

HT90 AND “SIMPLEST” NUMBER FIELDS

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ABSTRACT. A standard formula (1.1) leads to a proof of HT90, but requires proving the existence of θ such that $\alpha \neq 0$, so that $\beta = \alpha/\sigma(\alpha)$.

We instead impose the condition (M), that taking $\theta = 1$ makes $\alpha = 0$. Taking $n = 3$, we recover Shanks’s simplest cubic fields. The “simplest” number fields of degrees 3 to 6, Washington’s cyclic quartic fields, and a certain family of totally real cyclic extensions of $\mathbb{Q}(\cos(\pi/4m))$ all have defining polynomials whose zeroes satisfy (M).

Further investigation of (M) for $n = 4$ leads to an elementary algebraic construction of a 2-parameter family of octic polynomials with “generic” Galois group ${}_8T_{11}$. Imposing an additional algebraic condition on these octics produces a new family of cyclic quartic extensions. This family includes the “simplest” quartic fields and Washington’s cyclic quartic fields as special cases.

We obtain more detailed results on our octics when the parameters are algebraic integers in a number field. In particular, we identify certain sets of special units, including exceptional sequences of 3 units, and give some of their properties.

1. Introduction

Hilbert’s Theorem 90 (see, e.g., [16]) characterizes elements β of norm 1 in a cyclic extension L/\mathbf{k} of degree n with Galois group $G = \langle \sigma \rangle$; one has

$$(HT90) \quad \mathcal{N}_{L/\mathbf{k}}(\beta) = 1 \quad \text{if, and only if,} \quad \beta = \alpha/\sigma(\alpha) \quad \text{for some } \alpha \in L.$$

“If” is obvious. The usual proof of “only if” uses the formula

$$(1.1) \quad \alpha = \theta + \sigma(\theta)\beta + \sigma^2(\theta)\beta\sigma(\beta) + \cdots + \sigma^{n-1}(\theta)\beta\sigma(\beta) \cdots \sigma^{n-2}(\beta).$$

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If $\mathcal{N}_{L/\mathbf{k}}(\beta) = 1$, this formula makes $\alpha/\sigma(\alpha) = \beta$ a formal identity. One has to show that $\alpha \neq 0$ for some $\theta \in L$ to complete the proof.

In [6], Chapter XI, Theorem 2, there is a constructive proof of (HT90) for $n = 2$. It actually exhibits a nonzero α : Substituting $\theta = 1$ gives the simplified formula

$$\alpha = 1 + \beta.$$

This gives a nonzero α unless $\beta = -1$, and this case is handled separately.

We note that $\theta = 1$ is the *only* nonzero value guaranteed to be in *every* field. We make this choice for arbitrary n . We assume L/\mathbf{k} is a finite Galois extension with Galois group G , and $\sigma \in G$ is of order n . We impose the condition on $r \in L$ that

$$(M) \quad 1 + r + r\sigma(r) + \cdots + r\sigma(r) \cdots \sigma^{n-2}(r) = 0.$$

We call (M) the *Murphy condition*. Unfortunately, (M) alone does not force cyclic extensions. In Section 4 we see that when $n = 4$, L^σ can be a *proper* extension of \mathbf{k} , even with the additional conditions in Equation (1.4).

The following formal properties are immediate.

PROPOSITION 1.1. *Let \mathbf{k}, L, σ, r and n satisfy (M). Then*

- (a) (M) holds if r is replaced by $\sigma^i(r)$, $1 \leq i \leq n - 1$.
- (b) $r\sigma(r) \cdots \sigma^{n-1}(r) = 1$.

Proof. For (a), apply σ^i to (M). For (b), apply σ to (M), multiply by r , and subtract (M). □

If \mathbf{k} is a number field, and r is an *algebraic integer* for which (M) holds, then by Proposition 1.1(b), r and its σ -conjugates are *units*, which we call *Murphy's units*. They are quotients of unusual sets of conjugate associates (see the Remarks just before Section 3).

We take as a simple example the case $n = 3$. Then (M) becomes

$$(M3) \quad 1 + r + r\sigma(r) = 0.$$

Solving for $\sigma(r)$ and repeatedly applying σ , treating it as a field automorphism, we obtain the expressions

$$(C3) \quad \sigma(r) = \frac{-r - 1}{r}, \quad \sigma^2(r) = \frac{-1}{r + 1}, \quad \sigma^3(r) = r.$$

Setting

$$(Tr3) \quad r + \sigma(r) + \sigma^2(r) = A, \quad A \in \mathbf{k}$$

and clearing fractions, we find that any r for which (M3) and (Tr3) hold is a zero of

$$(P3) \quad p(x) = x^3 - Ax^2 - (A + 3)x - 1 \quad \text{for some } A \in \mathbf{k}.$$

By Proposition 2.3, *every* cyclic cubic field extension has a defining polynomial of this form. However, if we take $\mathbf{k} = \mathbb{Q}$ and $A \in \mathbb{Z}$, $p(x)$ is irreducible (mod 2),

so this restriction on the parameter produces a family of cyclic cubic fields in which 2 remains inert, which is clearly *not* true of all cyclic cubic fields. In [33], D. Shanks called the fields defined by (P3) with $A \in \mathbb{Z}$ the “simplest” cubic fields. Certain families of cyclic number fields of degrees 4, 5, and 6 have subsequently been dubbed “simplest.” They, too, have defining polynomials whose zeroes are units satisfying (M). We have the following result.

PROPOSITION 1.2. *Let \mathbf{k} be a field, $t \in \mathbf{k}$. In each of the following cases, $\sigma(x) \pmod{P(x)}$ makes (M) a formal identity, with n equal to the degree of $P(x)$.*

- (a) $P(x) = x^3 - tx^2 - (t + 3)x - 1$, $\sigma(x) = (-x - 1)/x$;
- (b) $P(x) = x^4 - tx^3 - 6x^2 + tx + 1$, $\sigma(x) = (-x - 1)/(x - 1)$;
- (c) $P(x) = x^4 + (t^2 + 2t + 4)x^3 + (t^3 + 3t^2 + 4t + 6)x^2 + (t^3 + t^2 + 2t + 4)x + 1$,
 $\sigma(x) = (-x^3 - (t^2 + 2t + 4)x^2 - (t^3 + 3t^2 + 4t + 5)x - t^3 - t^2 - 2t - 2)/t$;
- (d) $P(x) = x^5 - t^2x^4 - (2t^3 + 6t^2 + 10t + 10)x^3 - (t^4 + 5t^3 + 11t^2 + 15t + 5)x^2 + (t^3 + 4t^2 + 10t + 10)x - 1$, $\sigma(x) = ((-t - 1)x^4 + (t^3 + 2t^2 + 3t + 3)x^3 + (t^4 + 4t^3 + 9t^2 + 14t + 8)x^2 - (t^4 + 7t^3 + 19t^2 + 29t + 19)x - t^4 - 6t^3 - 16t^2 - 20t - 9)/(t^3 + 5t^2 + 10t + 7)$; and
- (e) $P(x) = x^6 - 2tx^5 - (5t + 15)x^4 - 20x^3 + 5tx^2 + (2t + 6)x + 1$, $\sigma(x) = (-2x - 1)/(x - 1)$.

Proof. For (a), (b), and (e), the fact that (M) becomes a formal identity is easily checked by hand. The others can be checked with symbolic algebra software. The author (with Phil Carmody’s guidance) used Pari-GP. \square

REMARKS. The term “formal identity” means that substituting $\sigma(x)$ and its compositional powers into (M) gives a quotient of polynomials in which $P(x)$ divides the numerator. Using the linear fractional transformations $\sigma(x)$ which are independent of t in (a), (b), and (e) and their compositional powers, (M) evaluates to 0 at any x for which all the terms in the sum are defined.

Shanks’s 1974 paper [33] seems to be the first to refer to certain families of number fields as “simplest.” The polynomials in (a) had previously appeared in H. Cohn’s 1956 paper [5]. They yield explicit systems of independent units, a property which Shanks sought. When this system is fundamental, the regulator is very small. A great deal of research has been done on these cubics.

In [40], L. C. Washington extended Uchida’s work in [38], to force class numbers divisible by n in the “simplest” cubic fields, and used elliptic curves to describe the 2-Sylow subgroup of their class groups, and to exhibit explicit quartic extensions of these fields.

In [28], Patrick Morton gave a parameterization of cyclic cubic fields based on automorphism polynomials, and obtained Shanks’s simplest cubics by change of parameter. Robin Chapman simplified Morton’s proofs in [2].

E. Thomas proved in [37] that if r is a zero of a Shanks's simplest cubic, then $\langle r, r + 1 \rangle$ is a system of fundamental units for the order $\mathbb{Z}[r]$.

The fact that r and $r + 1$ are both units means that $-r$ and $r + 1$ are *exceptional units* (two units adding to 1) by Nagell's definition in [30]. D. Buell and V. Ennola studied the only other family of totally real cubic fields with exceptional units in [1]. In [20], H. Lenstra defined *exceptional sequences* as finite sequences (of algebraic integers), the difference of any two of which is a unit. Here, $0, 1, r + 1$ is a "normalized" 3-term exceptional sequence.

The polynomials in (b) were constructed in [9], while those in (e) were introduced (in a slightly different form) in [8]. The quintics in (d) are of the form $-f(-x)$ where $f(x)$ is one of the quintics introduced by E. Lehmer in [19]. She observed in this paper that the zeroes of the cubic polynomials in [33] and the quartic and sextic polynomials in [9] and [8] are, in the case of a prime conductor, integer translates of Gaussian periods, and obtained quintic units with the same property. These extensions were studied further in [7], [32], and [36]. In [32] (Appendix), R. Schoof and L. C. Washington proved that the integer-translates property characterizes the "simplest" fields of degrees 2, 3, and 4 with prime conductor. In [17], A. Lazarus applied the term "simplest" to these fields of degrees 2 to 6, as well as to the candidate family of "simplest" octic fields constructed by Y. Y. Shen in [34].

