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HARTOGS FIGURE AND SYMPLECTIC NON-SQUEEZING

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To John D'Angelo

Abstract. We solve a problem on filling by Levi-flat hypersurfaces for a class of totally real 2-tori in a real 4-manifold with an almost complex structure tamed by an exact symplectic form. As an application, we obtain a simple proof of Gromov's non-squeezing theorem in dimension 4 and new results on rigidity of symplectic structures.

1. Introduction

Since Gromov's work [7] it is known that J-complex curves can be used in order to describe obstructions for symplectic embeddings. Following this theme, in this paper we apply classical complex analysis to symplectic rigidity. We obtain new results on non-existence of certain symplectic embeddings, in particular, we give a simple proof of Gromov's non-squeezing theorem in complex dimension 2. Our approach is based on a general result on Levi-flat fillings of totally real tori in an almost complex manifold with an exact symplectic form. This result is new even for manifolds with integrable almost complex structure.

Definition 1.1. Let G be a domain in $\mathbb{C}^2$ containing the origin. Denote by $\mathcal{O}_0^1(G)$ the set of closed complex purely one-dimensional analytic subsets in G passing through the origin. Denote by $E(X)$ the Euclidean area of $X \in \mathcal{O}_0^1(G)$. The holomorphic radius $\mathrm{rh}(G)$ of G is defined as

$$\mathrm{rh}(G) = \inf\left\{\lambda > 0 : \exists X \in \mathcal{O}_0^1(G), E(X) = \pi\lambda^2\right\}.$$

If the set in the right-hand part is empty, then we set $\mathrm{rh}(G) = +\infty$.

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EXAMPLE. Let $\mathbb{B}$ be the Euclidean ball of $\mathbb{C}^2$ and $r > 0$. Then $\mathrm{rh}(r\mathbb{B}) = r$. Indeed, the area $E(X)$ of $X \in \mathcal{O}_0^1(r\mathbb{B})$ is bounded from below by the area πr^2 of a section of the ball by a complex line through the origin (Lelong, 1950; see [2]).

Let (z_1, z_2), $z_j = x_j + iy_j$, be complex coordinates in $\mathbb{C}^2$. Let

$$\omega_{\mathrm{st}} = \frac{i}{2}\sum_{j=1}^{2} dz_j \wedge d\overline{z}_j$$

be the standard symplectic form on $\mathbb{C}^2$. A diffeomorphism $\phi : G_1 \to G_2$ between two domains $G_j \subset \mathbb{C}^2$ is called a *symplectomorphism* if $\phi^*\omega_{\mathrm{st}} = \omega_{\mathrm{st}}$. Let $\mathbb{D}$ denote the unit disc in $\mathbb{C}$. Our main result concerning symplectic rigidity is the following corollary.

THEOREM 1.2. *Let G_1 be a domain in $\mathbb{C}^2$ containing the origin and let G_2 be a domain in $R\mathbb{D} \times \mathbb{C}$ for some $R > 0$. Assume that there exists a symplectomorphism $\phi : G_1 \to G_2$. Then* $\mathrm{rh}(G_1) \le R$.

In view of the example above, we obtain Gromov's [7] non-squeezing theorem in $\mathbb{C}^2$.

COROLLARY 1.3. *Suppose that there exists a symplectomorphism between the ball $r\mathbb{B}$ and a domain contained in $R\mathbb{D} \times \mathbb{C}$. Then $r \le R$.*

As an alternative to the usual *complex bidisc*

$$\mathbb{D}^2 = \{(z_1, z_2) : |z_j| < 1, j = 1, 2\}$$

we introduce the *real bidisc* of the form

$$\mathbb{D}^2_{\mathbb{R}} = \{(z_1, z_2) : x_1^2 + x_2^2 < 1, y_1^2 + y_2^2 < 1\}.$$

The two bidiscs have the same volume. Are they symplectomorphic? We learned about this question from Sergey Ivashkovich [8]. We will prove that the answer is negative. Note that if a symplectomorphism $\mathbb{D}^2 \to \mathbb{D}^2_{\mathbb{R}}$ is smooth up to the boundary, then it maps the torus $\mathbb{T}^2 = \{(z_1, z_2) : |z_j| = 1, j = 1, 2\}$ to the torus $\mathbb{T}^2_{\mathbb{R}} = \{(z_1, z_2) : x_1^2 + x_2^2 = 1, y_1^2 + y_2^2 = 1\}$. However, it is not possible because $\mathbb{T}^2$ is Lagrangian, that is, $\omega_{\mathrm{st}}|_{\mathbb{T}^2} = 0$, while $\mathbb{T}^2_{\mathbb{R}}$ is not. This may lead one to a thought that the question is about exotic non-smooth maps. We show it is not the case. In fact, $\mathbb{D}^2_{\mathbb{R}}$ does not admit a symplectic embedding into a slightly larger complex bidisc. Furthermore, we obtain the following non-squeezing result.

COROLLARY 1.4. *There exists $R > 1$ such that there is no symplectomorphism between $\mathbb{D}^2_{\mathbb{R}}$ and a subdomain of $R\mathbb{D} \times \mathbb{C}$.*

We will show in the last section that $\mathrm{rh}(\mathbb{D}^2_{\mathbb{R}}) > 1$, then Corollary 1.4 will immediately follow from Theorem 1.2. In particular, we obtain the following corollary.

