psi$ and let $\nu := \frac{k-r}{2}$. Observe that for $\nu \leq s \leq k-\nu$, the eigenvalue $\varepsilon_1^2 \cdots \varepsilon_\nu^2 \varepsilon_{\nu+1} \cdots \varepsilon_k$ of $Z_{k,\nu,s}$ appears as the highest non-zero term in the character (Laurent polynomial in $\varepsilon_1, \ldots, \varepsilon_m$) of the action of $D(m)$ on the $\operatorname{Sp}(m)$-module $V^{k-s,s}$ with respect to the lexicographical order. Let $W_{k,r}$ be the irreducible representation of $\operatorname{Sp}(m)$ characterized by this property. So, given $(k,r) \in \Psi$, the smallest $\operatorname{Sp}(m)$-module of $V^{k-s,s}$ spanned by $Z_{k,\nu,s}$ is isomorphic to $W_{k,r}$ for $\nu \leq s \leq k-\nu$. Let $U_{k,r}$ be the vector space of V^k_ε spanned by the forms $Z_{k,\nu,s}$.

The following result is deduced from Proposition 2.4 in Fujiki [13] and its proof. It implies, in particular, that the family of equivalent classes of irreducible $\operatorname{Sp}(m)$-submodules of V^k is naturally in bijective correspondence to the pairs $(k,r) \in \Psi$. As Fujiki mentioned in his paper, it is likely true for all $k \leq 2m$.

Proposition 4.3. *Assume that $k \leq m$. Then V^k is the direct sum of the subspaces $V^t_0 \wedge V^{k-t}_\varepsilon$ with $0 \leq t \leq k$. The $W_{k,r}$-isotropic component of V^k_ε is isomorphic as $\operatorname{Sp}(m)$-module to $W_{k,r} \otimes U_{k,r}$, where $U_{k,r}$ is identified with $\{v\} \times U_{k,r}$ for some non-zero vector $v \in W_{k,r}$ and $\operatorname{Sp}(m)$ acts trivially on the second factor of $W_{k,r} \otimes U_{k,r}$. Moreover, the other isotropic components of V^k_ε vanish.*

For the reader's convenience, we summarize here Fujiki's arguments.

Proof. Let $\mathbb{H}$ be the real quaternion division algebra. We identify $\mathbb{H}^m$ equipped with the standard quaternion inner product to the underlying real Euclidean space $\mathbb{R}^{4m}$. Hence, the natural action of $\operatorname{Sp}(m)$ on $\mathbb{H}^m$ induces the natural inclusion $\operatorname{Sp}(m) \hookrightarrow \operatorname{SO}(4m, \mathbb{R})$. The standard action of $\operatorname{SO}(4m, \mathbb{R})$ on $\mathbb{R}^{4m}$ extends naturally to $V^k_\mathbb{R}$ and V^k. On the other hand, since $\mathbb{H}^* := \mathbb{H} \setminus \{0\} \simeq \operatorname{Sp}(1) \times_{\mathbb{Z}_2} \mathbb{R}^*$, the componentwise quaternionic multiplication $\mathbb{H}^* \times$

$\mathbb{H}^m \to \mathbb{H}^m$ also induces the natural inclusion $\mathrm{Sp}(1) \hookrightarrow \mathrm{SO}(4m,\mathbb{R})$. Consequently, the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ which is the complexification of the Lie algebra $\mathfrak{sp}(1)$ of $\mathrm{Sp}(1)$ acts $\mathbb{C}$-linearly on V^k by the Lie derivative.