In Proposition 4.17, we show that for $t \in \mathbf{k} - \{0, -2\}$, the polynomial $P(x)$ in (c), which falls out from (M) and the conditions in Equation (1.4) with $n = 4$, defines the same extension of \mathbf{k} as the polynomial

$$(1.2) \quad f_t(x) = x^4 - t^2x^3 - (t^3 + 2t^2 + 4t + 2)x^2 - t^2x + 1$$

which L. C. Washington constructed in [41]. In [29], Patrick Morton proved the equivalence of these fields with cyclic quartic fields whose Galois groups have a quadratic (rather than cubic) generating automorphism polynomial.

In [33], Shanks also proposed "simplest" quadratic fields defined by the polynomials $x^2 = ax + 1$, $a \in \mathbb{Z}$. These quadratic fields have a special significance with respect to (M). For if $v^2 - tv - 1 = 0$, $t \in \mathbb{Z} - \{0, -2\}$, the cyclic quartic field L defined by $f_t(x)$ contains v , and if $G(L/\mathbb{Q}) = \langle \sigma \rangle$, we have

$$(1.3) \quad 1 + v + v\sigma(v) + v\sigma(v)\sigma^2(v) = 1 + v + (-1) + (-1)v = 0.$$

That is, quadratic units of norm -1 (unlike those of norm $+1$), satisfy (M) with $n = 4$ when embedded in a cyclic quartic field (such as a Washington's cyclic quartic field). We give a generalization of this phenomenon based on (M), in the Remarks just before Section 3.

The "simplest" number fields of degrees 3, 4, and 6 were studied further by G. Lettl, A. Pethő, and P. Voutier in [21] and [22]. A. Lazarus studied the unit groups and class numbers of the "simplest" quartic fields in [17].

L. C. Washington used coverings of modular curves in [41] to construct his family of cyclic quartic fields. He observed that the "simplest" fields of degree 2, 3, 4, 5, and 6 can be constructed by the same method. He also observed

that the “simplest” fields of degrees 2, 3, 4, and 6 arise from curves of genus 0. However, $X_1(16)$ (used for the cyclic quartic fields in [41]), and $X_1(13)$ (used for the cyclic sextic fields in [18]), have genus 2.

In [26], S. Louboutin obtained explicit formulas for powers of Gaussian sums attached to the “simplest” fields of degrees 2 to 6, and used them to give efficient computations of their class numbers.

In Section 3, we construct a 1-parameter family of polynomials of degree n in $\mathbb{Q}(\cot(\pi/n))[x]$, whose zeroes are permuted cyclically by a linear fractional transformation λ of compositional order n for each $n > 1$, generalizing a construction in [34]. When $n = 4m$, $\sigma = \lambda^{-1}$ makes (M) a formal identity.

For $n > 3$, we cannot simply “solve” (M) for σ as we did in (C3); but we would like to obtain as general an algebraic map of compositional order n as possible, that makes (M) a formal identity. So, we impose purely algebraic conditions which always hold when $\mathbf{k}(r)/\mathbf{k}$ is cyclic of degree n with $G = \langle \sigma \rangle$:

$$(1.4) \quad \sum_{i=0}^{\frac{n}{d}-1} \sigma^i \left(\prod_{j=0}^{d-1} \sigma^{\frac{jn}{d}}(r) \right) \in \mathbf{k} \quad \text{for each divisor } d \text{ of } n.$$

These conditions do not depend on the choice of cyclic generator for $\langle \sigma \rangle$ but (as mentioned after Proposition 1.2), do admit extensions in which L^σ is a proper extension of \mathbf{k} .

In Section 4.1, using only (M) for $n = 4$, the condition that the map σ fixes the ground field elementwise, the conditions (1.4), and elementary algebra, we obtain a linear fractional map (C4) for σ , and a 2-parameter family of monic octics $T(m, A, x)$ ($m, A \in \mathbf{k}$), with “generic” Galois group $G \cong {}_8T_{11}$ (see below). We let L/\mathbf{k} denote the splitting field of $T(m, A, x)$.

L/\mathbf{k} contains (Section 4.4) the elementary Abelian extension $E = \mathbf{k}(s, w, y)$ of \mathbf{k} , where $s^2 = m^2 - 4$, $w^2 = (m + 2 + A)^2 - 4(m - 2)$, and $y^2 = A^2 - 4(m - 2)$. When $[E : \mathbf{k}] = 8$, $G \cong {}_8T_{11}$, and E contains 7 quadratic extensions of \mathbf{k} . In Theorem 4.14 we describe the subgroups of ${}_8T_{11}$ fixing each of these.

The description of the Galois group is greatly facilitated by the fact that the related octics $T(m, A, x)$ and $T(m, -m - 2 - A, x)$ have the same splitting field over \mathbf{k} (Theorem 4.13). When $[L : \mathbf{k}] = 8$, at least one of these octics is a defining polynomial for the splitting field.

The cyclic quartics for which our map σ makes (M) a formal identity with $n = 4$, occur in pairs. They are the quartic factors of $T(m, A, x)$ in $\mathbf{k}(sw)[x]$, so typically define cyclic quartic extensions of $\mathbf{k}(sw)$, not of \mathbf{k} . But when $sw \in \mathbf{k}$, they are in $\mathbf{k}[x]$, and we call them *Murphy’s twins*. They “generically” define distinct cyclic quartic extensions of \mathbf{k} , both containing $\mathbf{k}(s)$. The “simplest” quartic fields and Washington’s cyclic quartic fields are “degenerate” cases (with $\mathbf{k} = \mathbb{Q}$) for which $s = 0$ and $w = 0$, respectively. In these cases, the “twins” are identical. We construct other Murphy’s twins extensions of \mathbb{Q} in Section 6.3 using standard results on norms from real quadratic fields.

By “collapsing” other quadratic extensions of \mathbf{k} in E , we construct families of polynomials defining normal octic extensions with $G \cong D_8(8)$, Q_8 , and $C_4 \times C_2$; and quartics with $G \cong D_4$ and V_4 .

The “generic” Galois group ${}_8T_{11}$ is an order-16 transitive subgroup of A_8 ; ${}_8T_{11} \cong \text{GAP small group } \langle 16, 13 \rangle$. It has one maximal subgroup $\cong Q_8$ (quaternion group), three $\cong D_4$, and three $\cong C_4 \times C_2$. It has presentation

$$\langle a, b, c \rangle : a^4 = b^2 = c^2 = 1, ab = ba, ac = ca, (bc)^2 = a^2.$$

It is also known as the “almost extraspecial group of order 16.” Derek Holt [14] described almost extraspecial p -groups as *central products*. A central product is an “amalgamated product” (see [10]), in which the subgroups being identified are in the centers of the factors. In this particular case,

$${}_8T_{11} \cong C_4 \vee D_4 \cong C_4 \vee Q_8.$$

Our octics produce (Propositions 5.3 and 5.4) some unusual sets of units and associates. Specifically (Proposition 5.3(b)), if $m, A \in \mathcal{O}_{\mathbf{k}}$ for a number field \mathbf{k} , and $m - 2 \in \mathcal{O}_{\mathbf{k}}^\times$, then each zero of $T(m, A, x)$ is part of an *exceptional sequence of three units*; that is, three units, the difference of any two of which is also a unit. In particular, $T(1, A, x)$ and $T(3, A, x)$, $A \in \mathbb{Z}$, are one-parameter families in $\mathbb{Z}[x]$ whose zeroes have this property. An infinite subfamily of $T(3, A, x)$ gives (Equations (6.3a)–(6.4b)) Murphy’s twins cyclic quartic fields with exceptional sequences of three units. When $\mathbf{k} = \mathbb{Q}$ and $m, A \in \mathbb{Z}$, there are (Section 6.5) explicit sets of 3 independent units. If $T(m, A, x)$ defines one or more number fields whose units groups have rank 3, these can produce rather small regulators. We also obtain (Equation (6.5)) a unit index formula which may be of interest.

The terms “exceptional units,” “exceptional sequences,” and “cliques” (of units) arose as follows: In [30], Nagell called either of two units whose sum is 1, *exceptional units*. In [20], Lenstra used finite sequences of algebraic integers, the difference of any two of which is a unit, to construct Euclidean fields. He observed that the first two terms can be “normalized” to 0 and 1 by applying an appropriate affine transformation, and any further terms become exceptional units. Thus, Lenstra’s sets generalize the concept of exceptional units. In [24], A. Leutbecher and J. Martinet called them *exceptional sequences*. In [23], Leutbecher entitled Section 1 “The graph of exceptional units.” This applied an idea of Györy, that defining two elements of a commutative ring R as being “connected” when their difference is a unit, induces a graph structure on R (see, e.g., [25], reference [G3]). Leutbecher and G. Niklasch used the graph-theoretic term “cliques” in this context in [25].

2. Basic formal properties

Proposition 1.1 gave some very simple formal properties implied by (M). We give several more. The first of these applies to non-Abelian extensions.

PROPOSITION 2.1. *Let L/\mathbf{k} be a finite Galois extension with Galois group G , and assume $\sigma \in G$, r , and n satisfy (M). If $\gamma \in C_G(\langle \sigma \rangle)$, then (M) holds for $\gamma(r)$.*

Proof. Since $\gamma\sigma = \sigma\gamma$, we have $\gamma(\sigma^i(r)) = \sigma^i(\gamma(r))$ for all i . □

Next, (M) does not depend on the choice of generator for $\langle \sigma \rangle$.

PROPOSITION 2.2. *Assume \mathbf{k}, L, r, σ , and $n > 2$ satisfy (M). If $1 \leq k < n$, $(k, n) = 1$, $\eta = \sigma^k$, and $y_k = r\sigma(r) \cdots \sigma^{k-1}(r)$, then*

$$1 + y_k + y_k\eta(y_k) + \cdots + y_k\eta(y_k) \cdots \eta^{n-2}(y_k) = 0.$$

Proof. The proof is left as an exercise. □

The y_k are zeroes of the degree- n polynomials

$$(2.1) \quad f_k(x) = \prod_{i=1}^n (x - \sigma^i(y_k)).$$

We thus obtain a set of $\varphi(n)$ polynomials. Mutually inverse generators of $\langle \sigma \rangle$ give polynomials with mutually reciprocal zeroes.