COROLLARY 1.5. *There is no symplectomorphism between the real bidisc $\mathbb{D}^2_{\mathbb{R}}$ and the complex bidisc $\mathbb{D}^2$.*

The proof of Theorem 1.2 relies on filling by complex discs an analog of the Hartogs figure for an almost complex manifold. Let (M, J, ω) be a C^∞-smooth real 4-dimensional manifold with a symplectic form ω and an almost complex structure J. We suppose that J is *tamed* by ω (see [7]), that is, $\omega(V, JV) > 0$ for every nonzero tangent vector V; we call such M a *tame* almost complex manifold. Consider a relatively compact subdomain Ω in M with smooth *strictly pseudoconvex boundary.* This means that for every point $p \in b\Omega$ there exists an open neighborhood U and a smooth strictly J-plurisubharmonic function $\rho : U \to \mathbb{R}$ with non-vanishing gradient such that $\Omega \cap U = \{q \in U : \rho(q) < 0\}$. We do *not* require the existence of a global defining strictly plurisubharmonic function on $\overline{\Omega}$.

We now use the notation $Z = (z, w)$ for complex coordinates in $\mathbb{C}^2$.

DEFINITION 1.6. A C^∞-smooth embedding $H : \overline{\mathbb{D}} \times \overline{\mathbb{D}} \to \overline{\Omega}$ is called a Hartogs embedding if the following conditions hold:

(i) the map $f^0 : \mathbb{D} \ni z \mapsto H(z, 0)$ is J-complex and $f^0(\overline{\mathbb{D}}) \subset \Omega$;

(ii) for every $z \in \overline{\mathbb{D}}$, the map $h_z : \mathbb{D} \ni w \mapsto H(z, w) \in \Omega$ is J-complex; moreover, there exists $\delta > 0$ such that for every z with $1 - \delta \le |z| \le 1$, we have $h_z(b\mathbb{D}) \subset b\Omega$;

Denote $\Lambda^t = H(b\mathbb{D} \times tb\mathbb{D})$, $0 \le t \le 1$. Then Λ^t is a totally real torus in M. We will also denote by Π the Levi-flat hypersurface $\Pi = H(b\mathbb{D} \times \mathbb{D})$. Thus, the family of tori Λ^t and the hypersurface Π are canonically associated with a Hartogs embedding.

Our main technical tool is the following theorem.

THEOREM 1.7. *Let Ω be a relatively compact domain with smooth strictly pseudoconvex boundary in a tame almost complex manifold (M, J, ω) of complex dimension 2 and let $H : \overline{\mathbb{D}} \times \overline{\mathbb{D}} \to \overline{\Omega}$ be a Hartogs embedding. Assume that the symplectic form ω is exact in a neighborhood of the closure $\overline{\Omega}$. Then for every $0 < t \le 1$ there exists a unique one-parameter family of embedded J-complex discs $f : \mathbb{D} \to \Omega$ of class $C^\infty(\overline{\mathbb{D}})$ such that $f(b\mathbb{D}) \subset \Lambda^t$. They fill a smooth Levi-flat hypersurface $\Gamma^t \subset \Omega$ with boundary Λ^t. The family (Γ^t) foliates a subdomain in Ω whose boundary consists of the Levi-flat hypersurfaces Γ^1 and Π and the disc $f^0(\overline{\mathbb{D}})$.*

For simplicity, we assume that (M, J, ω) is C^∞, however the proof needs a finite smoothness. We construct the desired discs by a continuous deformation starting from the initial disc f^0. In particular, they are homotopic to f^0 in the space of J-complex discs in $\overline{\Omega}$ attached to Π. A similar approach was used by Bedford and Gaveau [1], Forstnerič [5], Gromov [7], and others in various situations. The statement of Theorem 1.7 can be slightly improved by introducing the map H with the given properties only on $b\mathbb{D} \times \mathbb{D}$. We keep the stated version for simplicity and convenience of presentation.

In his celebrated paper, Gromov [7] proved that for every compact *Lagrangian* submanifold Λ in $\mathbb{C}^n$, there exists a non-constant complex disc with boundary in Λ. In comparison, our Theorem 1.7 applies to non-Lagrangian tori and it gives information about the set swept out by the discs. In the case $M = \mathbb{C}^2$ with the standard complex structure, there are related results due to Duval and Gayet [4] and Forstnerič [5]. We stress that Theorem 1.7 is new even in the case M is a complex manifold, i.e., the structure J is integrable.

Recall that the classical Hartogs figure U is a neighborhood of $(\mathbb{D} \times \{0\}) \cup (b\mathbb{D} \times \mathbb{D})$ in $\mathbb{C}^2$. One can choose U as a union of complex discs $\{z\} \times r(z)\mathbb{D}$, $z \in \overline{\mathbb{D}}$, where $0 < r(z) \leq 1$ is smooth in z; then the embedding $H : \overline{\mathbb{D}}^2 \to U$ defined by $H(z, w) = (z, r(z)w)$ satisfies Definition 1.6 and smoothly parametrizes the Hartogs figure by the bidisc. Therefore, one can view Theorem 1.7 as a result on filling a Hartogs figure by complex discs. In the case J is integrable it can be used in the study of holomorphic extension problems and polynomial, holomorphic, and plurisubharmonic hulls. We also point out that the Hartogs embedding in Definition 1.6 is not necessarily biholomorphic, which brings additional flexibility to the method. In this paper, we focus on symplectic applications of the theorem.