A $\mathbb{H}^*$-module W is said to be of weight k, if $t \in \mathbb{R}^*$ acts on W via the multiplication by t^k. The family of equivalent classes of irreducible representations of $\mathbb{H}^*$ is naturally in bijective correspondence to the set of pairs

$$\big\{(k,r) \in \mathbb{N}^2, k \geq r \text{ and } k \equiv r \text{ modulo } 2\big\},$$

where k corresponds to the weight of the representation. Let $V_{k,r}$ denote the irreducible representation of $\mathbb{H}^*$ corresponding to the pair (k,r). The characterization of r will be given later on. By the definition of V_ε^k we see easily that it is a $\mathbb{H}^*$-module of weight k. Consequently, we obtain the following decomposition

$$V_\varepsilon^k = \bigoplus_{r:(k,r)\in\Psi} V_\varepsilon^{k;r},$$

where $V_\varepsilon^{k;r}$ is the $V_{k,r}$-isotropic component of V_ε^k. Since $\mathrm{Sp}(1)$ is the centralizer of $\mathrm{Sp}(m)$, it follows that the isotropic components of V_ε^k (with respect to a given irreducible representation of $\mathrm{Sp}(m)$) is equal to the direct sum of those of $V_\varepsilon^{k;r}$.

For $W := V_{k,r}$, consider the induced action on $W_\mathbb{C}$ of the natural action of $\mathfrak{sl}(2,\mathbb{C})$ on V^k. Let

$$H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

be the standard $\mathbb{C}$-basis of $\mathfrak{sl}(2,\mathbb{C})$. We have the following canonical Hodge decomposition

$$W = \bigoplus_{p+q=k} W^{p,q}, \qquad \overline{W}^{p,q} = W^{q,p}.$$

This coincides with the eigenspace decomposition of $W_\mathbb{C}$ with respect to the action of H, where $W^{p,q}$ corresponds to the eigenvalue $p-q$. Moreover,

- $W^{p,q} \neq 0$ if and only if $\frac{k-r}{2} \leq p,q \leq \frac{k+r}{2}$;
- $X : W^{p,q} \simeq W^{p+1,q-1}$ if p,q satisfy $\frac{k-r}{2} \leq p, p+1, q-1, q \leq \frac{k+r}{2}$;
- $X(W^{\frac{k+r}{2},\frac{k-r}{2}}) = 0$.

Recall from the discussion preceding the proposition that $Z_{k,\nu,s} \in V_\varepsilon^k$ with $\nu := \frac{k-r}{2}$. Now we will show that X defines a $\mathrm{Sp}(m)$ isomorphism from the smallest $\mathrm{Sp}(m)$-submodule of $V^{k-s,s}$ spanned by $Z_{k,\nu,s}$ to that spanned by $Z_{k,\nu,s-1}$ and that X maps $Z_{k,\nu,s}$ to $Z_{k,\nu,s-1}$. Indeed, the properties of X listed above show that $X : V^{0,1} \simeq V^{1,0}$ and $X(V^{1,0}) = 0$. Arguing as in Lemma 2.8 in [13], we obtain that

$$X(\overline{x}_i) = y_i, \qquad X(\overline{y}_i) = -x_i, \qquad X(x_i) = 0, \qquad X(y_i) = 0.$$

Since X acts on V_ε^k as a (Lie) derivative, a straightforward computation implies the above assertion. Next, we deduce from the equality $X(Z_{k,\nu,\nu}) = 0$ and the properties of X listed above that $Z_{k,\nu,\nu} \in V_\varepsilon^{k;r}$. Hence, the forms $Z_{k,\nu,s}$ and then the vector space $U_{k,r}$ are also contained in $V_\varepsilon^{k;r}$.

Consequently, by identifying $Z_{k,\nu,s}$ with $W_{k,r} \otimes Z_{k,\nu,s}$, we may consider $W_{k,r} \otimes U_{k,r}$, in a natural way, as a $\mathrm{Sp}(m)$-submodule of the $W_{k,r}$-isotropic component of V_ε^k. Namely, $\{v\} \times U_{k,r}$ is identified with $U_{k,r}$ for some non-zero vector $v \in W_{k,r}$ and $\mathrm{Sp}(m)$ acts trivially on the second factor of $W_{k,r} \otimes U_{k,r}$. This is, in fact, an equality, that is, we have that $V_\varepsilon^{k;r} = W_{k,r} \otimes U_{k,r}$, which implies, in turn, that

$$V_\varepsilon^k = \bigoplus_{r:(k,r)\in\Psi} W_{k,r} \otimes U_{k,r}.$$

This, combined with part 4 and part 5 of Proposition 2.4 in [13], gives the proposition.