EXAMPLES. Applying Equation (2.1) to $f_1(x) = P(x)$ and σ as in Proposition 1.2(d), we obtain the alternate defining polynomial

$$f_2(x) = x^5 + (t^3 + 3t^2 + 5t + 5)x^4 - (t^4 + 3t^3 + 7t^2 + 5t + 5)x^3 - (t^4 + 5t^3 + 17t^2 + 25t + 25)x^2 - (2t^2 + 5t + 10)x - 1.$$

Equation (2.1) gives $f_2(x)$, $f_3(x) = -x^5 f_2(1/x)$, and $f_4(x) = -x^5 f_1(1/x)$ in addition to $f_1(x)$.

The zeroes of the octic polynomials studied by Y. Y. Shen in [34],

$$P(a, x) = x^8 - ax^7 - 28x^6 + 7ax^5 + 70x^4 - 7ax^3 - 28x^2 + ax + 1, \quad a \in \mathbb{Z},$$

are permuted cyclically by the compositional powers of the algebraic map

$$\sigma : x \mapsto (-\xi x - 1)/(x - \xi), \quad \text{taking } \xi \pmod{\xi^2 - 2\xi - 1}.$$

It may be verified directly that (M) with $n = 8$ is a formal identity for the compositional powers of this algebraic map. Shen showed that $P(a, x)$ defines cyclic octic fields when $a^2 + 64 \in 2\mathbb{Z}^2$, and investigated the properties of these fields. For such a , $P(a, x)$ is irreducible in $\mathbb{Q}[x]$ but *not* in $\mathbb{Q}(\xi)[x]$. In this case, the *automorphism* defined by σ maps ξ to $-1/\xi$, so its compositional powers occur in a different order than those of the algebraic map. If we assume that $\mathbf{k} = \mathbb{Q}(\xi)$, $a \in \mathcal{O}_{\mathbf{k}}$, and $P(a, x)$ is irreducible in $\mathbf{k}[x]$, then it defines a cyclic octic extension of \mathbf{k} with Galois group $\langle \sigma \rangle$, and its zeroes are *units* which satisfy (M) with $n = 8$. The coefficients of $f_3(x)$ and $f_5(x)$ as in Equation (2.1) are not formally in $\mathbb{Z}[a]$; for instance, the coefficient of x^5 in $f_3(x)$ is $(12a^2 + 768)\xi + a^3 - 12a^2 + 57a - 768$.

From Proposition 1.1(b) and (HT90), we see that if L/\mathbf{k} is cyclic of degree n with Galois group $\langle \sigma \rangle$, any r for which (M) holds is of the form $\alpha/\sigma(\alpha)$, $\alpha \in L$. If $\alpha = 1/z$, we have $r = \sigma(z)/z$. Substituting this expression into (M), the products in each term telescope, giving

$$(z + \sigma(z) + \cdots + \sigma^{n-1}(z))/z = 0.$$

The numerator is $\text{Tr}_{L/\mathbf{k}}(z)$. Thus, in a cyclic extension, an element which satisfies (M) is of the form $\sigma(z)/z$ where z is *in the kernel of the trace*. For cyclic extensions of degree $n > 2$, primitive elements of this form always exist.

PROPOSITION 2.3. *Let L/\mathbf{k} be a cyclic extension of degree n with $G = \langle \sigma \rangle$. If $n > 2$, there is $r \in L$ with $L = \mathbf{k}(r)$ for which (M) holds.*

Proof. The kernel of the trace from L to \mathbf{k} is a \mathbf{k} -vector subspace V of L of dimension $n - 1$. The distinct elements $\sigma(z)/z$, $z \in V - \{0\}$, correspond in an obvious way to the 1-dimensional \mathbf{k} -subspaces of V .

If \mathbf{k} is a finite field with q elements, the number of such subspaces is

$$(q^{n-1} - 1)/(q - 1) = 1 + q + \cdots + q^{n-2}.$$

The usual formula for the number of monic irreducible polynomials of degree n in $\mathbf{k}[x]$ shows that there are fewer than

$$1 + q + \cdots + q^{n/\ell}$$

nonzero *nonprimitive* elements for L/\mathbf{k} , where ℓ is the least prime factor of n . If $n > 2$, the number of distinct elements $\sigma(z)/z$, $z \in V - \{0\}$, is thus too large for them all to be nonprimitive.

Now suppose \mathbf{k} is infinite and $n > 2$. Let $d \mid n$, $d > 1$, and suppose $z \in V$ with $\sigma(z)/z = w \in F$, where F is the intermediate field with L/F of degree d . Then $\mathcal{N}_{F/\mathbf{k}}(w) = \sigma^{n/d}(z)/z = W$; $\mathcal{N}_{L/\mathbf{k}}(W) = W^n = 1$; there are at most n such W . The distinct values of $\sigma^{n/d}(z)/z$ correspond to one-dimensional F -vector subspaces of L ; these are n/d -dimensional \mathbf{k} -vector spaces. Now $n/d < n - 1$ for $d > 1$ and $n > 2$. Thus, the values $r = \sigma(z)/z$, $z \in V$, for which $\mathbf{k}(r) \neq L$, correspond to the 1-dimensional subspaces of a finite union of *proper* \mathbf{k} -subspaces of V . It is well known that when \mathbf{k} is infinite, no \mathbf{k} -vector space is a finite union of proper subspaces, and the result follows. \square

REMARKS. If $r = \sigma(z)/z$ satisfies (M), then in Proposition 2.2, $y_k = \sigma^k(z)/z$.

When the ground field \mathbf{k} contains a primitive m th root of unity $\zeta \neq 1$, $F = \mathbf{k}(r)$ is a cyclic extension of degree d , and $\mathcal{N}_{F/\mathbf{k}}(r) = \zeta$, then embedding F in a cyclic extension of degree md over \mathbf{k} forces r to satisfy (M) with $n = md$, as occurs in Equation (1.3) with $m = 2$ and $\zeta = -1$.

If \mathbf{k} is a number field and r is a ‘‘Murphy’s unit,’’ we may take z so its σ -conjugates are *associate algebraic integers in the kernel of the trace*. Among cyclic cubic extensions of \mathbb{Q} , only Shanks’s simplest cubic fields possess such z .

If $P(r) = 0$ in Proposition 1.2(a), $r = \sigma(z)/z$ for $z = 2r^2 - (2t + 1)r - t - 4$. Noting that $(t^2 + 3t + 9)^2 = \text{disc}(x^3 - tx^2 - (t + 3)x - 1)$, we have

$$z^3 - (t^2 + 3t + 9)z + t^2 + 3t + 9 = 0.$$

3. Shen’s polynomials

In [34], Y. Y. Shen constructed polynomials of 2-power degree, generalizing the 1-parameter family of defining polynomials for the octic fields he investigated. His construction used the polynomials

$$(3.1) \quad Q_n(x) = \Re((x + i)^n) \quad \text{and} \quad V_n(x) = \Im((x + i)^n).$$

His work (Proposition 3(b) and (c) of [34]) shows that for $n = 2$ and $n = 4$, the polynomials

$$(3.2) \quad P_n(a, x) = Q_n(x) - \frac{a}{n}V_n(x)$$

give the usual defining polynomials for the “simplest” quadratic and quartic fields. He obtained his octic fields by investigating (3.2) for $n = 8$, and observed that taking $n = 2^k$ gives a 1-parameter family with similar properties.

Here, we generalize Shen’s construction to polynomials of degree n for all $n > 1$. Clearly

$$Q_n(x) = \sum_{0 \leq 2k \leq n} (-1)^k \binom{n}{2k} x^{n-2k},$$

$$V_n(x) = \sum_{0 \leq 2k < n} (-1)^k \binom{n}{2k+1} x^{n-2k-1}$$

so $Q_n(x)$ is monic of degree n , and $V_n(x)$ is of degree $n - 1$, with leading coefficient n . The gcd of the coefficients of $V_n(x)$ is $2^{v_2(n)}$, where $n/2^{v_2(n)}$ is an odd integer. The proof is left as an exercise. We take the following.

DEFINITION 3.1. With $Q_n(x)$ and $V_n(x)$ as in Equation (3.1), and $n \in \mathbb{Z}^+$, set

$$(3.3) \quad P_n(a, x) = Q_n(x) - \frac{a}{2^{v_2(n)}}V_n(x).$$

Then $P_n(a, x)$ is monic of degree n . For $0 \leq k \leq n$, the coefficient of x^{n-k} is in \mathbb{Z} if k is even, and in $\mathbb{Z} \cdot a$ if k is odd. Clearly $V_{2n}(x) = 2Q_n(x)V_n(x)$, so $V_{2^t}(x) = 2^t Q_1(x)Q_2(x) \cdots Q_{2^{t-1}}(x)$ for $t \in \mathbb{Z}^+$ by induction. Thus, when $n = 2^t$, $P_n(a, x)$ coincides with the degree- 2^t polynomial constructed in [34]. For each $n > 1$, Definition 3.1 gives a 1-parameter family of monic polynomials in $\mathbb{Z}[a][x]$. The useful formula

$$(3.4) \quad Q_n(\cot(\theta))/V_n(\cot(\theta)) = \cot(n\theta) \quad \text{for all } n \in \mathbb{Z}^+$$

follows directly from Equation (3.1). Taking $x = \cot(\theta)$, we see that the zeroes of $P_n(a, x)$ are permuted cyclically by the linear fractional transformation λ

in (3.6). The results in this section are thus very similar to those in [31], but are less general. We assume our parameter to be in the complex field \mathbb{C} . This allows us to “cheat” by invoking properties of $\cot(\theta)$ as a periodic meromorphic function, and to exploit the fact that λ corresponds to adding a division point to the argument of this function. Also, we assume the ground field (field of coefficients of λ) contains $K = \mathbb{Q}(\cot(2\pi/n))$ rather than $k = \mathbb{Q}(\cos(2\pi/n))$ as in [31]. Restricting our parameter to \mathcal{O}_K , produces polynomials of degree n in $K[x]$ which define cyclic extensions of K with several formal properties like those of the “simplest” quartic fields (Theorem 3.3). It is easily shown that $K = \mathbb{Q}(\cos(2\pi/N))$ where $N = \text{lcm}(n, 4)$. Thus, $K = k$ as in [31] precisely when $4 \mid n$. In this case, Proposition 3.5 shows that (M) holds with $\sigma = \lambda^{-1}$. The irreducible factors of $P_n(a, x)$ in $K[x]$ all have the same degree $d \mid n$. If d is a proper divisor of n , the “rescaled” polynomial of degree n/d splits into linear factors in $K[x]$ (Proposition 3.4).