This paper was written for a special volume in honor of our dear colleague Professor John D'Angelo on the occasion of his 60th birthday. The authors wish John good health, happiness, and new research accomplishments for years to come.

2. Almost complex manifolds

Let (M, J) be an almost complex manifold. We denote by J_{st} the standard complex structure of $\mathbb{C}^n$; the value of n will be clear from the context. A C^1-map $f : \mathbb{D} \to M$ is called a *J-complex* (or J-holomorphic) disc if $df \circ J_{\mathrm{st}} = J \circ df$.

In local coordinates $Z = (z, w) \in \mathbb{C}^2$, an almost complex structure J can be represented by a complex 2×2 matrix function A, so that a map $Z : \mathbb{D} \to \mathbb{C}^2$ is J-complex if and only if it satisfies the following partial differential equation

$$Z_{\overline{\zeta}} - A(Z)\overline{Z_\zeta} = 0. \tag{1}$$

The matrix $A(Z)$ is defined by

$$A(Z)V = \big(J_{\mathrm{st}} + J(Z)\big)^{-1}\big(J_{\mathrm{st}} - J(Z)\big)\overline{V}. \tag{2}$$

Indeed, one can see that the right-hand side of (2) is $\mathbb{C}$-linear in $V \in \mathbb{C}^n$ with respect to the standard structure J_{st}, hence $A(Z)$ is well defined (see, e.g., [13]). We call A *the complex matrix* of J. The ellipticity of (1) is equivalent to $\det(I - A\overline{A}) \neq 0$. In a fixed coordinate chart, the correspondence between almost complex structures J with $\det(J_{\mathrm{st}} + J) \neq 0$ and complex matrices with $\det(I - A\overline{A}) \neq 0$ is one-to-one [13].

Often we identify J-complex discs f and their images calling them just the discs. By the boundary of such a disc we mean the restriction $f|_{b\mathbb{D}}$, which we also identify with its image.

Let ρ be a function of class C^2 on M, let $p \in M$ and $V \in T_pM$. *The Levi form* of ρ at p evaluated on V is defined by the equality $L^J(\rho)(p)(V) := -d(J^* d\rho)(V, JV)(p)$. A real function ρ of class C^2 on M is called *J-plurisubharmonic* (resp. strictly J-plurisubharmonic) if $L^J(\rho)(p)(V) \geq 0$ (resp. > 0) for every $p \in M$, $V \in T_pM \setminus \{0\}$.

A smooth real hypersurface Γ in an almost complex manifold (M, J) is called *Levi-flat* if in a neighborhood U of every point $p \in \Gamma$ there exists a defining function with non-zero gradient whose Levi form vanishes for every tangent vector $V \in T_q\Gamma \cap JT_q\Gamma$ and every point $q \in U \cap \Gamma$. If the complex dimension of M is equal to 2, then by the Frobenius theorem, a hypersurface Γ is Levi-flat if and only if Γ is locally foliated by a real one-parameter family of J-complex discs.

3. Deformation

Returning to Theorem 1.7, we fix $\delta > 0$ that figures in Definition 1.6. Consider the annulus

$$\mathcal{A}_\delta = \big\{z \in \mathbb{C} : 1 - \delta \leq |z| \leq 1\big\}.$$

Introduce also the discs and the circles

$$G^t = \big\{w \in \mathbb{D} : |w| < t\big\}, \qquad \gamma^t = bG^t = \big\{w \in \mathbb{D} : |w| = t\big\}.$$

We recall the notations $\Lambda = H(b\mathbb{D} \times b\mathbb{D})$, $\Pi = H(b\mathbb{D} \times \overline{\mathbb{D}})$ and

$$\Lambda^t = \bigcup_{z \in b\mathbb{D}} H\big(\{z\} \times \gamma^t\big), \quad 0 \leq t \leq 1.$$

Then for $0 < t \leq 1$, Λ^t is a totally real torus, $\Lambda^1 = \Lambda$, and $\Lambda^0 = f^0(b\mathbb{D})$ is a circle.

We will consider J-complex discs with boundaries in Λ^t. By reflection principle [9], such discs are smooth up to the boundary.

Let $t_0 > 0$. Let $I(t_0)$ denote one of the intervals: $[0, t_0]$ or $[0, t_0)$. Let

$$\big\{f^{t,\tau} : \overline{\mathbb{D}} \to M : t \in I(t_0), \tau \in \mathbb{R}/2\pi\mathbb{Z}\big\}$$

be a continuous family of embedded J-complex discs, smooth in all the variables for $t > 0$.

Definition 3.1. We call the family $(f^{t,\tau})$ *an admissible deformation* (of the initial disc f^0) on $I(t_0)$ if it has the following properties.

(i) $f^{0,\tau} = f^0$.

(ii) $f^{t,\tau}(b\mathbb{D}) \subset \Lambda^t$; $f^{t,\tau_1}(b\mathbb{D}) \cap f^{t,\tau_2}(b\mathbb{D}) = \emptyset$ if $\tau_1 \neq \tau_2$; $\bigcup_{\tau\in\mathbb{R}/2\pi\mathbb{Z}} f^{t,\tau}(b\mathbb{D}) = \Lambda^t$; $\Gamma^t = \bigcup_{\tau\in\mathbb{R}/2\pi\mathbb{Z}} f^{t,\tau}(\overline{\mathbb{D}})$ is a smooth hypersurface with boundary Λ^t.

