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ON THE p-NORM OF THE BEREZIN TRANSFORM

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ABSTRACT. In this short note, the norm of Berezin transform, acting on $L^p(\mathbb{B}_n)$, is determined to be

$$\left\| B_{\mathbb{B}_n} : L^p(\mathbb{B}_n) \to L^p(\mathbb{B}_n) \right\| = \frac{1}{p} \prod_{k=1}^{n} \left(1 + \frac{1}{kp}\right) \frac{\pi}{\sin(\pi/p)}.$$

This extends a result of Dostanić (*J. Anal. Math.* **104** (2008) 13–23) to several complex variables.

1. Introduction

Let $\mathbb{D}$ be the open unit disc in the complex plane. By $L^p(\mathbb{D})$ we mean the Lebesgue space with respect to the normalized Lebesgue measure $dA = (1/\pi)dx\,dy$ on $\mathbb{D}$. The Bergman space $L^p_a(\mathbb{D})$ is the closed subspace of $L^p(\mathbb{D})$ consisting of holomorphic functions on $\mathbb{D}$. For $z \in \mathbb{D}$, the Bergman reproducing kernel is the function $K_z \in L^2_a(\mathbb{D})$ such that $f(z) = \langle f, K_z\rangle$ for every $f \in L^2_a(\mathbb{D})$, where $\langle \cdot, \cdot\rangle$ denote the inner product in $L^2(\mathbb{D})$. Explicitly,

$$K_z(w) = \frac{1}{(1-\bar{z}w)^2}, \quad w \in \mathbb{D}. \tag{1.1}$$

The normalized reproducing kernel k_z is defined by $k_z = K_z/\|K_z\|_2$.

For $f \in L^1(\mathbb{D})$ define

$$Bf(z) = \langle f k_z, k_z\rangle = \left(1-|z|^2\right)^2 \int_{\mathbb{D}} \frac{f(w)}{|1-z\bar{w}|^4}\, dA(w), \quad z \in \mathbb{D}. \tag{1.2}$$

The function Bf is called the Berezin transform of f. This transform was first introduced by F. A. Berezin [4] in the context of quantization of Kähler

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manifolds. It later turned out that the Berezin transform plays an important role in the theory of Toeplitz operators on the Bergman space. See [1], [2], [10], [13] for details.

It has long been a well-known fact that the Berezin transform B is bounded on $L^p(\mathbb{D})$ if and only if $p>1$ ([10, Proposition 2.2]), but only recently has its p-norm been calculated. In [7], Dostanić showed that

THEOREM A. *For* $1<p\le\infty$,

$$\big\|B: L^p(\mathbb{D})\to L^p(\mathbb{D})\big\| = \frac{1}{p}\left(1+\frac{1}{p}\right)\frac{\pi}{\sin(\pi/p)}. \tag{1.3}$$

When $p=\infty$, *the quantity on the right-hand side of* (1.3) *should be interpreted as* 1.

The purpose of this note is to extend the above result to the several complex variables setting.

Throughout we denote by $\mathbb{B}_n$ the open unit ball in $\mathbb{C}^n$. Let ν be the Lebesgue measure on $\mathbb{C}^n$, normalized so that $\nu(\mathbb{B}_n)=1$. For $f\in L^1(\mathbb{B}_n,\nu)$, the Berezin transform of f is defined by

$$B_{\mathbb{B}_n}f(z)=\big(1-|z|^2\big)^{n+1}\int_{\mathbb{B}_n}\frac{f(w)}{|1-\langle z,w\rangle|^{2(n+1)}}\,d\nu(w),\quad z\in\mathbb{B}_n. \tag{1.4}$$

See [17, p. 76], [3, p. 383] and [16] for more information on this transform.

Our main result is the following theorem.

THEOREM 1.1. *For* $1<p\le\infty$, *we have*

$$\big\|B_{\mathbb{B}_n}: L^p(\mathbb{B}_n)\to L^p(\mathbb{B}_n)\big\| = \frac{1}{p}\prod_{k=1}^{n}\left(1+\frac{1}{kp}\right)\frac{\pi}{\sin(\pi/p)}. \tag{1.5}$$

Again, when $p=\infty$, *the quantity on the right-hand side of* (1.5) *should be interpreted as* 1.

It is obvious that when $n=1$, we recover Theorem A.