The following properties are easily obtained.

PROPOSITION 3.1. *Let $n \in \mathbb{Z}^+$ and $a \in \mathbb{C}$.*

- (a) *If $a \neq \pm 2^{v_2(n)}i$, $P_n(a, x)$ has n distinct zeroes $x = \cot(\theta)$ for which $\cot(n\theta) = a/2^{v_2(n)}$. The zeroes are real if a is real. If $\cot(\theta_1)$ is one of them, all are given by $\cot(\theta_1 + k\pi/n)$, $0 \leq k \leq n - 1$.*
- (b) *The discriminant of $P_n(a, x)$ is $n(2^{n-2-2v_2(n)}n)^{n-1}(a^2 + 4^{v_2(n)})^{n-1}$.*
- (c) *If $P_n(a, \cot(\theta)) = 0$, then $Q_k(\cot(\theta)) - \cot(k\theta)V_k(\cot(\theta)) = 0$ for $k \in \mathbb{Z}^+$.*
- (d) *If $P_n(a, \cot(\theta)) = 0$ and $D \mid n$, $P_{n/D}(a/2^{v_2(D)}, x) = 0$ for*

$$x = \cot(D\theta + kD\pi/n), \quad 0 \leq k \leq n/D - 1.$$

Proof. For (a), let $z = a/2^{v_2(n)}$. For $z \in \mathbb{C} - \{\pm i\}$ there is θ for which $\cot(n\theta) = z$; we have $(0, \pi/n) \mapsto \mathbb{R}$. Then, $\cot(n\theta)$ has period π/n .

For (b), $\text{disc}(P_n(a, x)) = f(a) \in \mathbb{Z}[a]$. By construction, $P_n(a, x) = (x \mp i)^n$ when $a = \pm 2^{v_2(n)}i$. By (a), $f(a)$ has no linear factors in $\mathbb{C}[a]$ other than $a \pm 2^{v_2(n)}i$, so is a constant multiple of $(a^2 + 4^{v_2(n)})^{n-1}$. Taking $a = 0$, we see the constant is the discriminant of $Q_n(x)$ divided by $4^{(n-1)v_2(n)}$. The identity $Q'_n(x) = nQ_{n-1}(x)$ aids in evaluating the discriminant of $Q_n(x)$.

Finally, (c) and (d) follow easily from Equation (3.4). □

We recover the following well-known identities by comparing the coefficient of x^{n-1} and the constant term of $P_n(a, x)$ with the formula $\cot(n\theta) = a/2^{v_2(n)}$ in Proposition 3.1(a). They are valid if $n\theta$ is not an integer multiple of π .

$$(3.5a) \quad \text{If } n \in \mathbb{Z}^+, \text{ then } \sum_{k=0}^{n-1} \cot(\theta + k\pi/n) = n \cot(n\theta).$$

$$(3.5b) \quad \text{If } n \in \mathbb{Z}^+ \text{ is odd, then } \prod_{k=0}^{n-1} \cot(\theta + k\pi/n) = (-1)^{(n-1)/2} \cot(n\theta).$$

Let $n > 1$, $x = \cot(\theta)$ and $\xi = \cot(\pi/n)$. Then the map

$$(3.6) \quad T : \cot(\theta) \mapsto \cot(\theta + \pi/n) \text{ becomes } \lambda : x \mapsto \frac{\xi x - 1}{x + \xi}.$$

Since T has compositional order n , so does λ . Using Equation (3.4) and Proposition 3.1(a), we have

$$(3.7) \quad \sum_{k=0}^{n-1} \lambda^{(k)}(x) = nQ_n(x)/V_n(x),$$

where $\lambda^{(k)}$ is the k th compositional power of λ . Also,

COROLLARY 3.2. *Let $n \in \mathbb{Z}$, $n > 1$, and $a \in \mathbb{C}$. If $\xi = \cot(\pi/n)$, the zeroes of $P_n(a, x)$ are permuted cyclically by the linear fractional map λ in Equation (3.6).*

Proof. If $a \neq \pm 2^{v_2(n)}i$, this just restates Proposition 3.1(a). If $a = \pm 2^{v_2(n)}i$, the n -fold zero $\pm i$ of $P_n(a, x) = (x \mp i)^n$ is fixed by λ . □

Now, let $n \in \mathbb{Z}$, $n > 1$ and $K = \mathbb{Q}(\cot(\pi/n))$. We have the following result.

THEOREM 3.3. *Let $n \in \mathbb{Z}$, $n > 1$, $K = \mathbb{Q}(\cot(\pi/n))$, and $\alpha \in \mathcal{O}_K$.*

- (a) *The zeroes of $P_n(\alpha, x)$ are algebraic integers (units, if n is even).*
- (b) *If L/K is the splitting field of $P_n(\alpha, x)$, then L/\mathbb{Q} is totally real.*
- (c) *There is a divisor d of n such that, if $f(x)$ is any irreducible factor of $P_n(\alpha, x)$ in $K[x]$, $f(x)$ has degree d .*
- (d) *If $f(x)$ and $d \mid n$ are as in (c), then $G(L/K) = \langle \sigma \rangle$, where the restriction of σ to the zeroes of $f(x)$ is given by $\lambda^{(n/d)}$, λ as in Equation (3.6).*

Proof. For (a), $P_n(\alpha, x)$ is a monic polynomial in $\mathcal{O}_K[x]$ by Definition 3.1. If n is even, the constant term is $(-1)^{n/2}$.

For (b), the zeroes of $P_n(\alpha, x)$ are all real by part (a) of Proposition 3.1. Since K is totally real, the zeroes of $P_n(\alpha', x)$ are also all real, for each conjugate α' of α in K/\mathbb{Q} .

For (c), let $P_n(\alpha, r) = 0$. By Corollary 3.2, the rest of the zeroes are given by $\lambda^{(k)}(r)$, $1 \leq k \leq n - 1$, λ as in Equation (3.6). Now λ is defined by a matrix in $\mathbf{PGL}_2(K)$, so all zeroes of $P_n(\alpha, x)$ determine the same extension of K . Thus, its irreducible factors in $K[x]$ all have the same degree.

For (d), let $f(x)$ be a monic irreducible factor of $P_n(\alpha, x)$ in $K[x]$, and $d \mid n$ its degree. Let $f_k(x) = (x + \cot(k\pi/n))^d f(\lambda^{(k)}(x))$ “made monic” (divided by its lead coefficient).

If $f_{j+k}(x) = f_j(x)$, clearly $\lambda^{(k)}$ has compositional order d at most. So $f(x) = f_0(x)$, $f_1(x), \dots, f_{n/d-1}(x)$ are all distinct, and $f_{n/d}(x) = f(x)$. The rest is now self-evident. □

The integer d in Theorem 3.3(c) has some additional properties.

PROPOSITION 3.4. *Let n, α , and $P_n(\alpha, x)$ be as in Theorem 3.3, with d as in Theorem 3.3(c).*

- (a) *If $P_n(\alpha, \cot(\theta)) = 0$, then $\cot(d\theta) \in K$.*
- (b) *$P_{n/d}(\alpha/2^{v_2(d)}, x)$ splits into linear factors in $K[x]$.*
- (c) *d is the least positive integer for which (a) and (b) hold.*

Proof. Let $f(x)$ be the minimum polynomial for $\cot(\theta)$ in $K[x]$. By Theorem 3.3(d), the zeroes of $f(x)$ are $\cot(\theta + k\pi/d)$, $0 \leq k < d$. By the identity in Equation (3.5a), the sum of these zeroes is $d \cot(d\theta)$, proving (a). We have $f(x) = Q_d(x) - \cot(d\theta)V_d(x)$ by Proposition 3.1(c); d is the least positive integer with $\cot(d\theta) \in K$ since $f(x)$ is irreducible in $K[x]$. By the formula in Proposition 3.1(d), the zeroes of $P_{n/d}(\alpha/2^{v_2(d)}, x)$ are $\cot(d\theta + k\pi d/n)$, $0 \leq k < n/d$. These are all in K because $\cot(d\theta) \in K$ and $\cot(k\pi d/n) \in K$. \square

When $4 \mid n$, $\langle \lambda \rangle$ has generators which make (M) a formal identity.

PROPOSITION 3.5. *Let λ be as in Equation (3.6), $m \in \mathbb{Z}^+$, and $n = 4m$. If $S(x) = 1 + x + x\lambda^{-1}(x) + x\lambda^{-1}(x)\lambda^{-2}(x) + \dots + x\lambda^{-1}(x)\dots\lambda^{-(4m-2)}(x)$, then $S(x) \equiv 0$; that is, if $\sigma^k(r) = \lambda^{-k}(r)$, (M) becomes a formal identity.*

Proof. $S(x)$ is a rational function, which clearly has a partial fraction decomposition of the form

$$S(x) = A + Bx + \sum_{k=1}^{4m-2} \frac{c_k}{(x - \cot(k\pi/4m))},$$

where $A, B, c_k \in \mathbb{Q}(\cot(\pi/4m))$. Using $\lambda^{2m}(x) = -1/x$, we find

$$B = \sum_{k=0}^{2m-1} \prod_{j=1}^k (-\cot(j\pi/4m)).$$

Since $\cot(\pi/2 - \theta) = 1/\cot(\theta)$, the terms of index k and $2m - 1 - k$ are equal and opposite. Thus $B = 0$, so $S(x)$ is bounded as $x \rightarrow \infty$.

Now $\lambda^{-(4m-1)}(x) = \lambda(x)$. Using $\lambda^{j+2m}(x) = -1/\lambda^j(x)$, we find that $\lambda(x)S(x) = S(\lambda(x))$. Letting $x \rightarrow \infty$, we see $S(\lambda(x))$ remains bounded as $\lambda(x) \rightarrow \cot(\pi/4m)$, so $c_1 = 0$. Then, $\lambda^{(k)}(x)S(\lambda^{k-1}(x)) = S(\lambda^{(k)}(x))$, giving $c_k = 0$ for $1 \leq k \leq 4m - 2$. Thus, $S(x) = A$, a constant. Substituting $x = i$, which is fixed by λ , we find $S(i) = 0$. \square

Next, we evaluate the sums in Equation (1.4) with $\sigma = \lambda$. If d (and hence n) is even, the factors in each term occur in negative-reciprocal pairs, so the sum is $(-1)^{d/2}(n/d)$. If d is odd, we use Equation (3.5b) to evaluate each term, then Equation (3.5a) to add them; the result is $(-1)^{(d-1)/2}na/(d \cdot 2^{v_2(n)})$.