(iii) The set $H^{-1}(f^{t,\tau}(\overline{\mathbb{D}})) \cap (\mathcal{A}_\delta \times \overline{\mathbb{D}})$ is the graph of a non-vanishing smooth function $w^{t,\tau} : \mathcal{A}_\delta \to \overline{\mathbb{D}} \setminus \{0\}$ so that the map $w^{t,\tau}|_{b\mathbb{D}} : b\mathbb{D} \to \gamma^t$ has zero winding number. Furthermore, $f^{t,\tau}(\overline{\mathbb{D}}) \cap f^0(\overline{\mathbb{D}}) = \emptyset$ for $t > 0$.

(iv) (Normalization condition) For a fixed ζ_0 in the interior of $\mathcal{A}_\delta$, say $\zeta_0 = 1 - \delta/2$, we have $w^{t,\tau}(1) = te^{i\tau}$, $f^{t,\tau}(1) = H(1, w^{t,\tau}(1))$, and $f^{t,\tau}(\zeta_0) = H(\zeta_0, w^{t,\tau}(\zeta_0))$.

(v) Every J-complex disc f such that $f(b\mathbb{D}) \subset \Lambda^t$ and close to $f^{t,\tau}$ in $C^{1,\alpha}(\mathbb{D})$, coincides with $f^{t,\tau'}$ for some $\tau' \in \mathbb{R}/2\pi\mathbb{Z}$ close to τ after a reparametrization close to the identity; here $0 < \alpha < 1$ is fixed, say $\alpha = 1/2$. (Note that $f \in C^\infty(\overline{\mathbb{D}})$ by reflection principle [9].)

Since $b\Omega$ is strictly pseudoconvex, for every t and τ the discs $f^{t,\tau}$ are contained in Ω.

Consider the pull-back $H^*(J)$ of J to $\mathbb{D}^2$. It follows from Definition 1.6 (see, e.g., [14]) that the complex matrix A of $H^*(J)$ over $\mathcal{A}_\delta \times \mathbb{D}$ has the following special form:

$$A = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \tag{3}$$

with $|a| < 1$. Then (see, e.g., [14]) the functions $w^{t,\tau}$ satisfy the equation

$$w_{\overline{z}} + aw_z = b. \tag{4}$$

Conversely, the graph of every solution of (4) becomes a J-complex curve after a suitable reparametrization $z = z(\zeta)$.

We prove Theorem 1.7 by showing that there exists an admissible deformation on $[0, 1]$. Sometimes we will write f^t instead of $f^{t,\tau}$ if the value of τ is unimportant.

In [14], we obtain a result similar to Theorem 1.7 for $M = \mathbb{C}^2$ equipped with an almost complex structure whose matrix has a form even more general than (3). We can use that result in a neighborhood of the disc f^0. Then we obtain the following proposition.

Proposition 3.2. *For small $t_0 > 0$, there exists a unique admissible deformation on $I(t_0)$.*

Proposition 3.3.

(i) *If an admissible deformation $f^{t,\tau}$ on $I(t_0)$ exists, then it is unique.*

(ii) $f^{t_1,\tau_1}(\overline{\mathbb{D}}) \cap f^{t_2,\tau_2}(\overline{\mathbb{D}}) = \emptyset$ *unless $t_1 = t_2$ and $\tau_1 = \tau_2$.*

Proof. (i) By Proposition 3.2, two admissible deformations must coincide for small t. Then by the properties (iii)–(v) they have to be the same for all $t \in I(t_0)$.

(ii) Since $f^{t_1,\tau_1}(b\mathbb{D}) \cap f^{t,\tau_2}(b\mathbb{D}) = \emptyset$ for $0 \le t \le t_2$, then the intersection index of $f^{t_1,\tau_1}(\overline{\mathbb{D}})$ and $f^{t,\tau_2}(\overline{\mathbb{D}})$ is independent of t (see [11], [12]). Since $f^{t_1,\tau_1}(\mathbb{D}) \cap f^{0,\tau_2}(\mathbb{D}) = \emptyset$, then $f^{t_1,\tau_1}(\mathbb{D}) \cap f^{t_2,\tau_2}(\mathbb{D}) = \emptyset$, also. □

An admissible deformation defined on a closed interval $[0, t_0]$ can be extended to a larger interval. We include a slightly stronger version of that result.

