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METRIC CHARACTERIZATIONS II

DAVID P. BLECHER AND MATTHEW NEAL

Dedicated to the memory of William B. Arveson

ABSTRACT. The present paper is a sequel to our paper "Metric characterization of isometries and of unital operator spaces and systems." We characterize certain common objects in the theory of operator spaces (unitaries, unital operator spaces, operator systems, operator algebras, and so on), in terms which are purely *linear-metric*, by which we mean that they only use the vector space structure of the space and its matrix norms. In the last part, we give some characterizations of operator algebras (which are not linear-metric in our strict sense described in the paper).

1. Introduction

The present paper is a sequel to our paper "Metric characterization of isometries and of unital operator spaces and systems" [14]. The goal of both papers is to characterize certain common objects in the theory of operator spaces (unitaries, unital operator spaces, operator systems, operator algebras, and so on), in terms which are purely *linear-metric*, by which we mean that they only use the vector space structure of the space and its matrix norms, in the spirit of Ruan's matrix norm characterization of operator spaces [26], not mentioning products, involutions, or any kind of function such as linear maps on the space. In the present paper, we give new linear-metric characterizations of unital operator spaces (or equivalently, of 'unitaries' in an operator space). Some of our characterizations should be useful in future. Others may

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look cumbersome, but their virtue is that it is nice to know that such characterizations exist (only using the norm and/or vector space structure). An example of one of our new characterizations is the following theorem.

THEOREM 1.1. *If X is an operator space and $u \in X$ with $\|u\| = 1$ then (X, u) is a unital operator space (or equivalently, u is a unitary in X) iff*

$$\max\{\|u_n + i^k x\| : k = 0, 1, 2, 3\} \geq \sqrt{1 + \|x\|}, \quad n \in \mathbb{N}, x \in M_n(X).$$

Here u_n is the diagonal matrix $u \otimes I_n$ in $M_n(X)$ with u in each diagonal entry. Indeed in this result one only needs x of 'small norm,' where 'small' can differ for each n. Thus, only local information near each u_n is necessary in order to determine if u is unitary; and the above shows how this may be done. Another advantage of this approach is that it avoids 'matrices of matrices,' in contrast to our characterizations in [14]. We will give a convincing illustration of the use of this criterion after its proof (after Theorem 3.2 below).

We also give several other assorted results and observations, most of these being complements to various results in [14]. The structure of our paper is as follows: In Section 2, we present some matrix norm formulae that will be used later in the paper. In Section 3 (resp., Section 4), we give new linear-metric characterizations of unital operator spaces (resp., operator systems). In Section 3 we also relate unital operator spaces to our previous paper [15], by characterizing compact projections in a C^*-algebra in terms of unital operator spaces. In the remaining sections, we characterize operator algebra structures on operator spaces in various ways. For example, we give new variants of the characterization of operator algebras due to the first author with Ruan and Sinclair [17]. In various remarks scattered through our paper, we indicate where a result may be strengthened, or give counterexamples ruling out certain directions of enquiry.

Turning to definitions, all vector spaces are over the complex field $\mathbb{C}$. The letters H, K are usually reserved for Hilbert spaces. We write $\text{Ball}(X) = \{x \in X : \|x\| \leq 1\}$. We write $M_n(X)$ for the space of $n \times n$ matrices with entries in X. As always in operator space theory, $M_n(X)$ has a canonical norm which we write as $\|x\|_n$ or simply $\|x\|$. A given cone in a space X will sometimes be written as X_+, and $X_{\text{sa}} = \{x \in X : x = x^*\}$ assuming that there is an involution $*$ around. The reader may consult [9], or one of the other books on operator spaces, for more information if needed below. All normed (or operator) spaces are assumed to be complete. A *unital operator space* is a subspace of a unital C^*-algebra containing the identity [3]. More abstractly, a unital operator space is a pair (X, u) consisting of an operator space X containing a fixed element u such that there exists a Hilbert space H and a complete isometry $T : X \to B(H)$ with $T(u) = I_H$. In this case, we also say that u is a *unitary in X*. An *operator system* is a selfadjoint subspace of a unital C^*-algebra containing the identity. More abstractly, a unital operator space (X, u) for which there exists a linear complete isometry $T : X \to B(H)$ with $T(u) = I_H$ and $T(X)$ selfadjoint. An *operator algebra* is an operator space A which is an algebra such that there exists a completely isometric homomorphism from A into a C^*-algebra. An operator algebra is *unital* if it has an identity of norm 1.

A *TRO* (ternary ring of operators) is a closed subspace Z of a C^*-algebra, or of $B(K, H)$, such that $ZZ^*Z \subset Z$. We refer to, for example, [19], [9] for the basic theory of TROs. A *ternary morphism* on a TRO Z is a linear map T such that $T(xy^*z) = T(x)T(y)^*T(z)$ for all $x, y, z \in Z$. We write ZZ^* for the closure of the linear span of products zw^* with $z, w \in Z$, and similarly for Z^*Z. These are C^*-algebras. The *ternary envelope* of an operator space X is a pair $(\mathcal{T}(X), j)$ consisting of a TRO $\mathcal{T}(X)$ and a completely isometric linear map $j : X \to \mathcal{T}(X)$, such that $\mathcal{T}(X)$ is generated by $j(X)$ as a TRO (i.e., there is no closed subTRO containing $j(X)$), and which has the following property: given any completely isometric linear map i from X into a TRO Z which is generated by $i(X)$, there exists a (necessarily unique and surjective) ternary morphism $\theta : Z \to \mathcal{T}(X)$ such that $\theta \circ i = j$. If (X, u) is a unital operator space then its ternary envelope may be taken to be the C^*-*envelope* of, for example, [9, Section 4.3]; this is a C^*-algebra $C^*_e(X)$ with identity u. If X is an operator system then X is a selfadjoint unital subspace of $C^*_e(X)$.