The proof of the last identities has been carried out in part 3 of Proposition 2.4 in [13, pp. 121–122]. However, there is one point in Fujiki's argument which needs to be more explicit. Namely, the way Fujiki applies the classical invariant theory for $\mathrm{Sp}(m)$ (see the last lines in [13, p. 121]) should be written down more concretely for the reader's convenience. For the sake of simplicity, we will clarify his argument in a simpler setting. More specifically, we will prove that a form $\alpha \in V^{p,q}$ is $\mathrm{Sp}(m)$-invariant (i.e., $A^*\alpha = \alpha$ for $A \in \mathrm{Sp}(m)$) if and only if α is generated by σ, $\overline{\sigma}$, ω (i.e., $\alpha = h(\sigma, \overline{\sigma}, \omega)$ for a polynomial $h \in \mathbb{C}[t_1, t_2, t_3]$). Note that this proof also works in Fujiki's context of $\mathrm{Sp}(m)$-invariant tensors making the obviously necessary changes.

Let $\alpha \in V^{p,q}$ be $\mathrm{Sp}(m)$-invariant. Let $\binom{dx}{dy}$ be the $2m \times 1$ matrix consisting of the forms dx_i, dy_i. The matrix $\binom{d\overline{x}}{d\overline{y}}$ is defined in a similar way. Let $A \in \mathrm{Sp}(n)$. Since $A \in \mathrm{U}(2m)$, we have $\overline{A} = {}^tA^{-1}$ and

$$\begin{pmatrix} A^*dx \\ A^*dy \end{pmatrix} = A \begin{pmatrix} dx \\ dy \end{pmatrix}, \qquad \begin{pmatrix} A^*d\overline{x} \\ A^*d\overline{y} \end{pmatrix} = \overline{A} \begin{pmatrix} d\overline{x} \\ d\overline{y} \end{pmatrix} = {}^tA^{-1} \begin{pmatrix} d\overline{x} \\ d\overline{y} \end{pmatrix}.$$

We represent α as a polynomial f in $4m$ variables $(x, y; z, w)$, where

$$dx_i(x, y; z, w) = x_i, \qquad dy_i(x, y; z, w) = y_i,$$
$$d\overline{x}_i(x, y; z, w) = z_i, \qquad d\overline{y}_i(x, y; z, w) = w_i.$$

The above equalities, combined with the assumption $A^*\alpha = \alpha$, $A \in \mathrm{Sp}(m)$, implies the following invariant property of f:

$$f(x, y; z, w) = f\left(A \begin{pmatrix} x \\ y \end{pmatrix}; {}^tA^{-1} \begin{pmatrix} z \\ w \end{pmatrix}\right).$$

Note that $\mathrm{Sp}(m) = \mathrm{Sp}(2m, \mathbb{C}) \cap \mathrm{U}(2m)$. Consequently, we deduce from Lemma 7.1.A in [27] that the invariant property of f is also valid for all $A \in \mathrm{Sp}(2m, \mathbb{C})$. Recall that ${}^tAJA = J$ for $A \in \mathrm{Sp}(2m, \mathbb{C})$, where $J := \begin{pmatrix} 0 & \mathrm{id} \\ -\mathrm{id} & 0 \end{pmatrix}$

and id is the identity matrix in $\mathrm{GL}(m,\mathbb{C})$ and that $J^{-1}=-J$. Introduce a new polynomial $g\in\mathbb{C}[x,y;z,w]$ defined by

$$g(x,y;z,w):=f\left(x,y;J^{-1}\begin{pmatrix}z\\w\end{pmatrix}\right).$$

This, coupled with the invariant property of f and the above mentioned properties of J, imply that

$$g(x,y;z,w):=g\left(A\begin{pmatrix}x\\y\end{pmatrix};A\begin{pmatrix}z\\w\end{pmatrix}\right),\quad A\in\mathrm{Sp}(2m,\mathbb{C}).$$