Proposition 3.4. *Let $(f^{t,\tau})$ be an admissible deformation defined on $I(t_0) = [0, t_0)$. Suppose that for every $\tau \in \mathbb{R}/2\pi\mathbb{Z}$ there exists an increasing sequence (t^k), $k = 1, 2, \ldots$, with $t^k \to t_0$ such that $f^{t^k,\tau}$ converges in the $C^m(\overline{\mathbb{D}})$-norm for every m to a J-complex disc $f^{\infty,\tau}$. Then the deformation can be extended to $I(t_1)$ for some $t_1 > t_0$.*

Proof. It follows from Definition 3.1(iii) that $f^{\infty,\tau}|_{b\mathbb{D}}$ is an embedding. Since all $f^{t,\tau}$ are embeddings, then by the positivity of intersection indices [11], [12], the limiting disc $f^{\infty,\tau}$ remains an embedding. We obtain the discs $f^{t,\tau}$ for $t > t_0$ as small deformations of the discs $f^{\infty,\tau}$ by applying the results [6], [10] about small deformations of J-complex discs attached to totally real manifolds. In the case of complex dimension 2, such deformations are governed by the normal Maslov index (see, e.g., [6], [10]). The latter is equal to zero for the disc $f^{\infty,\tau}$ because of the winding number condition in Definition 3.1(iii). Hence, the family extends to an interval $I(t_1)$ for some $t_1 > t_0$.

It remains to show that the extended family on $I(t_1)$ still satisfies Definition 3.1, especially part (iii). For simplicity of notation, we temporarily omit τ in $f^{t,\tau}$ and write just f^t. The domain Ω naturally splits into two parts: $\Omega = \Omega_1 \cup \Omega_2$, where $\Omega_2 = H(\mathcal{A}_\delta \times \mathbb{D})$ and $\Omega_1 = \Omega \setminus \Omega_2$. Likewise, every disc $D^t = f^t(\mathbb{D})$ also splits into two parts: $D^t = D^t_1 \cup D^t_2$. Here D^t_2 is the subset of Ω_2 represented as the graph of the function $w^t := w^{t,\tau}$ over $\mathcal{A}_\delta$, and $D^t_1 = D^t \setminus D^t_2$. By Definition 3.1(iii), for every $0 < t < t_0$, we have $D^t_j = D^t \cap \Omega_j$, $j = 1, 2$. We claim that the latter still holds for every $0 < t < t_1$, in particular, $D^t_1 \subset \Omega_1$. Denote by t' the supremum of the set of all $t < t_1$ for which $D^t_j = D^t \cap \Omega_j$, $j = 1, 2$. Consider the Levi-flat hypersurface $\Pi_\delta := H(\{|z| = \delta\} \times \mathbb{D})$. Note that by the Hopf lemma D^t is transverse to Π_δ for $t < t'$ (see, e.g., [6]). If $t' < t_1$, then $D^{t'}_1$ touches Π_δ at an interior point which is impossible (see, e.g., [3]). Hence, $t' = t_1$, $D^t_1 \subset \Omega_1$ for all $0 < t < t_1$, and (iii) follows. The other conditions in Definition 3.1 are fulfilled automatically. □

4. Proof of the main result

We establish a priori estimates for any admissible deformation on an open interval.

LEMMA 4.1.

(i) *Let q and Q be bounded functions in the annulus $\mathcal{A}_\delta$ $(0<\delta<1)$, $|q|\le q_0<1$, $|Q|\le Q_0$, here q_0 and Q_0 are constants. Let $\varepsilon>0$ and $0<\delta'<\delta$. Let w be a solution of*

$$w_{\overline{z}} = q w_z + Q \tag{5}$$

in $\mathcal{A}_\delta$, such that $\varepsilon \le |w| \le 1/\varepsilon$ and $|w(z)|=1$ for $z\in b\mathbb{D}$. Then $\|w\|_{C^\alpha(\mathcal{A}_{\delta'})}\le C$; here $0<\alpha<1$ and $C>0$ depend on ε, δ, δ', q_0, and Q_0 only.

(ii) *Suppose in addition that $\|q\|_{C^{k,\beta}(\mathcal{A}_\delta)}+\|Q\|_{C^{k,\beta}(\mathcal{A}_\delta)}\le Q_0$, for some $0<\beta<1$ and $k\ge 0$. Then $\|w\|_{C^{k+1,\beta}(\mathcal{A}_{\delta'})}\le C$; here $C>0$ depends on β, k, ε, δ, δ', q_0, and Q_0 only.*

Proof. (i) We apply the reflection principle. Let $\mathcal{A}_\delta^*=\{z^*: z\in\mathcal{A}_\delta\}$, here $z^*:=1/\overline{z}$. Extend w and the coefficients q and Q to $\mathcal{A}_\delta^*$ by putting

$$w(z)=\big(w(z^*)\big)^*, \qquad q(z)=\overline{q(z^*)}\,z^2\big(z^*\big)^2, \qquad Q(z)=\overline{Q(z^*)}\big(z^*\big)^2 w(z)^2$$

for $z\in\mathcal{A}_\delta^*$. Then w is continuous in $G=\mathcal{A}_\delta\cup\mathcal{A}_\delta^*$ and satisfies (5) there. The coefficients satisfy $|q|\le q_0$ and $|Q|\le Q_0/\varepsilon^2$ in G.

We claim that w is uniformly bounded in C^α for some $0<\alpha<1$ after shrinking G. It suffices to prove this fact for $G=\mathbb{D}$. There exists a particular solution $w_0\in C^\alpha(\mathbb{D})$ of the non-homogeneous equation (5), say by Proposition 2.1(i) in [14]. Then $w=w_0+v$, where v is a solution of the homogeneous equation $v_z = q v_{\overline{z}}$. Then $v(z)=\phi(\xi(z))$, where $\xi:\overline{\mathbb{D}}\to\overline{\mathbb{D}}$ is a fixed Beltrami homeomorphism of class $C^\alpha(\mathbb{D})$, and ϕ is holomorphic in $\mathbb{D}$. Since ϕ is bounded, then the derivative ϕ' is also bounded in any smaller disc. Then $v(z)=\phi(\xi(z))$, whence $w=w_0+v$ is bounded in C^α in a smaller disc.