An element u in an operator space X is called a *coisometry* (resp., *isometry*) *in* X, if X may be linearly completely isometrically embedded in a TRO Z such that $uu^* = 1_{ZZ^*}$ (resp., $u^*u = 1_{Z^*Z}$). In this case Z may be taken to be the ternary envelope of X, or it may be taken to be $B(K, H)$ and the 1 in the last line replaced by the identity operator on the Hilbert space H (resp., K). Coisometries and isometries in X were characterized purely linear-metrically in [14] (see also Theorem 3.4 below). Also, u is a unitary in X iff it is both a coisometry and an isometry in X (see [14, Lemma 2.3]).

2. Some matrix norm formulae

We collect several known formulae for matrix norms that we use later in the paper.

Lemma 2.1. *Let A be a C^*-algebra (or operator space). We have*

$$\left\| \begin{bmatrix} a & b \\ b & a \end{bmatrix} \right\| = \max\{ \|a + b\|, \|a - b\| \}, \quad a, b \in A. \tag{2.1}$$

Proof. This is well known: the map taking the 2×2 matrix above to $(a + b, a - b)$, is a faithful $*$-homomorphism, from the C^*-algebra of such matrices into $A \oplus^\infty A$. □

LEMMA 2.2. *Let A be a C^*-algebra (or operator space). We have*

$$\left\| \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \right\| = \max\{\|a+ib\|, \|a-ib\|\}, \quad a,b \in A. \tag{2.2}$$

Proof. To see this, apply (2.1) with b replaced by ib, then multiply, first, the second row by $-i$, and second, the second column by i. ☐

Let X be an operator space, and $v \in X$. If $n \in \mathbb{N}$ and $x \in M_n(X)$, we write

$$t_x^v = \begin{bmatrix} v_n & x \\ 0 & v_n \end{bmatrix}.$$

If (X, v) is a unital operator space, and we identify $v = 1$, then we write t_x^v as t_x.

LEMMA 2.3. *If X is a unital operator space, then*

$$\|t_x\|^2 = \frac{1}{2}\left[2 + \|x\|^2 + \|x\|\sqrt{\|x\|^2+4}\right] \geq 1 + \|x\|, \quad n \in \mathbb{N}, x \in M_n(X). \tag{2.3}$$

Thus $\|t_x\| \geq \sqrt{1+\|x\|}$.

Proof. This is, for example, a consequence of the more general formula (2.1) in [11]. ☐

We will see in Theorem 3.1 that (the matricial version of) this condition characterizes unital operator spaces.

Let X be an operator space possessing a conjugate linear involution $* : X \to X$, and suppose that $v \in X$. If $n \in \mathbb{N}$ and $x = [x_{ij}] \in M_n(X)$ define $x^* = [x_{ji}^*]$, and

$$s_x^v = \begin{bmatrix} v_n & x \\ x^* & v_n \end{bmatrix}, \qquad r_x^v = \begin{bmatrix} v_n & x \\ -x^* & v_n \end{bmatrix}.$$

If (X, v) is a unital operator space, and we identify $v = 1$, then we write s_x^v as s_x and r_x^v as r_x.

LEMMA 2.4. *If X is an operator system, then*

$$\|s_x\| = 1 + \|x\|, \quad n \in \mathbb{N}, x \in M_n(X) \tag{2.4}$$

and

$$\|r_x\| = \sqrt{1 + \|x\|^2}, \quad n \in \mathbb{N}, x \in M_n(X). \tag{2.5}$$

Proof. The first is well known (and an easy exercise). The second is from [14], and is a simple application of the C^*-identity. ☐

In Proposition 4.2, we show that formula (2.5) characterizes operator systems.

3. New metric-linear characterizations of unital operator spaces

We now show that the inequality in Lemma 2.3 characterizes unital operator spaces. Before we prove this, note that $\|t_x^v\| \geq \sqrt{1+\|x\|}$ for all $x \in X$ iff

$$\left\| \begin{bmatrix} \lambda v_n & x \\ 0 & \lambda v_n \end{bmatrix} \right\| \geq \sqrt{|\lambda|^2 + |\lambda| \|x\|}$$

for all $\lambda \in \mathbb{C}$ and $x \in M_n(X)$. In fact it is enough to take $\lambda > 0$ here, and $\|x\| = 1$.

THEOREM 3.1. *If $v \in \mathrm{Ball}(X)$ then (X, v) is a unital operator space iff $\|t_x^v\| \geq \sqrt{1+\|x\|}$ for all $x \in M_n(X)$ of small norm, and all $n \in \mathbb{N}$.*

Proof. The one direction is Lemma 2.3. If the norm condition holds, then for all $x \in \mathrm{Ball}(M_n(X))$ (of small norm)

$$1 + \|x\| \leq \|t_x\|^2 = \|t_x^* t_x\| \leq 1 + \|x\|^2 + \|v^* x\|,$$

where we are writing v_n as v and t_x^v as t_x for brevity. Write $x = cy$ where $c > 0$ and $\|y\| = 1$, then $c \leq c^2 + c\|v^* y\|$, so that $1 \leq c + \|v^* y\|$. Hence, $1 \leq \|v^* y\|$ if $\|y\| = 1$ (letting $c \searrow 0$). This implies that $\|v^* x\| = \|x\|$ for all $x \in M_n(X)$. Similarly, by using $t_x t_x^*$ in the calculation above, we have $\|x v^*\| = \|x\|$. Now by the proof of Theorem 2.4 in [14], we see that v is a unitary in X. □

REMARK. It is not enough that $\|t_x^v\| \geq \sqrt{2}$ if $\|x\|_n = 1$; this does not characterize unital operator spaces. To see this take $X = H^c$, Hilbert column space.

THEOREM 3.2. *If X is an operator space and $v \in \mathrm{Ball}(X)$ then (X, v) is a unital operator space iff $\max\{\|v_n + i^k x\| : k = 0,1,2,3\} \geq \sqrt{1+\|x\|}$ for all $x \in M_n(X)$ of small norm, and all $n \in \mathbb{N}$.*

Proof. ($\Leftarrow$) Apply Theorem 3.1, replacing x two lines above by the 2×2 matrix with x in the 1–2 corner and zeroes elsewhere, and v_n there by v_{2n}, noting that $\|t_{i^k x}^v\| = \|t_x^v\|$.