So, we are able to apply the First Fundamental Theorem for $\mathrm{Sp}(2m,\mathbb{C})$ to g (see, e.g., Theorem 5.2.2 in [14]). Let $\widehat{\sigma}$ be the standard skew symmetric form in $\mathbb{C}^{2m}$. We infer that f is generated by $\widehat{\sigma}(x,y;x,y)$, $\widehat{\sigma}\big(x,y;J\binom{z}{w}\big)$, $\widehat{\sigma}\big(J\binom{z}{w};J\binom{z}{w}\big)$. In other words, $\{\sigma,\omega,\overline{\sigma}\}$ is a set of generators for the ring of all $\mathrm{Sp}(m)$-invariant forms. □

We deduce from this proposition the following lemma that we will use later.

Lemma 4.4. *Assume that $p+q\le m$. Then every representation $F\subset V^{p,q}$ contains a non-zero vector in $(V_0^{p+q-k}\wedge U_{k,r})\cap V^{p,q}$ for some $(k,r)\in\Psi$ depending on F with $k\le p+q$.*

Proof. Replacing F with a suitable subspace allows us to assume that F is isomorphic to $W_{k,r}$ for some $(k,r)\in\Psi$ with $k\le p+q$. We only have to show that F contains a non-zero vector in $V_0^{p+q-k}\wedge U_{k,r}$. Proposition 4.3 implies that the $W_{k,r}$-isotropic component of V^{p+q} is isomorphic to $W_{k,r}\otimes(V_0^{p+q-k}\wedge U_{k,r})$ where $V_0^{p+q-k}\wedge U_{k,r}$ is identified with $\{v\}\times(V_0^{p+q-k}\wedge U_{k,r})$ for some non-zero vector $v\in W_{k,r}$. The space F is identified with a subspace of $W_{k,r}\otimes(V_0^{p+q-k}\wedge U_{k,r})$ which, by Schur's lemma, is equal to $W_{k,r}\otimes\{u\}$ for some non-zero vector u in $V_0^{p+q-k}\wedge U_{k,r}$. It follows that F contains u. □

Now we define

$$\gamma_s:=*\big((\sigma\overline{\sigma})^{m-s}\omega^{2s}\big),$$

where $*$ is the Hodge star operator. The following lemma will be used repeatedly in our computation.

Lemma 4.5. *We have*

$$\gamma_s=\frac{m!(2s)!(m-s)!}{s!}.$$

In particular, we have

$$\gamma_s=\frac{m-s}{2(2s+1)}\gamma_{s+1}.$$

Proof. The form $(\sigma\overline{\sigma})^{m-s}\omega^{2s}$ is of maximal degree. So, we have

$$(\sigma\overline{\sigma})^{m-s}\omega^{2s} = \gamma_s(ix_1\overline{x}_1)(iy_1\overline{y}_1)\cdots(ix_m\overline{x}_m)(iy_m\overline{y}_m).$$

Write $x_{m+j} := y_j$ for $1 \leq j \leq m$. When we develop the expression $\sigma^{m-s}\overline{\sigma}^{m-s} \times \omega^{2s}$ any non-zero term has the form

$$(x_{j_1}y_{j_1})\cdots(x_{j_{m-s}}y_{j_{m-s}})(\overline{x}_{l_1}\overline{y}_{l_1})\cdots(\overline{x}_{l_{m-s}}\overline{y}_{l_{m-s}})(ix_{k_1}\overline{x}_{k_1})\cdots(ix_{k_{2s}}\overline{x}_{k_{2s}}),$$

where $\{j_1,\dots,j_{m-s}\}$, $\{l_1,\dots,l_{m-s}\}$ are two permutations of a set $J \subset \{1,\dots, m\}$ with $|J| = m-s$, K the complement of J in $\{1,\dots,m\}$ and $\{k_1,\dots,k_{2s}\}$ is a permutation of $K \cup (m+K)$. All these terms are equal to

$$(ix_1\overline{x}_1)(iy_1\overline{y}_1)\cdots(ix_m\overline{x}_m)(iy_m\overline{y}_m).$$