(ii) For simplicity, we first assume that w has a continuous logarithm in $\mathcal{A}_\delta$, which will be the case in our applications. Then $w=e^u$, $\operatorname{Re} u|_{b\mathbb{D}}=0$, and u satisfies in $\mathcal{A}_\delta$ the equation

$$u_{\overline{z}} = q u_z + Q w^{-1}.$$

Let χ be a smooth cut-off function on $\overline{\mathbb{D}}$ vanishing in a neighborhood of $(1-\delta)\overline{\mathbb{D}}$ and such that $\chi\equiv 1$ in $\mathcal{A}_{\delta'}$. Put $v=\chi u$. We can assume that q and Q are extended over all of $\mathbb{D}$. Then

$$v_{\overline{z}} = q v_z + \chi Q w^{-1} + u(\chi_{\overline{z}} - q\chi_z). \tag{6}$$

By part (i), after shrinking δ, we can assume $w^{-1}, u\in C^\alpha(\mathcal{A}_\delta)$. Then v satisfies (6) in $\mathbb{D}$ with boundary condition $\operatorname{Re} v|_{b\mathbb{D}}=0$. Without loss of generality, $v(1)=0$. The conclusion now follows by successively applying Proposition 2.1(ii) from [14].

In the general case, we can use the above argument with appropriate cut-off function to prove the result for every proper sector of the annulus $\mathcal{A}_\delta$. Since the whole annulus is a union of two such sectors, then the conclusion will hold for $\mathcal{A}_\delta$. $\square$

For an admissible deformation $(f^{t,\tau})$, we obtain a priori estimates of the derivatives of the functions $w^{t,\tau}$ from Definition 3.1(iii).

LEMMA 4.2. *Let* $0 < t_0 < 1$. *Let* $(f^{t,\tau})$ *be an admissible deformation on* $0 \le t < t_0$. *Then for every* $0 < \delta' < \delta$ *and* $m \ge 1$ *there exists* $C > 0$ *such that for every* $0 \le t < t_0$ *and* $\tau \in \mathbb{R}/2\pi\mathbb{Z}$ *we have* $\|w^{t,\tau}\|_{C^m(\mathcal{A}(\delta'))} \le C$.

In fact, the constant C is independent of t_0 but we do not need it in our application.

Proof of Lemma 4.2. Fix $t_1 < t_0$, say, $t_1 = t_0/2$. For $0 < t < t_1$ the desired estimate holds. We need to show that it also holds for $t_1 \le t < t_0$. From the properties of the admissible deformation it follows that the union of the discs $f^{t,\tau}$, $0 \le t < t_1$, $\tau \in \mathbb{R}/2\pi\mathbb{Z}$, cover an open neighborhood of $f^0(\overline{\mathbb{D}})$ in $H(\mathcal{A}_\delta \times \overline{\mathbb{D}})$. Since the discs $f^{t,\tau}$ do not intersect, then for $t_1 \le t < t_0$ the functions $w^{t,\tau}$ are uniformly separated from zero. Hence, the desired conclusion follows by Lemma 4.1. $\square$

The given symplectic form ω and almost complex structure J tamed by ω define a Riemannian metric

$$\mu(V,W) = \frac{1}{2}\big(\omega(V,JW) + \omega(W,JV)\big).$$

Let f be a J-complex disc in $\overline{\Omega}$. Let $E(f)$ denote the area of f with respect to μ. Then (see, e.g., [11])

$$E(f) = \int_{\mathbb{D}} f^*\omega. \tag{7}$$

We denote by $L(f)$ the length of the boundary of f, that is,

$$L(f) = \int_0^{2\pi} \left| \frac{df(e^{i\theta})}{d\theta} \right|_\mu d\theta,$$

here $|\bullet|_\mu$ is the norm defined by μ. Since the form ω is exact in a neighborhood of $\overline{\Omega}$, that is, $\omega = d\lambda$, then by Stokes' formula

$$E(f) = \int_{f(\mathbb{D})} \omega = \int_{f(b\mathbb{D})} \lambda \le CL(f), \tag{8}$$

where $C > 0$ depends only on Ω, λ, and μ. This inequality is a special case of the isoperimetric inequality for J-complex curves, see [7], [11].

By Lemma 4.2, the lengths of boundaries of $f^{t,\tau}$ are uniformly bounded. Hence we obtain an upper bound on areas of the discs from an admissible deformation.

Corollary 4.3. *Let $0 < t_0 < 1$. Let $(f^{t,\tau})$ be an admissible deformation on $0 \le t < t_0$. Then there exists a constant $C > 0$ such that $E(f^{t,\tau}) \le C$ for all t and τ.*

Proof of Theorem 1.7. Let $(f^{t,\tau})$ be an admissible deformation on $[0, t_0)$. Consider a sequence $t^k \to t_0$ as $k \to \infty$. Consider the sequence $(f^{t^k,\tau})$ for a fixed value of τ. For simplicity of notation we again omit τ in $f^{t,\tau}$ and write just f^t.