Let $n = 4m$, $a \in \mathbb{Z}$. Since $\cot(\pi/n) \in \mathbb{Q}(r, \lambda(r))$ if $P_n(a, r) = 0$, the splitting field of $P_n(a, x)$ over \mathbb{Q} has degree divisible by $\varphi(2m)$. If $\mathbb{Q}(r)/\mathbb{Q}$ is normal of degree n , then $\varphi(2m) \mid 4m$. If d is the degree over K , then d divides

$4m/\varphi(2m)$. If $n = 2^t$, $t \geq 4$, then $d \mid 4$, and it can be shown using Proposition 3.4(b) that $\mathbb{Q}(r)/\mathbb{Q}$ is a normal extension of degree $n = 2^t$ only if $a = \pm n$ and $\mathbb{Q}(r) = \mathbb{Q}(2 \cos(\pi/4n))$, or if $n = 16$ and $a = \pm 16 \cdot 239$.

4. The condition (M) with $n = 4$

Assuming that (M) and the conditions in Equation (1.4) hold, that the map σ fixes the ground field \mathbf{k} elementwise and that $1 + 1 \neq 0$ in \mathbf{k} , we only need elementary algebra, treating σ as a field isomorphism as needed, to produce a complete description of σ as an algebraic map, in terms of two parameters m and A in \mathbf{k} .

4.1. The formal construction. When $n = 4$, (M) becomes

$$(M4) \quad 1 + r + r\sigma(r) + r\sigma(r)\sigma^2(r) = 0.$$

The condition of Equation (1.4) with $d = 2$ is

$$r\sigma^2(r) + \sigma(r)\sigma^3(r) = m \in \mathbf{k}.$$

By Proposition 1.1, $r\sigma(r)\sigma^2(r)\sigma^3(r) = 1$, so taking $r\sigma^2(r) = u$, we have

$$(4.1a) \quad \sigma(u) = u^{-1}, \quad \text{where}$$

$$(4.1b) \quad u^2 - mu + 1 = 0, \quad m \in \mathbf{k}.$$

Substituting u for $r\sigma^2(r)$ in (M4), solving for $\sigma(r)$, repeatedly applying σ , and keeping in mind Equation (4.1a), we obtain the expressions

$$(C4) \quad \sigma(r) = \frac{-r - 1}{r + u}, \quad \sigma^2(r) = \frac{u}{r}, \quad \sigma^3(r) = \frac{-u^{-1}r - 1}{r + 1}, \quad \sigma^4(r) = r.$$

The product of the four expressions in (C4) is 1 as required by Proposition 1.1. The expression for $\sigma^2(r)$ is of the same form as that for a “relative” unit in a cyclic quartic extension, as given in [41], Section 2. These expressions are also strikingly similar to those in [23], Section 1.1.

Because of Equation (4.1a), for the map σ to fix \mathbf{k} elementwise it is necessary that either

$$(4.2) \quad m = \pm 2 \quad \text{or} \quad m^2 - 4 \notin \mathbf{k}^2.$$

Substituting (C4) into the condition in Equation (1.4) with $d = 1$,

$$(Tr4) \quad r + \frac{-r - 1}{r + u} + \frac{u}{r} + \frac{-u^{-1}r - 1}{r + 1} = A, \quad A \in \mathbf{k}.$$

Clearing fractions and collecting terms, we find that r is a zero of

$$(Q4) \quad p(x) = x^4 + (u - u^{-1} - A)x^3 + ((2 - A)u - A - 4)x^2 + (u^2 - 1 - Au)x + u^2,$$

where $m, A \in \mathbf{k}$, and Equations (4.1b) and (4.2) hold. By Equation (4.2), $p(x) \in \mathbf{k}[x]$ only if $m = \pm 2$. We treat those cases first.

4.2. The cases $m = \pm 2$. If $m = 2$ then $u = 1$, the formal expressions for $\sigma(r)$ and $\sigma^3(r)$ become -1 , and we obtain

$$p(x) = (x + 1)^2(x^2 - (A + 2)x + 1).$$

In this case the transformation σ is *not* a field automorphism of order 4, but if $(A + 2)^2 - 4 \notin \mathbf{k}^2$, the quadratic factor is irreducible in $\mathbf{k}[x]$, and the expression $1/r$ for $\sigma^2(r)$ does define an automorphism of order 2. The substitution $A \leftarrow -4 - A$ changes the signs of the zeroes of the quadratic factor.

If $m = -2$ then $u = -1$, and the expressions in (C4) become

$$(C4') \quad \sigma(r) = \frac{-r - 1}{r - 1}, \quad \sigma^2(r) = -\frac{1}{r}, \quad \sigma^3(r) = \frac{r - 1}{r + 1}, \quad \text{and} \quad \sigma^4(r) = r.$$

Substituting $u = -1$ into Equation (Q4) gives

$$(Q4') \quad p(x) = x^4 - Ax^3 - 6x^2 + Ax + 1.$$

Apart from parameter name, this is the $P(x)$ in Proposition 1.2(b) for the “simplest” quartic fields. If $\mathbf{k} = \mathbb{Q}$ and $A \in \mathbb{Z}$, $p(x)$ is irreducible in $\mathbb{Z}[x]$ unless $A \in \{-3, 0, 3\}$. In general, $p(x)$ is irreducible in $\mathbf{k}[x]$ when $A^2 + 16 \notin \mathbf{k}^2$. The Galois group acts on the zeroes as in (C4'). The polynomials $f_1(x) = p(x)$ and $f_3(x)$ in Proposition 2.2, are related by the change of parameter $A \leftarrow -A$.

4.3. Fundamental identities when $m^2 - 4 \notin \mathbf{k}^2$. When $m^2 - 4 \notin \mathbf{k}^2$, however, the $p(x)$ in (Q4) is *not* in $\mathbf{k}[x]$. Conjugating the coefficients in $\mathbf{k}(u)/\mathbf{k}$ gives a polynomial $\bar{p}(x) \neq p(x)$. Assuming σ is a field automorphism, and $p(r) = 0$, $\sigma(r)$ is a zero of $\bar{p}(x)$ rather than of $p(x)$. But *that* would imply an algebraic relation between $\bar{p}((-x - 1)/(x + u))$ and $p(x)$. And there is indeed such a relation:

THEOREM 4.1. *Let $m, A \in \mathbf{k}$, $m^2 - 4 \notin \mathbf{k}^2$, and $u, p(x)$ and $\bar{p}(x)$ be as above. Then*

$$(x + u)^4 \bar{p}\left(\frac{-x - 1}{x + u}\right) = (m - 2)p(x).$$

Proof. Straightforward polynomial algebra. □

Although $p(x) \notin \mathbf{k}[x]$, clearly

$$(4.3) \quad T(m, A, x) = p(x)\bar{p}(x)$$

is a monic octic in $\mathbb{Z}[m, A][x]$ (hence in $\mathbf{k}[x]$ since $m, A \in \mathbf{k}$), with constant term 1. Theorem 4.1 gives two important properties of $T(m, A, x)$:

COROLLARY 4.2. *Let m, A , and $p(x)$ be as above.*

- (a) $p(x)$ and $\bar{p}(x)$ have the same splitting field L over $\mathbf{k}(u)$, which is the splitting field of $T(m, A, x)$ over \mathbf{k} .
- (b) $\text{disc}(T(m, A, x)) \in \mathbf{k}^2$.

Proof. Adjoining to $\mathbf{k}(u)$ either the zero r of $p(x)$, or the zero $(-r - 1)/(r + u)$ of $\bar{p}(x)$, define the *same* extension of $\mathbf{k}(u)$. The join of the extensions defined by all the zeroes thus gives the same splitting field for both quartics.

For (b), a standard formula (see, for instance, [3], Corollary 3.3.6) gives

$$\text{disc}(T(m, A, x)) = \text{disc}(p(x)) \text{disc}(\bar{p}(x)) [\text{Res}(p(x), \bar{p}(x))]^2.$$

The arguments of the resultant are interchanged by conjugation in $\mathbf{k}(u)/\mathbf{k}$, but also by 4 (evenly many) row interchanges of the Sylvester’s matrix, which leaves its determinant unchanged. Therefore, the resultant is in \mathbf{k} .

If we differentiate the identity in Theorem 4.1, evaluate at the zeroes of $p(x)$, and multiply, we obtain $\text{disc}(p(x)) = u^6 \text{disc}(\bar{p}(x))$. The discriminants are conjugate in $\mathbf{k}(u)/\mathbf{k}$ and u has norm 1, so $\text{disc}(p(x)) = cu^3$ for some $c \in \mathbf{k}$. Thus, $\text{disc}(p(x)) \text{disc}(\bar{p}(x)) = c^2 \in \mathbf{k}^2$, and the proof is complete. \square

The situation is further simplified by the fact that $p(x)$ is a generalized reciprocal polynomial.

LEMMA 4.3. *If $m, A \in \mathbf{k}$, $u^2 - mu + 1 = 0$, and $p(x)$ is as in (Q4), then*

$$\frac{p(x)}{x^2} = \frac{p(u/x)}{(u/x)^2}.$$

Proof. Straightforward polynomial algebra. \square

We use Lemma 4.3 to split $p(x)$ into quadratic factors.

LEMMA 4.4. *With $p(x)$ as in Equation (Q4) and $Y = x + u/x$,*

- (a) $p(x) = x^2 F(Y)$, where $F(Y) = Y^2 + (-A + u - u^{-1})Y - A - 4 - Au$.
- (b) $\text{disc}(F(Y)) = (m + 2 + A)^2 - 4(m - 2)$.
- (c) *If $F(Y) = (Y - \alpha)(Y - \beta)$, then*

$$p(x) = q_1(x)q_2(x) = (x^2 - \alpha x + u)(x^2 - \beta x + u).$$

- (d) *Let $Q_1 = q_1(-1) = 1 + \alpha + u$, $Q_2 = q_2(-1) = 1 + \beta + u$. Then*
 - (i) $\text{disc}(x^2 - \alpha x + u) = \alpha^2 - 4u = Q_1(Q_1 + Q_2u - 2(u + 1))$.
 - (ii) $\text{disc}(x^2 - \beta x + u) = \beta^2 - 4u = Q_2(Q_2 + Q_1u - 2(u + 1))$.