($\Rightarrow$) Write v as 1 and $2v$ as 2. By (2.1) and (2.3), we have

$$\left\| \begin{bmatrix} 1 & x & 1 & x \\ x & 1 & -x & 1 \\ 1 & x & 1 & x \\ -x & 1 & x & 1 \end{bmatrix} \right\| = \max \left\{ \left\| \begin{bmatrix} 2 & 2x \\ 0 & 2 \end{bmatrix} \right\|, \left\| \begin{bmatrix} 0 & 0 \\ 2x & 0 \end{bmatrix} \right\| \right\} \geq 2\sqrt{1+\|x\|}.$$

However the norm of the big matrix here is also

$$\leq \left\| \begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix} \right\| + \left\| \begin{bmatrix} 1 & x \\ -x & 1 \end{bmatrix} \right\| \leq 2 \max\{\|v_n + i^k x\| : k = 0,1,2,3\},$$

by (2.1) and (2.2). □

REMARK. One may replace i^k in the theorem by the set of unimodular complex scalars (the proof is unchanged). There is also an equivalent rewriting of the 'max' condition in the last theorem in terms of the cone $\mathfrak{F}_X$ highlighted in [16]. If $z \in \mathfrak{F}_{M_n(X)}$, let $c_k(z) = v_n + i^k(v_n - z) \in \mathfrak{F}_{M_n(X)}, k = 0,1,2,3$. The condition then becomes: $\max\{\|c_k(z)\| : k = 0,1,2,3\} \geq \sqrt{1 + \|v_n - z\|}$, for all $z \in \mathfrak{F}_{M_n(X)}$.

We give some illustrations of the use of this criterion in practice.

EXAMPLE 1. To see immediately that c_0 is not a unital operator space: For any norm 1 element $\vec{x} \in c_0$, if $|x_n|$ is small enough then clearly $\|\vec{x} + i^k \vec{e}_n\|$ does not dominate $\sqrt{2}$ for any k (here $(\vec{e}_n)$ is the standard basis). So by Theorem 1.1, c_0 is not a unital operator space (with any operator space structure $\{\| \cdot \|_n\}_{n \geq 2}$).

EXAMPLE 2. A convincing and more nontrivial example is S^1_2, namely M_2 with the trace norm. This example is also interesting because its 'commutative variant' ℓ^1_2 is well known to be a unital operator space, as indeed also is ℓ^1 and more generally various 'Fourier algebras' $B(G)$ and their noncommutative variants (see [12, Section 3] and its methods). Suppose $a \in M_2$ with $\mathrm{trace}(|a|) = 1$. Then there exist unitaries u, v with $a = udv$, where d is a diagonal matrix with non-negative entries α, β with $\alpha + \beta = 1$. Let $x = tue_{21}v$, where e_{21} is the usual matrix unit, and t is a scalar. Then $\|a + i^k x\|_1 = \|d + i^k te_{21}\|_1$. Let $b = d + i^k te_{21}$, and by way of contradiction, assume $\mathrm{trace}(|b|) \geq \sqrt{1+t}$ for some k and some small $t > 0$. Let $c = b^*b$, and suppose that the eigenvalues of c are r, s. Then $\mathrm{trace}(c) = r + s$, and $\mathrm{trace}(|b|) = \sqrt{r} + \sqrt{s}$. Since $\mathrm{trace}(|b|) \geq \sqrt{1+t}$ we have $r + s + 2\sqrt{rs} \geq 1 + t$. However, since c has rows $(\alpha^2 + t^2, \overline{i^k} t\beta)$ and $(i^k t\beta, \beta^2)$, we see that $\mathrm{trace}(c) = r + s = \alpha^2 + \beta^2 + t^2$, and $\det(c) = rs = (\alpha\beta)^2$. Thus,

$$1 + t \leq \alpha^2 + \beta^2 + t^2 + 2\alpha\beta = (\alpha + \beta)^2 + t^2 = 1 + t^2,$$

a contradiction if $0 < t < 1$. By Theorem 3.2, a is not unitary, and so S^1_2 is not a unital operator space (with any operator space structure $\{\| \cdot \|_n\}_{n \geq 2}$). This example also illustrates one great advantage of this characterization over other ones: the criterion involves a linear combination of u and x rather than a matrix with these as entries. Indeed, we only needed 1×1 matrices in the computation above.

EXAMPLE 3. By the same argument, the 3 dimensional subspace L^1_2 of lower triangular matrices in S^1_2, and its two dimensional subspace with the diagonal scalar repeated, are not unital operator spaces. Note that L^1_2 and S^1_2, contain a two dimensional unital operator space, namely the diagonal, which is a copy of ℓ^1_2 (a unital operator space as we said above). This gives a glimpse of the delicacy of the process of adding an element or two to an operator space and trying to keep the space unital.

PROPOSITION 3.3. *Let X be an operator space with element $e \in \text{Ball}(X)$. Inside $M_2(X)$ consider the set $\mathcal{U}_e(X)$ of matrices $\begin{pmatrix} \lambda e & x \\ 0 & \lambda e \end{pmatrix}$ for $x \in X$, and λ scalar. Then (X, e) is a unital operator space iff $(\mathcal{U}_e(X), e \otimes I_2)$ is a unital operator space, and iff $(\mathcal{U}_e(X), e \otimes I_2)$ is a unital operator algebra with the canonical product.*

Proof. Suppose that $(\mathcal{U}_e(X), e \otimes I_2)$ is a unital operator space. If $x \in X, \|x\| = 1$, then

$$\sqrt{2} = \left\| \begin{bmatrix} e & 0 \\ 0 & e \\ 0 & x \\ 0 & 0 \end{bmatrix} \right\| = \left\| \begin{bmatrix} e \\ x \end{bmatrix} \right\|.$$

This is similarly true for matrices, and so by the main theorem in [14], e is an isometry. Similarly e is a coisometry, so (X, e) is a unital operator space. The rest is obvious. $\square$

Another characterization of unital operator spaces that is not linear-metric in the sense of our paper can be found in [20].