We will in fact deal with a family of integral operators as follows. For $\alpha>-1$, we define

$$S_\alpha f(z):=\int_{\mathbb{B}_n}\frac{(1-|w|^2)^\alpha}{|1-\langle z,w\rangle|^{n+1+\alpha}}f(w)\,d\nu(w)$$

for $z\in\mathbb{B}_n$ and $f\in L^1(\mathbb{B}_n,\nu)$. Note that the Berezin transform $B_{\mathbb{B}_n}=S^*_{n+1}$, the adjoint of S_{n+1}. These operators first appeared in [9] in connection with projections of Bergman type defined by

$$T_\alpha f(z)=\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}\int_{\mathbb{B}_n}\frac{(1-|w|^2)^\alpha}{(1-\langle z,w\rangle)^{n+1+\alpha}}f(w)\,d\nu(w).$$

It was shown in [9] that, if $\alpha>-1$, $1\le p<\infty$ and $p(\alpha+1)>1$, then S_α is a bounded linear operator on $L^p(\mathbb{B}_n)$. This in turn implies the L^p-boundedness of T_α. Moreover, in [9], Forelli and Rudin in fact proved (implicitly) that

$$\big\|S_\alpha : L^p(\mathbb{B}_n) \to L^p(\mathbb{B}_n)\big\| \le \frac{n!\Gamma(1/p)\Gamma(\alpha+1-1/p)}{\Gamma^2((n+1+\alpha)/2)}. \tag{1.6}$$

In this note, we show that in fact equality holds in (1.6).

THEOREM 1.2. *Suppose that $\alpha > -1$, $1 \le p < \infty$ and $p(\alpha+1) > 1$. Then we have*

$$\big\|S_\alpha : L^p(\mathbb{B}_n) \to L^p(\mathbb{B}_n)\big\| = \frac{n!\Gamma(1/p)\Gamma(\alpha+1-1/p)}{\Gamma^2((n+1+\alpha)/2)}. \tag{1.7}$$

In particular, when $n = 1$, we recover Theorem 1 in [7].

We mention other related works. In [6], Dostanić gave two-sided estimates of the norm of Cauchy transform on L^p spaces on bounded simply-connected domains in the complex plane. There is also a nice paper of similar nature by K. Zhu [18], where an asymptotic formula for the norm of the Bergman projection on L^p spaces of the unit ball is given. Also, although not directly related to our results, the determination of the exact L^p norm of singular integral operators has been studied extensively. Results of this type include Pichorides' determination of the p-norm of the Hilbert transform ([14]) and Iwaniec and Martin's work on the Riesz transform ([12]). Also, an outstanding open problem of the past three decades, known as the Iwaniec conjecture, is the computation of the p-norm of the Beurling—Ahlfors transform ([11]). For the present best known estimates on the L^p-norm of the Beurling–Ahlfors transform, see [5] and references therein.

2. Preliminaries

A number of hypergeometric functions will appear throughout. We use the classical notation ${}_2F_1(\alpha,\beta;\gamma;z)$ to denote

$$ {}_2F_1(\alpha,\beta;\gamma;z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{z^k}{k!} \tag{2.1}$$

with $\gamma \ne 0, -1, -2, \dots$, where

$$(\alpha)_0 = 1, \qquad (\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1) \quad \text{for } k \ge 1.$$

We list a few formulas for easy reference (see [8, Chapter II]):

$$ {}_2F_1(\alpha,\beta;\gamma;1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}, \quad \mathrm{Re}(\gamma-\alpha-\beta) > 0. \tag{2.2}$$

$$ {}_2F_1(\alpha,\beta;\gamma;z) = (1-z)^{\gamma-\alpha-\beta}{}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma;z). \tag{2.3}$$

$$ {}_2F_1(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\lambda)\Gamma(\gamma-\lambda)}\int_0^1 t^{\lambda-1}(1-t)^{\gamma-\lambda-1}{}_2F_1(\alpha,\beta;\lambda;tz)\,dt, \tag{2.4}$$

$$\mathrm{Re}\,\gamma > \mathrm{Re}\,\lambda > 0; |\arg(1-z)| < \pi; z \ne 1.$$

LEMMA 2.1. *Suppose* $\operatorname{Re}\delta > 0$ *and* $\operatorname{Re}(\lambda+\delta-\alpha-\beta) > 0$. *Then*

$$\int_0^1 t^{\lambda-1}(1-t)^{\delta-1}{}_2F_1(\alpha,\beta;\lambda;t)\,dt = \frac{\Gamma(\lambda)\Gamma(\delta)\Gamma(\lambda+\delta-\alpha-\beta)}{\Gamma(\lambda+\delta-\alpha)\Gamma(\lambda+\delta-\beta)}. \tag{2.5}$$