Proof. Straightforward polynomial algebra. \square

We then have

PROPOSITION 4.5. *Let $m, A \in \mathbf{k}$, $m^2 - 4 \notin \mathbf{k}^2$, and L/\mathbf{k} the splitting field of $T(m, A, x)$. Then $[L : \mathbf{k}]$ divides 16.*

Proof. Since $p(x)$ splits into quadratic factors in a quadratic extension (at most) of its coefficient field $\mathbf{k}(u)$, its splitting field over $\mathbf{k}(u)$ has degree dividing 8, and $\mathbf{k}(u)$ has degree 2 over \mathbf{k} . Corollary 4.2(a) then gives the result. \square

4.4. The quadratic and quartic factors of $T(m, A, x)$. Note that $\text{disc}(F(Y)) \in \mathbf{k}$ in Lemma 4.4(b), even though $F(Y) \notin \mathbf{k}[Y]$. The quadratic factors $q_1(x)$ and $q_2(x)$ of $p(x)$ in Lemma 4.4(c) (and, obviously, the corresponding factors of $\bar{p}(x)$) are in $\mathbf{k}(s, w)[x]$, where

$$(4.4) \quad s^2 = m^2 - 4 \quad \text{and} \quad w^2 = (m + 2 + A)^2 - 4(m - 2) = \text{disc}(F(Y)).$$

Taking $u = (m + s)/2$ and solving $F(Y) = 0$ by quadratic formula, we find the quadratic factors of $T(m, A, x)$ in $\mathbf{k}(s, w)[x]$ are

$$(4.5a) \quad q_1(x) = x^2 + (-A + s - w)x/2 + (m + s)/2,$$

$$(4.5b) \quad q_2(x) = x^2 + (-A + s + w)x/2 + (m + s)/2,$$

$$(4.5c) \quad q_3(x) = x^2 + (-A - s + w)x/2 + (m - s)/2, \quad \text{and}$$

$$(4.5d) \quad q_4(x) = x^2 + (-A - s - w)x/2 + (m - s)/2.$$

These choices give $q_1(x)q_2(x) = p(x)$ and $q_3(x)q_4(x) = \bar{p}(x)$. By regrouping the $q_i(x)$ in pairs, we see that $T(m, A, x)$ splits into quartic factors in $\mathbf{k}(s)[x]$, $\mathbf{k}(sw)[x]$, and $\mathbf{k}(w)[x]$, namely

$$(4.6a) \quad P_s(x) = p(x) = q_1(x)q_2(x) \quad \text{and} \quad \bar{P}_s(x) = \bar{p}(x) = q_3(x)q_4(x),$$

$$(4.6b) \quad P_{sw}(x) = q_1(x)q_3(x) \quad \text{and} \quad \bar{P}_{sw}(x) = q_2(x)q_4(x), \quad \text{and}$$

$$(4.6c) \quad P_w(x) = q_1(x)q_4(x) \quad \text{and} \quad \bar{P}_w(x) = q_2(x)q_3(x).$$

We have the following refinement of Theorem 4.1.

LEMMA 4.6. *Let $m, A \in \mathbf{k}$, $m^2 - 4 \notin \mathbf{k}^2$, and notation as above. Then*

- (a) $(x + u)^2 q_3(\frac{-x-1}{x+u}) = q_3(-1)q_1(x)$;
- (b) $\text{disc}(q_1(x)) \text{disc}(q_3(x)) = ((u - 1) \text{disc}(q_3(x))/q_3(-1))^2$;
- (c) $(u - 1) \text{disc}(q_3(x))/q_3(-1) = ((m + A - 2)s + (2 - m)w)/2$.

Proof. For (a), there clearly must be an algebraic relation between $q_1(x)$ and either $q_3((-x - 1)/(x + u))$ or $q_4((-x - 1)/(x + u))$. By Equations (4.5a) and (4.5c), the coefficient of x^3 in $q_1(x)q_3(x)$ is $-A$ in agreement with (Tr4). Routine though tedious algebra completes the verification.

For (b), differentiate (a), evaluate at the zeroes, and multiply.

Finally, (c) may be verified algebraically. □

REMARKS. The substitution $w \leftarrow -w$ replaces $(q_1(x), q_3(x))$ with $(q_2(x), q_4(x))$ in (a)–(c). Multiplying (a) by its counterpart gives Theorem 4.1.

To formulate s (thus $u = (m + s)/2$ and σ) in the arithmetic of $\mathbf{k}[x]$, let

$$(4.7a) \quad P_s(x) = p(x) = \mathcal{A}(x)s + \mathcal{B}(x),$$

where $\mathcal{A}(x), \mathcal{B}(x) \in \mathbf{k}[x]$, and take

$$(4.7b) \quad s \equiv -\mathcal{B}(x)/\mathcal{A}(x) \pmod{T(m, A, x)},$$

if $\text{Res}(\mathcal{A}(x), T(m, A, x)) \neq 0$.

Similarly, writing

$$(4.8a) \quad P_w(x) = \mathcal{C}(x)w + \mathcal{D}(x),$$

where $\mathcal{C}(x), \mathcal{D}(x) \in \mathbf{k}[x]$, we can take

$$(4.8b) \quad w \equiv -\mathcal{D}(x)/\mathcal{C}(x) \pmod{T(m, A, x)}$$

if $\text{Res}(\mathcal{C}(x), T(m, A, x)) \neq 0$.

We can express these resultants in terms of the “monster” resultant norm

$$(4.9) \quad \mu = \text{Res}(q_1(x), q_3(x)) \text{Res}(q_2(x), q_4(x)).$$

Direct calculation (using plenty of computing power) gives the results

$$(4.10a) \quad \text{Res}(\mathcal{A}(x), T(m, A, x)) = (m - 2)^2 \mu^2 / 256, \quad \text{and}$$

$$(4.10b) \quad \text{Res}(\mathcal{C}(x), T(m, A, x)) = \mu^2 / 256.$$

LEMMA 4.7. *Let $m, A \in \mathbf{k}$, and either $m = \pm 2$ or $m^2 - 4 \notin \mathbf{k}^2$, with the $q_i(x)$ as in Equations (4.5a)–(4.5d) and μ as in Equation (4.9). Then,*

$$\mu \neq 0 \quad \text{unless}$$

$$(m, A) = (2, -4); (-2, 4i) \text{ or } (-2, -4i) \text{ if } -1 \in \mathbf{k}^2; \text{ or } (2/3, -4/3) \text{ if } -2 \notin \mathbf{k}^2.$$

Proof. If $\text{Res}(q_1(x), q_3(x)) = 0$, then $q_1(x)$ and $q_3(x)$ have a common factor in $\mathbf{k}(s, w)[x]$, which is also a factor of every $\mathbf{k}(s, w)$ -linear combination of $q_1(x)$ and $q_3(x)$. Then, by Lemma 4.6(a), $q_u(x) = x^2 + (1 + u)x + 1$ or $\bar{q}_u(x) = x^2 + (u^{-1} + 1)x + 1$ has a common factor of degree ≥ 1 with $q_1(x)$ and $q_3(x)$, because any common zero of $q_1(x)$ and $q_3(x)$ is a fixed point of (at least) one of the linear fractional transformations $\sigma(x)$ or $\sigma^3(x)$ in Equation (C4).

There can be no common factor of degree ≥ 1 unless $\{q_1(x), q_u(x), q_3(x)\}$ or $\{q_1(x), \bar{q}_u(x), q_3(x)\}$ is a linearly dependent set in the $\mathbf{k}(s, w)$ vector space V with basis $\{1, x, x^2\}$. Let $v = (q_1(x), q_u(x), q_3(x)) \in V^3$, and let M be the 3×3 matrix whose i, j entry is the coefficient of x^{3-j} in v_i . The entries of v are linearly dependent when the “test value” $\mathbf{tv} = \text{Det}(M)$ is 0. We have

$$\mathbf{tv} = (-2m - A)s/2 + (m - 2)w/2 - (m^2 - 4)/2.$$

If $m = 2$, all three terms are 0. Then $w^2 = (A + 4)^2$. Taking $w = A + 4$ and $s = 0$, $q_1(x) = x^2 - (A + 2)x + 1$ and $q_3(x) = (x + 1)^2$ as in Section 4.2. Then $\text{gcd}(q_1(x), q_3(x)) = 1$ unless $A = -4$, when $q_1(x) \equiv q_3(x) = (x + 1)^2$.

If $m = -2$, $s = 0$ and $\mathbf{tv} = 0$ when $w = m + 2 = (-2) + 2 = 0$. But also $w^2 = A^2 + 16$, which is 0 only for $A = \pm 4i$, when $q_1(x) \equiv q_3(x) = (x \mp i)^2$.

If $m \neq \pm 2$, then $s \notin \mathbf{k}$. If $(-2m - A) \neq 0$, $(-2m - A)s/2 \notin \mathbf{k}$. Then $\mathbf{tv} \notin \mathbf{k}$ unless $(-2m - A)s/2 + (m - 2)w/2 = 0$. But then $\mathbf{tv} \neq 0$ since $m \neq \pm 2$.