As pointed out in [14], any theorem characterizing unital operator spaces 'linear-metrically,' is also a characterization of unitaries in X, that is, of elements of X that are a unitary in some TRO containing X. We give a slight refinement of the main result in [14], which we will need later.

THEOREM 3.4. *Let X be an operator space, and fix $m, n \in \mathbb{N}$. An element $u \in M_{mn}(X)$ is a coisometry (resp., an isometry) in $M_{mn}(X)$ (in the sense defined at the end of the introduction) iff $\|[u_k \;\; x]\|^2 = 1 + \|x\|^2$ (resp., $\|[u_k \;\; x]^t\|^2 = 1 + \|x\|^2$) for all $k \in \mathbb{N}$ and $x \in M_{km}(X)$ (resp., $x \in M_{kn}(X)$). Indeed, it suffices to consider norm one matrices x here.*

Proof. We just sketch this, since it is similar to the proof of the main theorem in [14], which the reader might follow along with. It also uses facts about the ternary envelope of $M_{mn}(X)$ from, for example, [19] or [4, Appendix A.13(ii)], such as if Z is the ternary envelope of X, then $M_{mn}(Z)$ is the ternary envelope of $M_{mn}(X)$. We just prove the coisometry case, the other is similar. Let $c = (uu^*)^{\frac{1}{2}} \in M_m(Z^*Z)$. Then $\|cx\| = \|u^*x\| = 1$ if $x \in M_m(X)$ with $\|x\| = 1$, as in the proof of Theorem 2.4 in [14], which we are following. As in that proof, left multiplication by c on $M_m(Z)$ is an isometry, since it restricts to an isometry on $M_m(X)$. By [14, Theorem 2.1], c is a coisometry in $M_m(Z^*Z)$. Since $c \geq 0$ and $uu^* \geq 0$, by the unicity of positive square roots we must have $c = I$, and $uu^* = I$. Thus, u is a coisometry. $\square$

We next relate unital operator spaces to our previous paper [15], by characterizing compact projections in a C^*-algebra in terms of unital operator spaces. We first mention some background facts from [23]. Let p be an open projection in the sense of Akemann [1], [2] or [7], in the bidual of an approximately unital operator algebra A, and let $q = 1 - p$. We recall that q is compact iff $q = aq$ for some $a \in \mathrm{Ball}(A)$ (in fact it is enough that $a \in A$). We write $A_p = \{a \in A : a = ap\}$ and ${}_pA = \{a \in A : a = pa\}$. It is easy to see that the bidual of $X = A/({}_pA + A_p)$ is the unital operator algebra $qA^{**}q$. Indeed consider the complete quotient map $x \mapsto qxq$ from A^{**} onto $qA^{**}q$. Its kernel is easily seen to be $pA^{**} + A^{**}p$. In particular, the latter space is weak* closed. Thus $A^{**}/(pA^{**} + A^{**}p) \cong qA^{**}q$ completely isometrically. Next, note that the weak* closure of ${}_pA + A_p$ equals $pA^{**} + A^{**}p$ (using the fact that the latter space is weak* closed). Thus, we have

$$\big(A/({}_pA + A_p)\big)^{**} \cong A^{**}/\big(pA^{**} + A^{**}p\big) \cong qA^{**}q$$

completely isometrically.

PROPOSITION 3.5. *Suppose that B is a C^*-algebra, and that $q = 1 - p$ is a closed projection in B^{**}. Then q is compact if and only if $X = B/({}_pB + B_p)$ is a unital operator space (i.e., iff it possesses a unitary in X in the sense of the introduction).*

Proof. Let $i : X \to qB^{**}q$ be the canonical complete isometry induced by the canonical map from X into its bidual, and the identification in the last centered line above the proposition. Explicitly, $i([a]) = qaq$ for $a \in A$.

For one direction of the result, if q is compact, so that $q = aq$ for an $a \in \mathrm{Ball}(A)$, let $e = [a] = a + ({}_pA + A_p) \in X$, and note that $i(e)$ is the identity $qaq = q$ of $qA^{**}q$. So X is a unital operator space.

For the other direction, by [7, Lemma 5.3], the TRO Z generated by $i(X)$ inside $qB^{**}q$ is a ternary envelope of X, so that $i(v)$ is a coisometry in Z by [14, Lemma 2.3], where v is the identity of X. Thus, $i(v)$ is a partial isometry in $qB^{**}q$. Also, $i(v)i(v)^*i(x) = i(x)$ since $i(x) \in Z$ and $i(v)$ is a coisometry in Z. By weak* density of $i(X)$ in $qB^{**}q$, we have $i(v)i(v)^* = q$. Next note that by a result of Kirchberg (see the remark after Corollary 1.3 of [23]), there exists $a \in \mathrm{Ball}(B)$ with $a + ({}_pB + B_p) = v$. Then $qaq = i(v)$ so that

$$i\big(\big[a^*a\big]\big) = qa^*aq = qa^*qaq + qa^*(1-q)aq = q + qa^*(1-q)aq.$$

Taking norms, $1 \geq 1 + \|qa^*(1-q)aq\|$, so that $qa^*(1-q)aq = 0$. Hence $i([a^*a]) = qa^*aq = q$, from which it is clear that q is compact. $\square$

REMARK. One may weaken the condition that X is unital, to that X possesses an isometry or coisometry in the sense of the introduction. The proof above still works. We also suspect that the result is also true for general operator algebras (using the compact projections of [15]).