So, γ_s is the number of such terms. A simple computation on the number of J and the numbers of permutations gives

$$\gamma_s = \binom{m}{m-s}(m-s)!(m-s)!(2s)!.$$

The lemma follows. □

We first take granted the following claim.

CLAIM. *Every form Ω as in Proposition 4.1 is Lefschetz.*

End of the proof of Proposition 4.1. The proof uses a decreasing induction on v. Applying the claim to $\Omega\wedge\omega^{2r}$ with $0 \leq r \leq q$, we deduce that $\Omega\wedge\omega^{2r}$ is a Lefschetz form for the bidegree $(p-r, q-r)$. Recall that the Hodge–Riemann cone is open. So, for $v = m-p-1$, since ω^{2m-p-q} is Hodge–Riemann, Ω is Hodge–Riemann when t is large enough. It follows from the claim applied to $\Omega\wedge\omega^{2r}$ that Ω is Hodge–Riemann for the bidegree (p,q) for every $t \geq 0$.

Assume now the case where v is replaced with $v+1$, that is,

$$\Omega' := \big(\sigma\overline{\sigma} + t\omega^2\big)(\sigma\overline{\sigma})^{m-p-v-2}\omega^{p-q+2v+2}$$

is Hodge–Riemann for the bidegree (p,q) and for every $t \geq 0$. Since this is true for $t = 0$, we deduce by continuity that Ω is Hodge–Riemann for the bidegree (p,q) and for t large enough. Therefore, the claim implies that Ω is Hodge–Riemann for the bidegree (p,q) and for every $t \geq 0$. This also ends the proof. □

We now give the proof of the claim. It is divided into two cases.

Case 1. Assume that $q = 0$. Consider a non-zero form $\alpha \in V^{p,0}$. It is enough to check that $i^{p^2}\alpha\overline{\alpha}\Omega$ is a non-zero positive form. For this purpose, we can assume that $\Omega = (\sigma\overline{\sigma})^r\omega^{2m-p-2r}$ with $0 \leq r \leq m-p$.

By Fujiki's theorem [13, Proposition 2.6], the map $\beta \mapsto \beta\sigma^r$ is injective on $V^{p,0}$ when $r \leq m-p$. Therefore, $\alpha\sigma^r$ is a non-zero form in $V^{p+2r,0}$. So, we can choose $(1,0)$-forms β_j such that $\alpha\sigma^r\beta_1\cdots\beta_{2m-p-2r}$ does not vanish.

Since this form is of bidegree $(2m,0)$, it is a multiple of $x_1\cdots x_m y_1\cdots y_m$. Therefore, it is not difficult to see that

$$i^{p^2}\alpha\overline{\alpha}(\sigma\overline{\sigma})^r(i\beta_1\overline{\beta}_1)\cdots(i\beta_{2m-p-2r}\overline{\beta}_{2m-p-2r})$$

is a non-zero positive form. This implies the result because $i\beta_j\overline{\beta}_j\leq c\omega$ for c a large enough positive constant.

Case 2. Assume that $q=1$. Let $F\subset V^{p,1}$ be the set of α such that $\alpha\Omega=0$. Suppose in order to get a contradiction that $F\neq 0$. Since Ω is invariant under $\mathrm{Sp}(m)$, F is a representation of $\mathrm{Sp}(m)$. By Lemma 4.4, there are integers $k\leq p+1$, $\nu=0,1$ and forms $P_s\in V_0^{p-k+s,1-s}$ for $\max\{\nu,k-p\}\leq s\leq 1$ such that

$$\alpha=\sum_s P_s Z_{k,\nu,s}$$

is a non-zero form in F.