Proof. Note that, under the assumption of the lemma, both sides of (2.4) are continuous at $z=1$. The lemma then follows by letting $z \to 1$ in (2.4) and applying (2.2). □

The following lemma is contained implicitly in the proof of Theorem 1.4.10 in [15] (see the formula in page 19, line 5 of [15]):

LEMMA 2.2. *For* $\alpha \in \mathbb{R}$ *and* $\gamma > -1$, *we have*

$$\int_{\mathbb{B}_n} \frac{(1-|w|^2)^\gamma}{|1-\langle z,w\rangle|^{2\alpha}}\,dV(w) = \frac{\Gamma(n+1)\Gamma(1+\gamma)}{\Gamma(n+1+\gamma)}\,{}_2F_1\big(\alpha,\alpha;n+1+\gamma;|z|^2\big).$$

The following result, usually called Schur's test, is a very effective tool in proving the L^p-boundedness of integral operators. See, for example, [19].

LEMMA 2.3. *Suppose that* (X,μ) *is a* σ*-finite measure space and* $K(x,y)$ *is a nonnegative measurable function on* $X\times X$ *and* T *the associated integral operator*

$$Tf(x) = \int_X K(x,y)f(y)\,d\mu(y).$$

Let $1<p<\infty$ *and* $1/p+1/q=1$. *If there exist a positive constant* C *and a positive measurable function* u *on* X *such that*

$$\int_X K(x,y)u(y)^q\,d\mu(y) \le Cu(x)^q$$

for almost every x *in* X *and*

$$\int_X K(x,y)u(x)^p\,d\mu(x) \le Cu(y)^p$$

for almost every y *in* X, *then* T *is bounded on* $L^p(X,\mu)$ *with* $\|T\| \le C$.

3. The proofs

Proof of Theorem 1.2. We first deal with the case $p=1$. Note that in this case the assumption $p(\alpha+1)>1$ implies $\alpha>0$.

It is clear that

$$\big\|S_\alpha : L^1(\mathbb{B}_n) \to L^1(\mathbb{B}_n)\big\| \le \sup_{w\in\mathbb{B}_n}\big(1-|w|^2\big)^\alpha \int_{\mathbb{B}_n} \frac{d\nu(z)}{|1-\langle z,w\rangle|^{n+1+\alpha}}.$$

By Lemma 2.2 and (2.3), we find that

$$\begin{aligned}\int_{\mathbb{B}_n} \frac{d\nu(z)}{|1-\langle z,w\rangle|^{n+1+\alpha}} &= {}_2F_1\left(\frac{n+1+\alpha}{2},\frac{n+1+\alpha}{2};n+1;|w|^2\right)\\ &= \big(1-|w|^2\big)^{-\alpha}{}_2F_1\left(\frac{n+1-\alpha}{2},\frac{n+1-\alpha}{2};n+1;|w|^2\right).\end{aligned}$$

Note that the last hypergeometric function is increasing on the interval $[0,1)$, since its Taylor coefficients are all positive. Hence,

$$\begin{aligned}\big\|S_\alpha : L^1(\mathbb{B}_n)\to L^1(\mathbb{B}_n)\big\| &\le {}_2F_1\left(\frac{n+1-\alpha}{2},\frac{n+1-\alpha}{2};n+1;1\right)\\ &= \frac{\Gamma(n+1)\Gamma(\alpha)}{\Gamma^2((n+1+\alpha)/2)},\end{aligned}$$

where the last equality follows from (2.2).