Since the areas of all discs f^t are bounded, then by Gromov's [7] compactness theorem (see also [9], [11]), after passing to a subsequence if necessary, the sequence f^{t^k} converges to a J-complex disc f^∞ uniformly on every compact subset of $\overline{\mathbb{D}} \setminus \Sigma$. Here Σ is a finite set, where bubbles arise. The map f^∞ is smooth on $\overline{\mathbb{D}}$ and $f^\infty(b\mathbb{D}) \subset \Lambda^{t_0}$. A bubble is a non-constant J-complex sphere (a non-constant J-complex map from the Riemann sphere to M) or a non-constant J-complex disc with boundary in Λ^{t_0}; disc-bubbles arise only at the boundary points of $\mathbb{D}$. We will prove that $\Sigma = \emptyset$, that is, there are no bubbles. Then Gromov's compactness theorem will imply the convergence in every $C^m(\overline{\mathbb{D}})$-norm.

Since the form ω is exact, then by Stokes' formula every J-complex sphere in Ω has zero area. Hence, there are no spherical bubbles, and Σ can contain only points of $b\mathbb{D}$, where disc-bubbles arise. The sequence $f^{t^k}(\mathbb{D})$ converges to a finite union of $f^\infty(\mathbb{D})$ and disc-bubbles in the Hausdorf metric. It follows from the normalization condition of Definition 3.1(iv) that the disc $f^\infty(\mathbb{D})$ does not degenerate to a single point.

Let F be the limit of the sequence D^{t^k} in the Hausdorf metric. Recall the decomposition $\Omega = \Omega_1 \cup \Omega_2$ that we used in the proof of Proposition 3.4. Since $D_1^t \subset \Omega_1$ for all $0 < t < t_0$, then $F \cap \Omega_2$ coincides with the Hausdorf limit of the sequence $D_2^{t^k}$. By Lemma 4.2 and Ascoli's theorem (after passing to a subsequence if necessary), the sequence w^{t^k} converges in $C^m(\mathcal{A}_\delta)$, $m \ge 0$, to some function $w^\infty \in C^m(\mathcal{A}_\delta)$. But then the graph of w^∞ necessarily coincides with an open piece of $f^\infty(\mathbb{D})$. Hence boundary bubbles do not arise either.

Thus, for every τ there is a subsequence of $f^{t^k,\tau}$ converging in every C^m-norm. Then by Proposition 3.4 the admissible deformation $(f^{t,\tau})$ extends past t_0. Hence, there is an admissible deformation on the whole interval $[0,1]$, and the proof of Theorem 1.7 is complete. □

5. Non-squeezing

We first establish the following proposition.

Proposition 5.1. *Let G be a bounded domain in $\mathbb{C}^2$ and let $R > 0$. Suppose $\overline{G} \subset R\mathbb{D} \times \mathbb{C}$ and $\overline{G} \cap (R\mathbb{D} \times \{0\}) = \emptyset$. Let J be an almost complex structure on $\mathbb{C}^2$ tamed by $\omega_{\rm st}$ and let $J = J_{\rm st}$ on $\mathbb{C}^2 \setminus \overline{G}$. Then the domain*

$R\mathbb{D} \times (\mathbb{C} \setminus \{0\})$ is foliated by a real one-parameter family of Levi-flat hypersurfaces Γ^t, $t > 0$, with boundary $b\Gamma^t = \Lambda^t = Rb\mathbb{D} \times tb\mathbb{D}$. Every hypersurface Γ^t in turn is foliated by embedded J-complex discs attached to Λ^t, and every such disc has Euclidean area πR^2.

Proof. Without loss of generality, assume $R = 1$. Consider the Euclidean ball $s\mathbb{B}$ in $\mathbb{C}^2$ for $s > 0$ big enough so that $\overline{G} \subset s\mathbb{B}$. The boundary of $s\mathbb{B}$ is a strictly pseudoconvex hypersurface with respect to J. We apply Theorem 1.7 to the family of tori $\Lambda^t = b\mathbb{D} \times tb\mathbb{D}$ in $\Omega = s\mathbb{B}$. As a result, we obtain a foliation by hypersurfaces Γ^t. Every hypersurface Γ^t in turn is foliated by embedded J-complex discs $f^{t,\tau} = (z^{t,\tau}, w^{t,\tau})$.

Since $J = J_{\mathrm{st}}$ on $\mathbb{C}^2 \setminus \overline{G}$, then for big t, we have $f^{t,\tau}(\zeta) = (\zeta, te^{i\tau})$. Hence the hypersurfaces Γ^t cover the set $\mathbb{D} \times (\mathbb{C} \setminus r\mathbb{D})$, where r is large. By continuity, they cover the whole set $\mathbb{D} \times (\mathbb{C} \setminus \{0\})$ as stated.