If $\nu\neq 0$, we can write $\alpha=x_1\overline{y}_1\alpha'$ with α' independent of the variables x_1, y_1. The equation $\alpha\Omega=0$ is equivalent to the equation $\alpha'\Omega'=0$ where Ω' is obtained from Ω by deleting the terms which depend on x_1, y_1. This reduces the problem to the case of lower dimension and lower degrees. More precisely, the last equation contradicts the result obtained in Case 1. Now, assume that $\nu=0$. Write for simplicity Z_s instead of $Z_{k,\nu,s}$. Observe that $p+q-k$ is even.

Using the notation $u:=m-p-v-1$ gives $m=u+v+p+1$ and

$$\Omega=\big(\sigma\overline{\sigma}+t\omega^2\big)(\sigma\overline{\sigma})^u\omega^{p-1+2v}.$$

There are three subcases to consider.

Subcase 2(a). Assume that $k=p-1$. We have

$$\alpha=\lambda_1\sigma Z_1+\lambda_2\omega Z_0\quad\text{with } (\lambda_1,\lambda_2)\in\mathbb{C}^2\setminus\{0\},$$

where

$$Z_1=\sum_{j=1}^{k}x_1\cdots x_{j-1}\overline{y}_j x_{j+1}\cdots x_k\quad\text{and}\quad Z_0=x_1\cdots x_k.$$

We will consider the expansion of $\alpha\Omega$ in coordinates x_i,y_i for $i\leq k$. Then, the equation $\alpha\Omega=0$ induces some equations on forms depending only on the other coordinates, that is, equations on forms on $\mathbb{C}^{2m-2k}$. In order to simplify the notation, in this space σ and ω will denote also the standard symplectic and Kähler forms. We will consider Ω as a polynomial in σ, $\overline{\sigma}$, ω and we will also consider derivatives in that variables. The constants γ_s are defined as in Lemma 4.5 but for $\mathbb{C}^{2m-2k}$ instead of $\mathbb{C}^{2m}$.

Consider the coefficient of $(x_1y_1\overline{y}_1)\cdots(x_ky_k\overline{y}_k)$ in $\alpha\Omega$. This is a form bidegree $(2m-2k-1,2m-2k-1)$ in $\mathbb{C}^{2m-2k}$. Here is the kind of argument that we use repeatedly in the computation. In order to obtain the coefficient of $(x_1y_1\overline{y}_1)\cdots(x_ky_k\overline{y}_k)$, for example, in

$$\begin{aligned}\sigma x_1\cdots x_{j-1}\overline{y}_j x_{j+1}\cdots x_k(\sigma\overline{\sigma})^{u+1}\omega^{k+2v}\\ = x_1\cdots x_{j-1}\overline{y}_j x_{j+1}\cdots x_k\overline{\sigma}^{u+1}\sigma^{u+2}\omega^{k+2v}\end{aligned}$$

we have to take x_jy_j from a factor σ and $y_l\overline{y}_l$ with $l\neq j$ from a factor ω. Now, since the coefficient of $(x_1y_1\overline{y}_1)\cdots(x_ky_k\overline{y}_k)$ in $\alpha\Omega$ vanishes, we obtain the following equation on forms on $\mathbb{C}^{2m-2k}$ where the first factor k represents the number of choices for j and i^{k-1}, i^k come from the factors ω, that is, $i=\partial\omega/\partial(y_l\overline{y}_l)$

$$i^{k-1}k\frac{\partial^k(\sigma\Omega)}{\partial\sigma\,\partial^{k-1}\omega}\lambda_1+i^k\frac{\partial^k(\omega\Omega)}{\partial^k\omega}\lambda_2=0.$$