To prove the reverse inequality, consider, for fixed $\varepsilon>0$, the function

$$f_\varepsilon(z)=\frac{\Gamma(n+\varepsilon)}{\Gamma(n+1)\Gamma(\varepsilon)}\big(1-|z|^2\big)^{\varepsilon-1}.$$

It is easy to check that $\|f_\varepsilon\|_1=1$. Again by Lemma 2.2,

$$\begin{aligned}S_\alpha f_\varepsilon(z) &= \frac{\Gamma(n+\varepsilon)}{\Gamma(n+1)\Gamma(\varepsilon)}\int_{\mathbb{B}_n}\frac{(1-|w|^2)^{\alpha+\varepsilon-1}}{|1-\langle z,w\rangle|^{n+1+\alpha}}\,d\nu(w)\\ &= C(\varepsilon)\,{}_2F_1\left(\frac{n+1+\alpha}{2},\frac{n+1+\alpha}{2};n+\alpha+\varepsilon;|z|^2\right),\end{aligned}$$

where

$$C(\varepsilon):=\frac{\Gamma(n+\varepsilon)\Gamma(\alpha+\varepsilon)}{\Gamma(\varepsilon)\Gamma(n+\alpha+\varepsilon)}.$$

It follows that

$$\begin{aligned}\|S_\alpha f_\varepsilon\|_1 &= C(\varepsilon)\int_{\mathbb{B}_n}{}_2F_1\left(\frac{n+1+\alpha}{2},\frac{n+1+\alpha}{2};n+\alpha+\varepsilon;|z|^2\right)d\nu(z)\\ &= C(\varepsilon)\int_0^1 nr^{n-1}{}_2F_1\left(\frac{n+1+\alpha}{2},\frac{n+1+\alpha}{2};n+\alpha+\varepsilon;r\right)dr\\ &\ge nC(\varepsilon)\int_0^1 r^{n+\alpha+\varepsilon-1}{}_2F_1\left(\frac{n+1+\alpha}{2},\frac{n+1+\alpha}{2};n+\alpha+\varepsilon;r\right)dr.\end{aligned}$$

Hence, an application of Lemma 2.1 yields

$$\big\|S_\alpha : L^1(\mathbb{B}_n)\to L^1(\mathbb{B}_n)\big\|\ge\|S_\alpha f_\varepsilon\|_1\ge\frac{n\Gamma(n+\varepsilon)\Gamma(\alpha+\varepsilon)}{\Gamma^2((n+1+\alpha)/2+\varepsilon)}.$$

Finally, by letting $\varepsilon\to 0^+$, we obtain

$$\big\|S_\alpha : L^1(\mathbb{B}_n)\to L^1(\mathbb{B}_n)\big\|\ge\frac{\Gamma(n+1)\Gamma(\alpha)}{\Gamma^2((n+1+\alpha)/2)}.$$

Now, assume that $1<p<\infty$ and $p(\alpha+1)>1$. For the upper bound of $\|S_\alpha : L^p(\mathbb{B}_n)\to L^p(\mathbb{B}_n)\|$, we appeal to Schur's test (Lemma 2.3). Set

$$u(z)=\big(1-|z|^2\big)^{-1/(pq)},$$

where q is the conjugate exponent of p. It then suffices to show

$$\int_{\mathbb{B}_n} \frac{(1-|w|^2)^\alpha}{|1-\langle z,w\rangle|^{n+1+\alpha}} u(w)^q \, d\nu(w) \le \frac{n!\Gamma(1/p)\Gamma(\alpha+1-1/p)}{\Gamma^2((n+1+\alpha)/2)} u(z)^q \tag{3.1}$$

for all $z \in \mathbb{B}_n$ and

$$\int_{\mathbb{B}_n} \frac{(1-|w|^2)^\alpha}{|1-\langle z,w\rangle|^{n+1+\alpha}} u(z)^p \, d\nu(z) \le \frac{n!\Gamma(1/p)\Gamma(\alpha+1-1/p)}{\Gamma^2((n+1+\alpha)/2)} u(w)^p \tag{3.2}$$

for all $w \in \mathbb{B}_n$. We only prove the first inequality, the other one follows the same lines. By Lemma 2.2 and (2.3), we have

$$\begin{aligned}
&\int_{\mathbb{B}_n} \frac{(1-|w|^2)^\alpha}{|1-\langle z,w\rangle|^{n+1+\alpha}} u(w)^q \, d\nu(w) \\
&\quad = \frac{\Gamma(n+1)\Gamma(\alpha+1-1/p)}{\Gamma(n+\alpha+1-1/p)} \\
&\qquad \times {}_2F_1\left(\frac{n+1+\alpha}{2}, \frac{n+1+\alpha}{2}; n+1+\alpha-\frac{1}{p}; |z|^2\right) \\
&\quad = \frac{\Gamma(n+1)\Gamma(\alpha+1-1/p)}{\Gamma(n+\alpha+1-1/p)} \big(1-|z|^2\big)^{-1/p} \\
&\qquad \times {}_2F_1\left(\frac{n+1+\alpha}{2}-\frac{1}{p}, \frac{n+1+\alpha}{2}-\frac{1}{p}; n+1+\alpha-\frac{1}{p}; |z|^2\right).
\end{aligned}$$