4. Characterizations of operator systems and C^*-algebras

This section can be viewed as some remarks that naturally belong with the sections on operator systems in [14]. We begin with a characterization of the 'positive' part of a unital operator space:

LEMMA 4.1. *If A is a unital operator space or approximately unital operator algebra, then an element $x \in \text{Ball}(A)$ is in the positive cone of $A \cap A^*$ iff $\|1 - zx\| \le 1$ for all $z \in \mathbb{C}$ with $|1 - z| \le 1$.*

Proof. This follows from the argument for Lemma 8.5 in [16]. $\square$

The last result easily leads to a metric-linear characterization of operator systems: they are the unital operator spaces X spanned by the positive cone of $A \cap A^*$, the latter characterized in Lemma 4.1. We now give another metric-linear characterization of operator systems which should have been stated in [14]. We use the notation above Lemma 2.4.

PROPOSITION 4.2. *If X is an operator space possessing a conjugate linear involution $* : X \to X$, and an element $v \in \text{Ball}(X)$ with $v = v^*$, then there exists a $*$-linear complete isometry $T : X \to Y$ onto an operator system Y with $T(v) = 1$, iff*

$$\left\|r_x^{v_n}\right\| = \sqrt{1 + \|x\|^2}, \quad n \in \mathbb{N}, x \in M_n(X).$$

Proof. The one direction follows from (2.5). For the other, first note that we have for any $x \in X$ that $\|x^*\| \le \|r_x^v\| = \sqrt{1 + \|x\|^2}$. Replacing x by tx for a positive scalar t, we obtain $\|x^*\| \le \sqrt{\frac{1}{t^2} + \|x\|^2}$. Letting $t \to \infty$ shows that $*$ is contractive, hence isometric since $*$ has period 2. Similarly, $\|x^*\| = \|x\|$ if $x \in M_n(X)$, so that X is a selfadjoint operator space by the discussion a few paragraphs above Proposition 1.1 in [8].

If $\|x\| = 1$, and we write the first row of r_x^v as a, and the second of r_x^v as b, then the norm of each of these is $\le \sqrt{2}$. However,

$$4 \le \|r_x\|^2 = \left\|a^* a + b^* b\right\| \le \|a\|^2 + \|b\|^2 \le 4.$$

Thus $\|a\| = \sqrt{2}$. Similarly, the second column of r_x^v has norm $\sqrt{2}$. This works analogously at the matrix level, and we may now appeal to the main theorem in [14] to see that (X, v) is a unital operator space. Finally, we appeal to 3 (c) of [14, Remark 3.5] to see that (X, v) is an operator system. $\square$

REMARKS. (1) The following discussion rules out a possible simplification of the last characterization of operator systems. In [14], we proved that if X is a 'selfadjoint function space' with a selfadjoint 'unitary' u, then (X, u) is a 'function system' (the 'commutative' variant of an operator system). The analogous thing for operator systems is not true. Indeed consider the selfadjoint operator space $X = \{[x_{ij}] \in M_2 : x_{11} = 0, x_{12} = x_{21}\}$. It is easy to see that X generates M_2, so that M_2 is the ternary envelope of X. However if $u = E_{21} + E_{12}$ then $uX^*u = uXu$ is not contained in X. It follows from the discussion at the start of Section 4 in [14] that (X, u) is not an operator system. This example also shows that a selfadjoint operator space which is completely isometric to a unital operator algebra, need not be completely isometric to a C^*-algebra.

(2) One may ask if the equation (2.4) characterizes operator systems. That is, if u is a selfadjoint unitary in a selfadjoint subspace $X \subset B(H)$, then does the condition $\|s_x^u\| = 1 + \|x\|$ for all $n \in \mathbb{N}$ and $x \in M_n(X)$, force (X, u) to be an operator system? Indeed a variant of the proof of Proposition 4.2 shows that the equation $\|s_x^u\| = 1 + \|x\|$ above forces (X, u) to be a unital operator space. We leave this question to the interested reader, suspecting that it is not hard to find a counterexample, and that it is also not hard to find other simple conditions to add to (2.4) to yield a characterization of operator systems.

The following is a new metric-linear characterization of unital C^*-algebras among the operator systems, up to complete isometry. The metric-linear characterizations of C^*-algebras in [14] referred to unitaries being spanning, which is avoided here. The result is certainly not best possible, but the point again is that it is nice to know that formulae exist that essentially *only* refer to the norm.

THEOREM 4.3. *An operator system A has a product with respect to which it is a C^*-algebra (with the same operator space structure) if and only if for all $x, y \in A$, there exist elements $b, z \in A$ such that*

$$M_+ = \frac{\begin{bmatrix} y & 0 & 1 & x & b & z \\ x & b & z & y & 0 & 1 \end{bmatrix}}{\left\| \begin{bmatrix} y & 0 & 1 & x & b & z \\ x & b & z & y & 0 & 1 \end{bmatrix} \right\|}, \qquad M_- = \frac{\begin{bmatrix} y & 0 & 1 & x & b & z \\ x & b & z & -y & 0 & -1 \end{bmatrix}}{\left\| \begin{bmatrix} y & 0 & 1 & x & b & z \\ x & b & z & -y & 0 & -1 \end{bmatrix} \right\|}$$

satisfy $\|[M_+ \otimes I_m, w]\| = \sqrt{2}$ and $\|[M_- \otimes I_m, w]\| = \sqrt{2}$, for all $m \in \mathbb{N}$ and all contractions $w \in M_{2m}(A)$.

Proof. Suppose the condition involving M_+ and M_- holds for all $x, y \in A$ and let S denote the C^*-envelope of A. Abusing notation, consider A as canonically embedded in S. By Theorem 3.4 the above condition guarantees that M_+ and M_- are coisometries in the ternary envelope $M_{2,6}(S)$ of $M_{2,6}(A)$. This implies that $(xy^* + z) \pm (yx^* + z^*) = 0$ in S, so that $z = -xy^*$ lies in S. Hence A is a subalgebra of its C^*-envelope, and hence A coincides with its C^*-envelope.