Multiplying this equation with ω in order to get forms of maximal degree and using the $*$-operation, we obtain

$$\begin{aligned}\Big[k(u+2)\frac{(k+2v)!}{(2v+1)!}\gamma_{v+1}+k(u+1)\frac{(k+2v+2)!}{(2v+3)!}\gamma_{v+2}t\Big]\lambda_1\\ +i\Big[\frac{(k+2v+1)!}{(2v+1)!}\gamma_{v+1}+\frac{(k+2v+3)!}{(2v+3)!}\gamma_{v+2}t\Big]\lambda_2=0.\end{aligned}$$

Using the last assertion in Lemma 4.5 for $\mathbb{C}^{2m-2k}$, we obtain the equation

$$a_{11}\lambda_1+ia_{12}\lambda_2=0,$$

where

$$a_{11}:=k(u+2)(v+1)(u+1)+k(u+1)(k+2v+1)(k+2v+2)t$$

and

$$a_{12}:=(k+2v+1)(v+1)(u+1)+(k+2v+1)(k+2v+2)(k+2v+3)t.$$

Now, consider the coefficient of $x_1(x_2y_2\overline{y}_2)\cdots(x_ky_k\overline{y}_k)$ in $\alpha\Omega$. This is a form of maximal bidegree in $\mathbb{C}^{2m-2k}$. Observe that the first term in Z_1 does not contribute to this coefficient. Therefore, we obtain the following equation where the factor $k-1$ represents the number of the other terms in Z_1

$$i^{k-2}(k-1)\frac{\partial^{k-1}(\sigma\Omega)}{\partial\sigma\,\partial^{k-2}\omega}\lambda_1+i^{k-1}\frac{\partial^{k-1}(\omega\Omega)}{\partial^{k-1}\omega}\lambda_2=0.$$

Using $*$-operation gives

$$\begin{aligned}\Big[(k-1)(u+2)\frac{(k+2v)!}{(2v+2)!}\gamma_{v+1}+(k-1)(u+1)\frac{(k+2v+2)!}{(2v+4)!}\gamma_{v+2}t\Big]\lambda_1\\ +i\Big[\frac{(k+2v+1)!}{(2v+2)!}\gamma_{v+1}+\frac{(k+2v+3)!}{(2v+4)!}\gamma_{v+2}t\Big]\lambda_2=0.\end{aligned}$$

Using the last assertion in Lemma 4.5 for $\mathbb{C}^{2m-2k}$, we obtain the equation

$$a_{21}\lambda_1+ia_{22}\lambda_2=0,$$

where

$$a_{21} := (k-1)(u+2)(v+2)(u+1) + (k-1)(u+1)(k+2v+1)(k+2v+2)t$$

and

$$a_{22} := (k+2v+1)(v+2)(u+1) + (k+2v+1)(k+2v+2)(k+2v+3)t.$$

Since the above equations have a non-trivial solution (λ_1, λ_2), we have

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0.$$

A simple computation gives

$$At^2 + Bt + C = 0,$$

where

$$\begin{aligned} A &:= (k+2v+3)(k+2v+2)^2(k+2v+1)^2(u+1), \\ B &:= (k+2v+2)(k+2v+1)(u+1) \\ &\qquad \times \left(4uv^2 + 2kuv + 10uv + 7u + ku + 6v^2 + kv + 17v + 13 - k^2\right), \\ C &:= (v+2)(v+1)(u+2)(u+1)^2(k+2v+1). \end{aligned}$$

Since $m \geq 2p$, we have $u + v \geq k$. Therefore, A, B, C are positive. This is a contradiction since $t \geq 0$. Hence, Subcase 2(a) cannot happen.