Note that the last hypergeometric function is increasing on the interval $[0,1)$, since its Taylor coefficients are all positive. Thus, this hypergeometric function is bounded from above by

$$\begin{aligned}
&{}_2F_1\left(\frac{n+1+\alpha}{2}-\frac{1}{p}, \frac{n+1+\alpha}{2}-\frac{1}{p}; n+1+\alpha-\frac{1}{p}; 1\right) \\
&\quad = \frac{\Gamma(n+1+\alpha-1/p)\Gamma(1/p)}{\Gamma^2((n+1+\alpha)/2)}.
\end{aligned}$$

This proves (3.1), which in turn gives

$$\big\|S_\alpha : L^p(\mathbb{B}_n) \to L^p(\mathbb{B}_n)\big\| \le \frac{n!\Gamma(1/p)\Gamma(\alpha+1-1/p)}{\Gamma^2((n+1+\alpha)/2)}. \tag{3.3}$$

We now proceed to show

$$\big\|S_\alpha : L^p(\mathbb{B}_n) \to L^p(\mathbb{B}_n)\big\| \ge \frac{n!\Gamma(1/p)\Gamma(\alpha+1-1/p)}{\Gamma^2((n+1+\alpha)/2)}. \tag{3.4}$$

For fixed $\varepsilon > 0$, define

$$\begin{aligned}
g_\varepsilon(w) &= C_1(\varepsilon)\big(1-|w|^2\big)^{(\varepsilon-1)/p}, \\
h_\varepsilon(z) &= C_2(\varepsilon)\big(1-|z|^2\big)^{(\varepsilon-1)/q} |z|^{2(\alpha+1)+2(\varepsilon-1)/p},
\end{aligned}$$

where

$$C_1(\varepsilon)=\left\{\frac{\Gamma(\varepsilon)\Gamma(n+1)}{\Gamma(n+\varepsilon)}\right\}^{-1/p}, \tag{3.5}$$

$$C_2(\varepsilon)=\left\{\frac{n\Gamma(\varepsilon)\Gamma(n+q(\alpha+1)+(\varepsilon-1)q/p)}{\Gamma(n+(\varepsilon-1)q/p+q(\alpha+1)+\varepsilon)}\right\}^{-1/q}. \tag{3.6}$$

Easy calculations show that $\|g_\varepsilon\|_p=\|h_\varepsilon\|_q=1$.

By applying Lemma 2.2 and integrating in polar coordinates, we obtain

$$\begin{aligned}
&\int_{\mathbb{B}_n}\left\{\int_{\mathbb{B}_n}\frac{(1-|w|^2)^\alpha}{|1-\langle z,w\rangle|^{n+\alpha+1}}g_\varepsilon(w)d\nu(w)\right\}\overline{h_\varepsilon(z)}\,d\nu(z)\\
&\quad=C_1(\varepsilon)C_2(\varepsilon)\frac{\Gamma(n+1)\Gamma(\alpha+1+(\varepsilon-1)/p)}{\Gamma(n+\alpha+1+(\varepsilon-1)/p)}\\
&\qquad\times\int_{\mathbb{B}_n}\left(1-|z|^2\right)^{(\varepsilon-1)/q}|z|^{2(\alpha+1+(\varepsilon-1)/p)}\\
&\qquad\times{}_2F_1\left(\frac{n+1+\alpha}{2},\frac{n+1+\alpha}{2};n+1+\alpha+\frac{\varepsilon-1}{p};|z|^2\right)d\nu(z)\\
&\quad=nC_1(\varepsilon)C_2(\varepsilon)\frac{\Gamma(n+1)\Gamma(\alpha+1+(\varepsilon-1)/p)}{\Gamma(n+\alpha+1+(\varepsilon-1)/p)}\\
&\qquad\times\int_0^1 r^{n+\alpha+(\varepsilon-1)/p}(1-r)^{(\varepsilon-1)/q}\\
&\qquad\times{}_2F_1\left(\frac{n+1+\alpha}{2},\frac{n+1+\alpha}{2};n+1+\alpha+\frac{\varepsilon-1}{p};r\right)dr\\
&\quad=nC_1(\varepsilon)C_2(\varepsilon)\frac{\Gamma(n+1)\Gamma(\alpha+1+(\varepsilon-1)/p)\Gamma(\varepsilon/q+1/p)\Gamma(\varepsilon)}{\Gamma^2((n+\alpha+1)/2+\varepsilon)},
\end{aligned}$$