So if $m \neq \pm 2$, $\mathbf{t}\mathbf{v} \neq 0$ unless $A = -2m$. Again $w = m + 2$, but also $w^2 = (m + 2 + (-2m))^2 - 4(m - 2) = m^2 - 8m + 12$. The two conditions on w are only satisfied simultaneously when $m = 2/3$ and $A = -4/3$. Substituting $(s, w) \leftarrow (-s, -w)$ changes v to $v' = (q_3(x), \bar{q}_u(x), q_1(x))$, whose “test value” $\mathbf{t}\mathbf{v}'$ is 0 for the same (m, A) . Substituting $(s, w) \leftarrow (s, -w)$ gives the argument for $\text{Res}(q_2(x), q_4(x))$. If $m = 2/3$, $m^2 - 4 \notin \mathbf{k}^2$ when $-2 \notin \mathbf{k}^2$. \square

REMARK. Direct calculation shows that

$$\mathbf{t}\mathbf{v} \cdot \mathbf{t}\mathbf{v}' = (2 - m) \text{Res}(q_1(x), q_3(x)).$$

LEMMA 4.8. *Let $m, A \in \mathbf{k}$, $m^2 - 4 \notin \mathbf{k}^2$, and s and w as in Equation (4.4). If $T(m, A, r) = 0$, then $\mathbf{k}(s, w) \subset \mathbf{k}(r)$.*

Proof. We may take $r \equiv x \pmod{T(m, A, x)}$. Since $m \neq \pm 2$, s and w are given by Equations (4.7b) and (4.8b) unless $(m, A) = (2/3, -4/3)$, by Lemma 4.7. This case may be verified directly. \square

PROPOSITION 4.9. *Let $m, A \in \mathbf{k}$, $m^2 - 4 \notin \mathbf{k}^2$. Then $P_w(x)$ and $\bar{P}_w(x)$ have the same splitting field over $\mathbf{k}(w)$.*

Proof. This follows from an argument similar to that for Corollary 4.2(a), using Lemma 4.8. \square

Brute-force algebra gives the simple expressions

$$(4.11a) \quad \text{disc}(q_1(x)) \text{disc}(q_2(x)) = (A^2 - 4(m - 2)) \cdot (u - 1)^2,$$

$$(4.11b) \quad \text{Res}(q_1(x), q_2(x)) = w^2u,$$

$$(4.11c) \quad \text{disc}(q_1(x)) \text{disc}(q_4(x)) = (A^2 - 4(m - 2)) \cdot Q_1^2, \quad \text{and}$$

$$(4.11d) \quad \text{Res}(q_1(x), q_4(x)) = s^2Q_1 = (m^2 - 4)Q_1,$$

where $Q_1 = q_1(-1) = (m + 2 + A + w)/2$ is as in Lemma 4.4(d).

From Equations (4.11a) and (4.11c), we see that the splitting field of $T(m, A, x)$ contains $\mathbf{k}(y)$, where

$$(4.12) \quad y^2 = A^2 - 4(m - 2).$$

We have the following result.

PROPOSITION 4.10. *Let $m, A \in \mathbf{k}$, $m^2 - 4 \notin \mathbf{k}^2$. The zeroes of $T(m, A, x)$ are simple unless $w^2y^2 = 0$ or $(m, A) = (2/3, -4/3)$.*

Proof. Equations (4.11a)–(4.11d) show that if $m \neq \pm 2$, $w^2 \neq 0$, and $y^2 \neq 0$, then none of the $q_i(x)$ have repeated factors, and $\text{Res}(q_i(x), q_j(x)) \neq 0$ unless $\{i, j\} = \{1, 3\}$ or $\{2, 4\}$. For these, apply Lemma 4.7. \square

When $w = 0$, $q_1(x) \equiv q_2(x)$ and $q_3(x) \equiv q_4(x)$, so $P_w(x) \equiv P_{sw}(x)$ and $T(m, A, x) = P_{sw}^2$ in $\mathbf{k}[x]$. We deal with this case in Section 4.7. We see from Lemma 4.6(a) and its counterpart for $q_2(x)$ and $q_4(x)$, that except for the

four pairs (m, A) in Lemma 4.7, if $y^2 \neq 0$ the zeroes of $P_{sw}(x)$ and $\bar{P}_{sw}(x)$ are simple, and the linear fractional transformation σ in Equation (C4) (defined via Equation (4.7b)) permutes the zeroes of each cyclically.

The expression in Lemma 4.6(c) for a square root of the discriminant norm leads to an irreducibility criterion for $P_{sw}(x)$ and $\bar{P}_{sw}(x)$ in $\mathbf{k}(sw)[x]$.

THEOREM 4.11. *Let $m, A \in \mathbf{k}$, $m^2 - 4 \notin \mathbf{k}^2$. Then*

(a) $P_{sw}(x)$ and $\bar{P}_{sw}(x)$ are both irreducible in $\mathbf{k}(sw)[x]$ if and only if

$$[\mathbf{k}(s, w) : \mathbf{k}(sw)] = 2 \quad \text{and} \quad A^2 - 4(m - 2) \neq 0.$$

(b) If $P_{sw}(x)$ and $\bar{P}_{sw}(x)$ are irreducible in $\mathbf{k}(sw)[x]$, they define cyclic quartic extensions of $\mathbf{k}(sw)$. In each case, the action of the Galois group on the zeroes is given by (C4), defined via Equation (4.7b).

Proof. First, $P_{sw}(x) = q_1(x)q_3(x)$ in $\mathbf{k}(s, w)[x]$, so is reducible in $\mathbf{k}(sw)[x]$ if $\mathbf{k}(sw) = \mathbf{k}(s, w)$. So suppose $[\mathbf{k}(s, w) : \mathbf{k}(sw)] = 2$. Conjugation in $\mathbf{k}(s, w)/\mathbf{k}(sw)$ changes the sign of the square root of $\text{disc}(q_1(x))\text{disc}(q_3(x))$ in Lemma 4.6(c), so neither $\text{disc}(q_1(x))$ nor $\text{disc}(q_3(x))$ is a square in $\mathbf{k}(s, w)$ unless their product is 0. Similarly for $q_2(x), q_4(x)$ and $\bar{P}_{sw}(x)$. Multiplying these square roots gives $(m - 2)(A^2 - 4(m - 2))$ as a square root of $\text{disc}(q_1(x))\text{disc}(q_2(x))\text{disc}(q_3(x))\text{disc}(q_4(x))$. Now $m \neq 2$ by hypothesis, so if $A^2 - 4(m - 2) \neq 0$, both $P_{sw}(x)$ and $\bar{P}_{sw}(x)$ are irreducible in $\mathbf{k}(sw)[x]$. But if $A^2 - 4(m - 2) = 0$, at least one of $P_{sw}(x)$ and $\bar{P}_{sw}(x)$ has a repeated factor, so is reducible in $\mathbf{k}(sw)[x]$.

Part (b) follows from the discussion after Equations (4.11a)–(4.11d). □

We then obtain an irreducibility criterion for $T(m, A, x)$:

COROLLARY 4.12. *Let $m, A \in \mathbf{k}$. Then $T(m, A, x)$ is irreducible in $\mathbf{k}[x]$ if and only if $s^2 \notin \mathbf{k}^2$, $w^2 \notin \mathbf{k}^2$, and $s^2w^2 \notin \mathbf{k}^2$, i.e. if and only if $[\mathbf{k}(s, w) : \mathbf{k}] = 4$.*

Proof. Clearly, $T(m, A, x)$ is irreducible in $\mathbf{k}[x]$ if and only if $P_{sw}(x)$ and $\bar{P}_{sw}(x)$ are irreducible in $\mathbf{k}(sw)[x]$ (which requires $[\mathbf{k}(s, w) : \mathbf{k}(sw)] = 2$), and $[\mathbf{k}(sw) : \mathbf{k}] = 2$. We only need to show that, if $A^2 - 4(m - 2) = 0$, then $[\mathbf{k}(s, w) : \mathbf{k}] < 4$. This is trivial if $m = 2$. Otherwise, $A \neq 0$, so we can write

$$w^2 = (m + 2)(m + 2 + 2A) = (2s/A)^2(A/2 + 2)^2 \in (\mathbf{k}(s))^2. \quad \square$$

4.5. Pairs of related octics. Equations (4.4) and (4.11a)–(4.11d) show the splitting field of $T(m, A, x)$ contains $E = \mathbf{k}(s, w, y)$, where

$$s^2 = m^2 - 4, \quad w^2 = (m + A + 2)^2 - 4(m - 2), \quad \text{and} \quad y^2 = A^2 - 4(m - 2).$$

Note that the substitution $A \leftarrow -m - 2 - A$ has compositional order 2, and interchanges w^2 and y^2 . (This substitution gives the changes of parameter in Section 4.2.) We call $T(m, A, x)$ and $T(m, -m - 2 - A, x)$ *related octics*. The following result indicates how closely they are related.

THEOREM 4.13. *Let $m, A \in \mathbf{k}$, and $s^2 = m^2 - 4 \notin \mathbf{k}^2$. Then $T(m, A, x)$ and $T(m, -m - 2 - A, x)$ have the same splitting field over \mathbf{k} .*

Proof. We consider the corresponding quadratic factors

$$q_1(x) = x^2 + (-A + s - w)x/2 + (m + s)/2 \quad \text{as in Equation (4.5a), and}$$

$$\psi_1(x) = x^2 + (m + 2 + A + s - y)x/2 + (m + s)/2$$

with y as in Equation (4.12).

If $w^2y^2 \neq 0$, Equations (4.11a), (4.11c), and Lemma 4.6 show that the $q_i(x)$ all define the *same* extension of E . Similarly for the corresponding $\psi_i(x)$. Greatly facilitated by the substitution $A = -(m + 2)/2 + \mathbf{v}$ and Phil Carmody’s guidance with Pari-GP, we found an algebraic square root of $\text{disc}(q_1(x)) \text{disc}(\psi_1(x))$ in E , namely

$$c_0 + c_1s + c_2w + c_3y + c_4sw + c_5sy + c_6wy + c_7swy,$$

where

$$\begin{aligned} c_0 &= (3m^2 - 20m + 12 + 4\mathbf{v}^2)/16, & c_1 &= (m - 6)/4, \\ c_2 &= (-3m + 2 + 2\mathbf{v})/8, & c_3 &= (-3m + 2 - 2\mathbf{v})/8, & c_4 &= -1/4, \\ c_5 &= -1/4, & c_6 &= -1/4, & \text{and } c_7 &= 0. \end{aligned}$$

If $w^2 = 0$, at least one of $\text{disc}(q_1(x)) \text{disc}(\psi_1(x))$ and $\text{disc}(q_1(x)) \text{disc}(\psi_2(x))$ will be nonzero unless $y^2 = 0$ also. (Replacing $\psi_1(x)$ with $\psi_2(x)$ has the effect of replacing y with $-y$ in the algebraic square root.) But $w^2 = y^2 = 0$ only when $m = 6$ and $A = -4$, for which the two related octics are identical, $T(6, -4, x) = (x^2 + 2x - 1)^4$. \square

4.6. The Galois group of $T(m, A, x)$. We can now prove the main results about the splitting field L/\mathbf{k} of $T(m, A, x)$.