Conversely, suppose A is linearly completely isometric to a C^*-algebra B via a map $\Psi : A \to B$. Then $v = \Psi(1)$ is a unitary in B in the ordinary sense (as follows from, e.g., [14, Theorem 2.1]. Also B with product xv^*y and involution vx^*v is a C^*-algebra with identity v which is unitally completely order isomorphic to A. Thus, A with its original identity has a product with respect to which it is a C^*-algebra. In this C^*-algebra, let $z = -xy^*$ and let $b = \sqrt{\|xx^* + yy^* + zz^*\| \cdot 1 - xx^* - yy^* - zz^*}$. It is now easy to check that M_+ and M_- are coisometries. The result then follows from Theorem 3.4. $\square$

REMARK. An operator system linearly completely isometric to a TRO or unital operator algebra has a product with respect to which it is a C^*-algebra (with the same operator system structure as the original one). Indeed, the last theorem is true if we replace 'C^*-algebra' in the statement with 'TRO,' or 'unital operator algebra,' or 'unital C^*-algebra' or 'unital C^*-algebra with 1 mapping to 1.' The proofs of these are usually the same as the proof of 4.3, except in the 'unital operator algebra' case where one should also use [14, Proposition 4.2].

It is not true however that an operator system which is linearly completely isometric to an operator algebra, needs to be completely isometric to a C^*-algebra. Thus, one cannot characterize C^*-algebras as operator systems with a general operator algebra product. For a counterexample, consider the operator system in [10, Proposition 2.1], which by that result and Sakai's theorem cannot be completely isometric to a C^*-algebra. However the multiplication $(x, y) \mapsto xky$ for a fixed contraction k in the image of the compact operators in X, makes X an operator algebra by Remark 2 on p. 194 of [17].

5. Characterizations of operator algebra products

In the last sections of our paper, we will consider characterizations of operator algebras. The first point to be made is that although we have not found one as yet, there *ought* to be a purely linear-metric characterization of unital operator algebras. Indeed, we know from the noncommutative Banach–Stone theorem that the identity in a unital operator algebra A determines the product (this is true even if the identity is one-sided [5, Corollary 5.3]). Moreover, if we have forgotten the product on a unital operator algebra A it can be recovered from the unital operator space structure by the methods of, for example, [6, Section 6]. These methods certainly yield a characterization of unital operator algebras using only the 'unital operator space data,' but they are not quite 'linear-metric' in our strict sense, since they refer to certain linear maps on A, for example. The second point is that it is still open as to whether there is a truly metric condition on a bilinear map $m : X \times X \to X$ on an operator space X characterizing when m is a (nonunital) operator algebra product. We offer in the remainder of the paper two partial contributions to these subjects. In most of the present section, we focus on this second point in the case that X posseses an isometry or coisometry (which need not be even a one-sided identity for the ensuing operator algebra product). Thus, we are giving variants and extensions of the characterization of operator algebras from [17]. In Section 6, we address the first point with a linear-metric characterization of operator algebras which does use elements of a containing C^*-algebra. In [25] a holomorphic characterization of operator algebras is given (generalizing the holomorphic characterization of C^*-algebras from [24]).

Lemma 5.1. *If an operator algebra A contains a left identity u of norm* 1, *then u is a coisometry in A in the sense of the introduction.*

Proof. This follows from, for example, [21] or the considerations involved in [5, Theorem 4.4], but we give a quick proof of it using the main theorem in [14] (or Theorem 3.4 above), as a nice application of that result. Note that u is a projection in any containing C^*-algebra, and so if $x = ux \in A$ has norm 1 then

$$\big\|[ux]\big\|^2 = \big\|uu^* + xx^*\big\| = \big\|u + uxx^*u\big\| = 1 + \|ux\|^2 = 1 + \|x\|^2 = 2.$$

Similarly for matrices, so that u is a coisometry in A by Theorem 3.4. □

For some of the characterizations of operator algebras below, we will use the quasimultiplier formulation of operator algebras [22]. For the readers convenience, we include a simple unpublished proof of this that was in a preliminary version of [9], and which was presented, for example, at the Banach Algebras 2007 conference. Here $I(X)$ is the injective envelope of X, which is a TRO containing the ternary envelope of X as a subTRO (see, e.g., [19] or [9, Section 4]).

THEOREM 5.2 (Kaneda–Paulsen). *Let X be an operator space. The algebra products on X for which there exists a completely isometric homomorphism from X onto an operator algebra, are in a bijective correspondence with the elements $z \in \mathrm{Ball}(I(X))$ such that $Xz^*X \subset X$. For such z the associated operator algebra product on X is xz^*y.*

Proof. The one direction, and the last statement, follows from Remark 2 on p. 194 of [17], viewing $I(X)$ as a TRO in $B(H)$, and $V = z^*$. For the other direction, if X is a subalgebra of $B(H)$ say, then by the theory of the injective envelope (see, e.g., [19] or [9, Section 4]) we can view $X \subset I(X) \subset B(H)$, and there exists a completely contractive projection P from $B(H)$ onto $I(X)$. Set $z = P(1)$. For $x, y \in X$ we have $xy = P(x1^*y) = P(xP(1)^*y)$, by Youngson's theorem [9, Theorem 4.4.9], the proof of which asserts that the last quantity is the ternary product xz^*y in $I(X) = \mathrm{Ran}(P)$. The bijectivity follows from, for example, [9, Proposition 4.4.12] and its 'right-hand version': if $Xz^*X = (0)$ then $Xz^* = (0) = Xz^*z$, so $z^*z = 0 = z$. □

If the operator space has more structure, then one can say more (see [21]).

COROLLARY 5.3. *Let (X, u) be a unital operator space. The algebra products on X for which there exists a completely isometric homomorphism from X onto an operator algebra, are in a bijective correspondence with the elements $w \in \mathrm{Ball}(X)$ such that $XwX \subset X$ (multiplication taken in the C^*-envelope $C^*_{\mathrm{e}}((X, u))$). For such w the associated operator algebra product on X is xwy.*

Proof. This follows immediately from Theorem 5.2, since in this setting $I(X)$ may be taken to be a C^*-algebra, containing $C^*_{\mathrm{e}}(X)$ as a C^*-subalgebra, with common identity u. If z is as in Theorem 5.2, then $w = z^* = uz^*u \in Xz^*X \subset X$. □

Remark. The elements w in the unital operator space X in the corollary constitute the unit ball of the operator algebra $D = \{a \in C^*_{\mathrm{e}}(X) : XaX \subset X\}$, which is a subalgebra of $C^*_{\mathrm{e}}(X)$. Thus, D could justly be called *the operator algebra of operator algebra products on X*. It would be quite desirable to find a linear-metric characterization of D as a subset of X.