where the last equality follows from (2.5). Having in mind that

$$\begin{aligned}
&\left\|S_\alpha:L^p(\mathbb{B}_n)\to L^p(\mathbb{B}_n)\right\|\\
&\quad=\sup_{\substack{\|f\|_p=1\\\|g\|_q=1}}\left\{\left|\int_{\mathbb{B}_n}\left(\int_{\mathbb{B}_n}\frac{(1-|w|^2)^\alpha}{|1-\langle z,w\rangle|^{n+\alpha+1}}f(w)\,d\nu(w)\right)\overline{g(z)}\,d\nu(z)\right|\right\},
\end{aligned}$$

this implies

$$\begin{aligned}
&\left\|S_\alpha:L^p(\mathbb{B}_n)\to L^p(\mathbb{B}_n)\right\|\\
&\quad\geq\frac{\Gamma(n+1)\Gamma(\alpha+1+(\varepsilon-1)/p)\Gamma(\varepsilon/q+1/p)}{\Gamma^2((n+\alpha+1)/2+\varepsilon)}\\
&\qquad\times\left\{\frac{\Gamma(n+\varepsilon)}{\Gamma(n)}\right\}^{1/p}\left\{\frac{\Gamma(n+(\varepsilon-1)q/p+q(\alpha+1)+\varepsilon)}{\Gamma(n+q(\alpha+1)+(\varepsilon-1)q/p)}\right\}^{1/q}.
\end{aligned}$$

Equation (3.4) now follows by letting $\varepsilon\to 0^+$ and the proof is complete. $\square$

Proof of Theorem 1.1. Note that $B_{\mathbb{B}_n} = S_{n+1}^*$. It follows from Theorem 1.2 that

$$\begin{aligned}\left\|B_{\mathbb{B}_n} : L^p(\mathbb{B}_n) \to L^p(\mathbb{B}_n)\right\| &= \left\|S_{n+1} : L^q(\mathbb{B}_n) \to L^q(\mathbb{B}_n)\right\| \\ &= \frac{1}{n!}\Gamma\left(\frac{1}{q}\right)\Gamma\left(n+2-\frac{1}{q}\right) \\ &= \frac{1}{n!}\Gamma\left(1-\frac{1}{p}\right)\Gamma\left(n+1+\frac{1}{p}\right).\end{aligned}$$

Then, repeated use of $\Gamma(x+1) = x\Gamma(x)$, together with the well-known formula

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(x\pi)},$$

completes the proof. □

4. The polydisc case

There is yet another multidimensional extension of Theorem A, that is, to the polydisc case. Let $\mathbb{D}^n$ be the unit polydisc in $\mathbb{C}^n$, that is, the cartesian product of n copies of $\mathbb{D}$. Denote by dm the normalized Lebesque volume measure on the polydisk $\mathbb{D}^n$. For $f \in L^1(\mathbb{D}^n, m)$, the Berezin transform of f is define by

$$B_{\mathbb{D}^n} f(z) = \int_{\mathbb{D}^n} \prod_{j=1}^n \frac{(1-|z_j|^2)^2}{|1-z_j\bar{w}_j|^4} f(w)\, dm(w). \tag{4.1}$$

THEOREM 4.1. *For* $1 < p \leq \infty$, *we have*

$$\left\|B_{\mathbb{D}^n} : L^p\left(\mathbb{D}^n\right) \to L^p\left(\mathbb{D}^n\right)\right\| = \left\{\frac{1}{p}\left(1+\frac{1}{p}\right)\frac{\pi}{\sin(\pi/p)}\right\}^n.$$

The proof is almost the same as (and slightly simpler than) that of Theorem 1.1, so we omit it.

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