We now claim that the discs $f^{t,\tau}$ have area π. Indeed, let $\lambda = (i/2)(z\, d\overline{z} + w\, d\overline{w})$. Then $\omega_{\mathrm{st}} = d\lambda$. Consider the parametrization of Λ^t given by $z = e^{i\phi}$, $w = te^{i\psi}$. Then the restriction $\theta = \lambda|_{\Lambda^t}$ has the form

$$\theta = \frac{1}{2}\big(d\phi + t^2\, d\psi\big).$$

By Stokes' formula (8)

$$E\big(f^{t,\tau}\big) = \int_{b\mathbb{D}} f^*\theta = \frac{1}{2}\int_{b\mathbb{D}} \big(z^{t,\tau}\big)^*(d\phi) + \frac{t^2}{2}\int_{b\mathbb{D}} \big(w^{t,\tau}\big)^*(d\psi) = \pi$$

since the winding numbers of $z^{t,\tau}$ and $w^{t,\tau}$ are equal to 1 and 0, respectively for all t and τ. $\square$

In particular, we immediately obtain the following corollary.

COROLLARY 5.2. *Let J be a smooth almost complex structure in $R\mathbb{D} \times \mathbb{C}$ tamed by ω_{st} and such that $J - J_{\mathrm{st}}$ has compact support in $R\mathbb{D} \times \mathbb{C}$. Then for every $p \in R\mathbb{D} \times \mathbb{C}$ there exists a J-complex disc $f : \overline{\mathbb{D}} \to R\overline{\mathbb{D}} \times \mathbb{C}$ such that $f(0) = p$, $f(b\mathbb{D}) \subset Rb\mathbb{D} \times \mathbb{C}$, and $E(f) = \pi R^2$.*

Proof of Theorem 1.2. Pushing forward the standard complex structure J_{st} by ϕ yields an almost complex structure $J = \phi_*(J_{\mathrm{st}})$ on G_2 tamed by ω_{st}. Consider an exhaustion sequence of subdomains $K_n \subset G_1$ such that every K_n is relatively compact in K_{n+1}. Note that an almost complex structure J is tamed by ω_{st} if and only if its complex matrix A at every point has Euclidean norm $\|A\| < 1$; the set of such matrices is convex. Therefore, for every n, there exists an almost complex structure $\tilde{J}$ on $R\mathbb{D} \times \mathbb{C}$, which is tamed by ω_{st}, coincides with J on $\phi(K_n)$, and coincides with J_{st} outside $\phi(K_{n+1})$.

Let $p = \phi(0)$. By Corollary 5.2, there exists a proper $\tilde{J}$-complex disc $D := f(\mathbb{D})$ in $R\mathbb{D} \times \mathbb{C}$ passing through p with $E(D) = \pi R^2$. Therefore, $E(\phi(K_n) \cap D) \le \pi R^2$. Let $X_n = \phi^{-1}(D) \cap K_n$. Since the map ϕ is a symplectomorphism, then $E(X_n) \le \pi R^2$. Since $\phi : K_n \to \phi(K_n)$ is biholomorphic with respect to

$J_{\rm st}$ and $\tilde J$, then $X_n \in \mathcal{O}_0^1(K_n)$. Since (K_n) is an exhaustion sequence for G_1, then by Bishop's convergence theorem (see, e.g., [2]), there is a subsequence of (X_n) converging to a set $X \in \mathcal{O}_0^1(G_1)$ with $E(X) \le \pi R^2$. Hence, $\mathrm{rh}(G_1) \le R$. $\square$

Proof of Corollary 1.4. By Theorem 1.2, it suffices to prove that $\mathrm{rh}(\mathbb{D}^2_{\mathbb{R}}) > 1$. By Bishop's convergence theorem, there exists $X \in \mathcal{O}_0^1(\mathbb{D}^2_{\mathbb{R}})$ such that $E(X) = \pi(\mathrm{rh}(\mathbb{D}^2_{\mathbb{R}}))^2$. The Euclidean unit ball $\mathbb{B}$ is contained in $\mathbb{D}^2_{\mathbb{R}}$ and their boundaries $b\mathbb{B}$ and $b\mathbb{D}^2_{\mathbb{R}}$ meet along two circles:

$$S_1 = \big\{(z_1, z_2) : x_1^2 + x_2^2 = 1, y_1 = y_2 = 0\big\},$$
$$S_2 = \big\{(z_1, z_2) : x_1 = x_2 = 0, y_1^2 + y_2^2 = 1\big\}.$$

Suppose that the boundary $bX = \overline{X} \setminus X$ of X is contained in $S_1 \cup S_2$. Then X is a complex one-dimensional analytic subset in $\mathbb{C}^2 \setminus (S_1 \cup S_2)$. By the reflection principle for analytic sets [2], X extends as a complex 1-dimensional analytic set to a neighborhood of $S_1 \cup S_2$. Then by the uniqueness theorem X is contained in the complex algebraic curve $(z_1^2 + z_2^2)^2 = 1$. But the latter does not pass through the origin, a contradiction. Therefore, the closure $\overline{X}$ intersects the sphere $b\mathbb{B}$ at a point p which is not in $S_1 \cup S_2$. Since X is closed in $\mathbb{D}^2_R$, the point p is an interior point for X. The unit sphere $b\mathbb{B}$ is a strictly pseudoconvex hypersurface. By the maximum principle [2] applied to the plurisubharmonic function $|z_1|^2 + |z_2|^2$ on the analytic set X, the latter can not be contained in $\mathbb{B}$. Hence X contains an open piece outside the closed ball $\overline{\mathbb{B}}$. Hence, $E(X) > E(X \cap \mathbb{B}) \ge \pi$, and $\mathrm{rh}(\mathbb{D}^2_{\mathbb{R}}) > 1$ as desired. $\square$

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