We recall the spaces $\mathcal{M}_\ell(X)$ and $\mathcal{M}_r(X)$, of left and right multipliers of X, which were introduced in [4]. Such multipliers of X were 'metric-linearly' characterized by the first author, Effros and Zarikian (see [9, Theorem 4.5.2]). For example, if $T : X \to X$ is linear, then $T \in \mathrm{Ball}(\mathcal{M}_\ell(X))$ iff

$$\left\| \begin{bmatrix} T(a_{ij}) \\ b_{ij} \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} a_{ij} \\ b_{ij} \end{bmatrix} \right\|, \quad [a_{ij}], [b_{ij}] \in M_n(X), n \in \mathbb{N}.$$

Lemma 5.4. *If u is a coisometry in an operator space X, then the map $\theta : \mathcal{M}_\ell(X) \to X$ defined by $T \mapsto T(u)$, is a complete isometry from $\mathcal{M}_\ell(X)$ into X. Indeed if $T \in \mathcal{M}_\ell(X)$ then the $\mathcal{M}_\ell(X)$ norm of T equals $\|T\|_{\mathrm{cb}} = \|T\| = \|T(u)\|$. The range of θ is the set $X_\ell(u)$ defined to be $\{x \in X : xu^*X \subset X, x = xu^*u\}$, product taken in a ternary envelope Z of X, and $\mathcal{M}_\ell(X)$ may be identified with $\{xu^* \in ZZ^* : x \in X_\ell(u)\}$. If u is a unitary and Z is the C^*-envelope $C^*_{\mathrm{e}}(X)$ of (X, u), then $\mathcal{M}_\ell(X)$ is identified with $\{a \in X : aX \subset X\}$ (product taken in $C^*_{\mathrm{e}}(X)$).*

Proof. If u is a coisometry, so that $uu^* = 1_{ZZ^*}$, then by the theory of one-sided multipliers of operator spaces we may view $\mathcal{M}_\ell(X) \subset ZZ^*$ (see, e.g., the 5th and 6th last lines of p. 302 in [4]). Then θ is simply right multiplication by u. It follows that θ is a complete isometry, since it is a complete contraction with completely contractive left inverse $x \mapsto xu^*$. Since it is well known that the $\mathcal{M}_\ell(X)$ norm of T dominates $\|T\|_{\mathrm{cb}}$, the asserted norm equalities hold. Clearly $\mathcal{M}_\ell(X) = \{xu^* \in ZZ^* : x \in \mathrm{Ran}(\theta)\}$. If $T \in \mathcal{M}_\ell(X)$ then $Tuu^*X = TX \subset X$, so $\theta(T) \in X_\ell(u)$. Conversely, if $x \in X_\ell(u)$, then $xu^* \in \mathcal{M}_\ell(X)$ so that $x = xu^*u \in \mathrm{Ran}(\theta)$. Hence, $\mathrm{Ran}(\theta) = X_\ell(u)$. The rest is obvious. □

Remark. Write $Z_2(u)$ for the Pierce 2-space of u, this is a C^*-algebra in the natural 'Pierce' product (see, e.g., p. 230–231 in [13], or [18]). The set $X_\ell(u)$ above equals the set of elements $z \in Z_2(u)$ such that left multiplication in the Pierce product by z maps X into X (i.e., $zu^*X \subset X$). It is easy to see that $X_\ell(u)$ is a unital subalgebra of the C^*-algebra $Z_2(u)$ in the Pierce product.

Theorem 5.5. *Let u be a coisometry in an operator space X. Suppose that $m : X \times X \to X$ is a bilinear map such that $m(x, \cdot) \in \mathcal{M}_\ell(X)$ for all $x \in X$. We also suppose that $m(\cdot, u) \in \mathrm{Ball}(\mathcal{M}_r(X))$ (resp., $m(\cdot, u) \in \mathcal{M}_r(X)$). Then m (resp., m multiplied by some positive scalar) is an associative product such that X with this product is an operator algebra (i.e., there exists a completely isometric homomorphism from X onto an operator algebra). Conversely, every operator algebra product m on X satisfies all the conditions above.*

Proof. In the respective case, we can multiply m by a positive scalar to ensure that $m(\cdot, u) \in \mathrm{Ball}(\mathcal{M}_r(X))$. Let Z be the ternary envelope of X, and view $X \subset Z$ and $\mathcal{M}_\ell(X) \subset ZZ^*$ as in the proof of Lemma 5.4. Define $v(x) = m(x, \cdot) \in ZZ^*$, so that $m(x, y) = v(x)y$. Similarly, we can view $m(\cdot, u)$ as a contractive right multiplier of Z^*Z, hence as a contraction R in $(Z^*Z)^{**}$. Thus

$$m(x, y) = v(x)y = v(x)uu^*y = m(x, u)u^*y = x\bigl(Ru^*\bigr)y, \quad x, y \in X.$$

Now the result follows from Remark 2 on p. 194 of [17]. □

REMARKS. (1) In the previous theorem, the element u need not be related to any identity, or one-sided identity, for the ensuing operator algebra product.

(2) Theorem 5.5 answers the last question in [22] for operator spaces containing a coisometry or isometry, and in fact in this case gives a stronger result than the one discussed there.

(3) If u is a unitary in X, then the 'respectively' assertion of Theorem 5.5 is true with the positive scalar mentioned there equal to 1, if we also ask that m be contractive as a bilinear map. This follows from a slight modification of the proof, using the other-handed version of Lemma 5.4 (indeed, the multiplier norm of a right multiplier T of a unital operator space (X, u) equals $\|T(u)\|$).