Subcase 2(b). Assume now that $k = p+1$. Then $\alpha = \lambda Z_1$ with $\lambda \in \mathbb{C}^*$ and

$$Z_1 = \sum_{j=1}^{k} x_1 \cdots x_{j-1} \overline{y}_j x_{j+1} \cdots x_{p+1}.$$

Consider the coefficients of

$$(x_1\overline{x}_1\overline{y}_1)(x_2y_2\overline{y}_2)(x_3y_3\overline{y}_3)\cdots(x_{p+1}y_{p+1}\overline{y}_{p+1})$$

in $\alpha\Omega = 0$. We obtain

$$i^{p+1}\frac{\partial^{p+1}\Omega}{\partial\omega^{p+1}} + i^{p-1}p\frac{\partial^{p+1}\Omega}{\partial\sigma\,\partial\overline{\sigma}\,\partial\omega^{p-1}} = 0.$$

This gives us

$$\begin{aligned} &(u+1)v(v+1) + (p+2v)(p+2v+1)(v+1)t \\ &\qquad - p(u+1)^2(v+1) - p(p+2v)(p+2v+1)ut = 0. \end{aligned}$$

Define

$$t' := \frac{(p+2v)(p+2v+1)t}{(u+1)(v+1)}.$$

We obtain

$$v + (v+1)t' - p(u+1) - put' = 0.$$

Recall that $u = m - p - v - 1$. So, the above expression is non-zero for all $t \in \mathbb{R}_+$ if and only if

$$\left(v - \frac{p(m-p)}{p+1}\right)\left(v - \frac{p(m-p-1)-1}{p+1}\right) > 0.$$

Since the last inequality is true by the hypothesis that

$$v > v_1 \geq \frac{p(m-p)}{p+1},$$

we get the desired contradiction. This completes the proof in this subcase.

Subcase 2(c). Assume now that $k < p-1$. If $p-1-k = 2s$, then we write $\alpha = \sigma^s \beta$, and replace α, Ω, p, v with $\beta, (\sigma\overline{\sigma})^s\Omega, p-2s, v+s$, we can reduce the problem to the last case with lower degree p. Therefore, it suffices to verify that last inequality in Subcase 2(b) still holds for the new values of p, v after this reduction. So, it is enough to check the condition $v > v_1'$, where v_1' is the maximum of the function

$$\left[0, \frac{p}{2}\right] \ni s \mapsto \frac{(p-2s)(m-p+2s)}{p+1-2s} - s.$$

Setting $x := p+1-2s$, the above function can be rewritten as

$$\phi(x) := \frac{2(x-1)(m+1-x)+x^2}{2x} - \frac{p+1}{2}, \quad x \in [0, \infty).$$

This function attains its maximum at $x := \sqrt{2(m+1)}$ and it is not difficult to check that $v_1' = v_1$. The proof is thereby completed.

Remark 4.6. When $p = q = 1$ and $\alpha = Z_1$ as in Subcase 2(b) and $\beta = x_1\overline{y}_1$, a straighforward computation shows that $\alpha\overline{\alpha}\omega^{2m-2} < 0$ whereas $\alpha\overline{\alpha}(\sigma\overline{\sigma})^{m-1} > 0 > \beta\overline{\beta}(\sigma\overline{\sigma})^{m-1}$. Consequently, both positive forms ω^{2m-2} and $(\sigma\overline{\sigma})^{m-1}$ have the same primitive space $P^{1,1}$, which is, by Proposition 4.3, the $\mathrm{Sp}(m)$-submodule of $V^{1,1}$ spanned by α and β. However, $Q_{\omega^{2m-2}}$ is positive-definite on $P^{1,1}$ whereas $Q_{(\sigma\overline{\sigma})^{m-1}}$ is not semi-definite on $P^{1,1}$. Another consequence is that by continuity, there is an integer v with $0 \leq v \leq m-2$ and $t \in \mathbb{R}_+$ such that the corresponding form Ω in Proposition 4.1 is not Hodge–Riemann. In general (e.g., for tori), the Hodge–Riemann form $Q_{\omega^{2m-2}}$ and the Beauville–Bogomolov form $Q_{(\sigma\overline{\sigma})^{m-1}}$ do not have the same signature.

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