COROLLARY 5.6. *Let u be a coisometry in an operator space X. Suppose that $m : X \times X \to X$ is a bilinear map such that $m(x, \cdot) \in \mathcal{M}_\ell(X)$ for all $x \in X$. We also suppose that $m(x, u) = x$ for all $x \in X$. Then m is an associative product, and X with this product is completely isometrically isomorphic to an operator algebra with a two-sided identity (namely, u). Conversely, every unital operator algebra satisfies all the conditions above.*

Proof. By the theorem, X with product m is an operator algebra. Since u is a right identity it is an isometry by the 'other-handed' variant of Lemma 5.1. By [14, Lemma 2.3], u is a unitary in A, hence is a unitary in the ternary envelope Z. Then Z is a C^*-algebra with product xu^*y. In the proof of the last theorem, $R = 1$, and $m(x, y) = xu^*y$, and now it is clear that u is a two-sided identity for m. □

Corollary 5.6 takes longer to state than the characterization of unital operator algebras from [17]. However the latter characterization is in terms of a product of *two large matrices*, whereas the condition that $m(x, \cdot) \in \mathcal{M}_\ell(X)$ in Corollary 5.6, is as discussed above Lemma 5.4, essentially the requirement that

$$\left\| \begin{bmatrix} m(x, a_{ij}) \\ b_{ij} \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} a_{ij} \\ b_{ij} \end{bmatrix} \right\|, \quad [a_{ij}], [b_{ij}] \in M_n(X),$$

for $n \in \mathbb{N}$ and $x \in \mathrm{Ball}(X)$. We emphasize that this uses a small (1×1) matrix (namely x) and one large $n \times (2n)$ matrix, and in particular uses at most one operation in each entry of the matrix, as opposed to the many operations (sums and products) that appear in the entries of a product of two large matrices. And, of course, it is worth repeating that the requirement that u be a coisometry in Corollary 5.6 (and the results before it), is equivalent to the metric condition $\|[u_n \;\; x]\| = \sqrt{2}$ for $n \in \mathbb{N}$ and every matrix $x \in M_n(X)$ of norm 1.

6. Metric characterizations of operator algebras referencing a containing C^*-algebra

In 3(c) of [14, Remark 3.5] the authors gave a metric-linear characterization, related to our norm formulae for r_x above, of the adjoint x^* of any operator x in an operator system X. (In particular, if x and z are contractive operators then $z = x^*$ if and only if for all $t \in \mathbb{R}$

$$\left\| \begin{bmatrix} t \cdot 1 & x \\ -z & t \cdot 1 \end{bmatrix} \right\| \leq \sqrt{1+t^2}.)$$

Knowing this, we may freely reference adjoints in the following metric characterization of operator algebras. The notation 2 in the matrices below (and in the next corollary) refers to 2 times the identity of B.

Theorem 6.1. *Suppose A is a subspace of a unital C^*-algebra B. Then A is closed under multiplication if and only if for each pair of elements $x \in A$ and $y \in A^*$ with $\|y\| \leq 1$, there exists an element $z \in A$ such that for all $b \in B$,*

$$\left\| \begin{bmatrix} 0 & y & 1 & 0 \\ 2 & x & z & b \end{bmatrix} \right\| = \|[2, x, z, b]\|.$$

Proof. If A is closed under multiplication, we may choose $z = -xy^*$. Multiplying the above matrix by its adjoint we see that the condition holds. Conversely, suppose the condition holds. Let $b = \sqrt{\|xx^* + zz^*\| \cdot 1 - xx^* - zz^*}$. Multiplying the above matrix by its adjoint we see that

$$\left\| \begin{bmatrix} yy^* + 1 & yx^* + z^* \\ xy^* + z & (4 + \|xx^* + zz^*\|) \cdot 1 \end{bmatrix} \right\| = \|[2 \cdot 1, x, z, b]\|^2 = 4 + \|xx^* + zz^*\|.$$

This implies that

$$\sqrt{\|xy^* + z\|^2 + (4 + \|xx^* + zz^*\|)^2} \leq 4 + \|xx^* + zz^*\|,$$

hence $\|xy^* + z\| = 0$ and $xy^* = -z \in A$. □

Clearly many other algebraic conditions A might satisfy can be characterized by a variant of this theorem. For example, a pair $x, y \in A$ satisfies $xy = 1$ if and only for all $b \in B$, the displayed condition is satisfied with $z = 1$. Orthogonality, commutivity, normality and any other algebraic condition may be similarly characterized. We emphasize the following corollary.

Corollary 6.2. *Suppose that A is a subspace of a unital C^*-algebra B.*

(1) *If $x \in A$, then $xA \subset A$ if and only if for all $y \in \mathrm{Ball}(A^*)$ there exists an element $z \in A$ such that for all $b \in B$,*

$$\left\| \begin{bmatrix} 0 & y & 1 & 0 \\ 2 & x & z & b \end{bmatrix} \right\| = \big\| [2, x, z, b] \big\|.$$

(2) *If $y \in \mathrm{Ball}(A)$, then $Ay \subset A$ if and only if for all $x \in A$ there exists an element $z \in A$ such that for all $b \in B$,*

$$\left\| \begin{bmatrix} 0 & y^* & 1 & 0 \\ 2 & x & z & b \end{bmatrix} \right\| = \big\| [2, x, z, b] \big\|.$$

(3) *If $x \in A$, then $AxA \subset A$ if and only if for all $y \in \mathrm{Ball}(A^*)$ there exists an element $z \in \{b \in B : Ab \subset A\}$ such that for all $b \in B$, the equality in* (1) *holds.*

Note that A is a unital operator space and B its C^*-envelope, for example, then the last result characterizes left, right, and quasi-multipliers of A (see the last assertion of Lemma 5.4 and Corollary 5.3).

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David P. Blecher, Department of Mathematics, University of Houston, Houston, TX 77204-3008, USA

E-mail address: dblecher@math.uh.edu

Matthew Neal, Department of Mathematics, Denison University, Granville, OH 43023, USA

E-mail address: nealm@denison.edu