THEOREM 4.14. *Let $m, A \in \mathbf{k}$, and $[E : \mathbf{k}] = 8$ where $E = \mathbf{k}(s, w, y)$. Then*

- (a) $T(m, A, x)$ is irreducible in $\mathbf{k}[x]$ with Galois group $G \cong {}_8T_{11}$.
- (b) The fixed field of the quaternion subgroup is $\mathbf{k}(swy)$.
- (c) The fixed fields of the subgroups $\cong D_4$ are $\mathbf{k}(s)$, $\mathbf{k}(w)$, and $\mathbf{k}(y)$.
- (d) The fixed fields of the subgroups $\cong C_4 \times C_2$ are $\mathbf{k}(sw)$, $\mathbf{k}(sy)$, and $\mathbf{k}(wy)$.

Proof. $T(m, A, x)$ is irreducible in $\mathbf{k}[x]$ by Corollary 4.12. If L/\mathbf{k} is its splitting field, then $E \subset L$, $[L : \mathbf{k}] \mid 16$ by Proposition 4.5, and L contains a C_4 extension of $\mathbf{k}(sw)$ by Theorem 4.11, so $L \neq E$. Thus $|G| = 16$, and $G \subset A_8$ by Corollary 4.2(b). This narrows the possibilities for G to ${}_8T_9$, ${}_8T_{10}$, and ${}_8T_{11}$. Of these, only ${}_8T_{11}$ has a quaternion subgroup.

Let H be the subgroup of G fixing $\mathbf{k}(swy)$. Then $|H| = 8$. Since the $q_i(x)$ remain irreducible in $E[x]$, it follows that $P_s(x)$, $P_{sw}(x)$, and $P_w(x)$ remain irreducible in $\mathbf{k}(s, swy)[x]$, $\mathbf{k}(sw, swy)[x]$, and $\mathbf{k}(w, swy)[x]$, respectively. By

Lemma 4.6 and Equations (4.11a) and (4.11c), their discriminants are not squares in the coefficient fields, and L is normal of degree 4 over each, so

$$G(L/\mathbf{k}(s, swy)) \cong G(L/\mathbf{k}(sw, swy)) \cong G(L/\mathbf{k}(w, swy)) \cong C_4.$$

Now the elements of order 4 in these groups can only map $q_1(x)$ to $q_2(x)$, $q_3(x)$, and $q_4(x)$, respectively. Thus, H has 6 elements of order 4, so $H \cong Q_8$, proving (a) and (b). By Corollary 4.2, $G(L/\mathbf{k}(s)) \cong D_4$. By Proposition 4.9, $G(L/\mathbf{k}(w)) \cong D_4$ also; and $G(L/\mathbf{k}(y)) \cong D_4$ (use the related octic), establishing (c). The other three maximal subgroups of ${}_8T_{11}$ are $\cong C_4 \times C_2$, so (d) follows by the Galois correspondence. \square

REMARKS. The degree of E/\mathbf{k} can be determined by testing whether s^2 , w^2 , y^2 , s^2w^2 , s^2y^2 , w^2y^2 , and $s^2w^2y^2$ are squares in \mathbf{k} .

When $[E : \mathbf{k}] = 8$, the factors $P_{sw}(x)$ and $\bar{P}_{sw}(x)$ of $T(m, A, x)$ define the fixed fields of a conjugacy class of order-2 subgroups of G . The corresponding factors of the related octic define the fixed fields of a *different* conjugacy class of order-2 subgroups.

Taking $\sqrt{a} = sw$, $\sqrt{b} = sy$, $\sqrt{c} = s$, and $d = \text{disc}(q_1(x))$ in [27], Appendix, we have $\text{disc}(q_1(x))\text{disc}(q_2(x)) = k_a$, $\text{disc}(q_1(x))\text{disc}(q_3(x)) = k_c$, and $\text{disc}(q_1(x))\text{disc}(q_4(x)) = k_{ac}$. Using Equations (4.11a), (4.11c), and Lemma 4.6, the obstruction $(a, b)(c, c)$ to Galois group $\mathbf{DC} \cong {}_8T_{11}$ then evaluates to 1.

COROLLARY 4.15. *Let $m, A \in \mathbf{k}$, $m^2 - 4 \notin \mathbf{k}^2$, $[E : \mathbf{k}] = 4$ where $E = \mathbf{k}(s, w, y)$, and L/\mathbf{k} the splitting field of $T(m, A, x)$. Then $[L : \mathbf{k}] = 8$, and*

$$G(L/\mathbf{k}) \cong \begin{cases} Q_8, & \text{if } swy \in \mathbf{k}, \\ D_4 \cong D_8(8), & \text{if } w \in \mathbf{k} \text{ or } y \in \mathbf{k}, \text{ and} \\ C_4 \times C_2, & \text{if } sw \in \mathbf{k}, sy \in \mathbf{k}, \text{ or } wy \in \mathbf{k}. \end{cases}$$

Proof. At least one of the related octics is irreducible in $\mathbf{k}[x]$ by Corollary 4.12, and the rest is clear from Theorem 4.14. \square

We formulate the condition $w \in \mathbf{k}$ (disregarding $[E : \mathbf{k}]$) by taking

$$(4.13) \quad \begin{aligned} m &\in \mathbf{k}, & d &\in \mathbf{k}^\times, \\ A &= -m - 2 - d - (m - 2)/d, & \text{and } w &= (m - 2)/d - d. \end{aligned}$$

Substituting into Equation (4.6c), we obtain

$$(4.14) \quad \begin{aligned} P_w(x) &= x^4 + (m + 2d + 2)x^3 + ((d + 2)m + d^2 + 2d + 2)x^2 \\ &\quad + ((d + 1)m + 2)x + 1. \end{aligned}$$

If $[E : \mathbf{k}] = 4$, the related octic has Galois group $D_8(8)$, and is a defining polynomial for the splitting field L , while $P_w(x)$ as in Equation (4.14) and $\bar{P}_w(x)$ are quartics in $\mathbf{k}[x]$ with $G \cong D_4$ which define the same conjugacy class of (nonnormal) quartic intermediate fields of L/\mathbf{k} . Note that replacing

d by $(m - 2)/d$ interchanges $P_w(x)$ and $\bar{P}_w(x)$. If $d = (m - 2)/d = t$, then $P_w(x) = \bar{P}_w(x) = P_t(x)$ as in Section 4.7. We give two other “degenerate” cases of Equations (4.13) and (4.14). In both cases, the “generic” Galois group is V_4 .

When $d = -2$, $A = -m - 2 - A$, so the related octics are identical. (This happens for the pair $(m, A) = (2/3, -4/3)$ in Lemma 4.7.) In this case, $\text{disc}(P_w(x)) \in \mathbf{k}^2$. We find $L = \mathbf{k}(\sqrt{(m - 2)^2 - 16}, \sqrt{(m - 4)^2 - 4})$.

If $d = -1$ then $A = -3$, and $\text{disc}(P_w(x)) \in \mathbf{k}^2$ when $m = 4 - t - t^2$, $t \in \mathbf{k}$. In this case, $L = \mathbf{k}(\sqrt{t^2 - 4}, \sqrt{(t + 1)^2 - 4})$.

If $swy \in \mathbf{k}$ or $wy \in \mathbf{k}$ and $[E : \mathbf{k}] = 4$, the related octics are distinct defining polynomials for the splitting field. We do not have a complete description of either $swy \in \mathbf{k}$ or $wy \in \mathbf{k}$. With respect to m , $\text{Res}(w^2, y^2) = (A + 4)^2(A^2 + 16)$. Taking $A = -4$, we find the following.

PROPOSITION 4.16. *Let $t \in \mathbf{k}$.*

- (a) *If $t^2 + 4 \notin \mathbf{k}^2$, $t^2 - 4 \notin \mathbf{k}^2$, and $t^4 - 16 \notin \mathbf{k}^2$, then the related octics $T(2 - t^2, -4, x)$ and $T(2 - t^2, t^2, x)$ are irreducible in $\mathbf{k}[x]$. Both have the same splitting field L with $G(L/\mathbf{k}) \cong C_4 \times C_2$.*
- (b) *If $t^2 + 4 \notin \mathbf{k}^2$, $t^2 + 8 \notin \mathbf{k}^2$, and $(t^2 + 6)^2 - 4 \notin \mathbf{k}^2$, then the related octics $T(-t^2 - 2, -4, x)$ and $T(-t^2 - 2, t^2 + 4, x)$ are irreducible in $\mathbf{k}[x]$. Both have the same splitting field L with $G(L/\mathbf{k}) \cong Q_8$.*

Proof. Check that $[E : \mathbf{k}] = 4$; then $wy \in \mathbf{k}$ in (a), and $swy \in \mathbf{k}$ in (b). \square

REMARKS. If $(m, A) = (-t^2 + 2 - 8i, 4i)$ (or $(-t^2 + 2 + 8i, -4i)$), $t \in \mathbb{Z}[i]$, then $T(m, A, x)$ and the related octic are “typically” distinct defining polynomials for a $C_4 \times C_2$ extension of $\mathbb{Q}(i)$ ($wy \in \mathbf{k}$).

With respect to A , $\text{Res}(w^2, y^2) = (m - 2)^2(m - 6)^2$. Taking $m = 6$, we find that if $A = 8a_n - 4$ where $a_n + b_n\sqrt{2} = (3 + 2\sqrt{2})^n$ and $n \in \mathbb{Z}^+$, then $T(6, A, x)$ and $T(6, -8 - A, x)$ are defining polynomials for a quaternion field in a family reminiscent of the cyclic octic fields in [34].

If $sw \in \mathbf{k}$ and $[\mathbf{k}(s, y) : \mathbf{k}] = 4$, $P_{sw}(x)$ and $\bar{P}_{sw}(x)$ are Murphy’s twins in $\mathbf{k}[x]$, and define distinct cyclic quartic extensions of \mathbf{k} . The related octic is then a defining polynomial for their join. We deal with Murphy’s twins for $\mathbf{k} = \mathbb{Q}$ and $m, A \in \mathbb{Z}$, in Section 6.3.

4.7. Washington’s cyclic quartic fields. We close this section with a discussion of the “degenerate” case $w^2 = 0$. It produces the polynomials $P(x)$ in Proposition 1.2(c). These are alternate defining polynomials for the cyclic quartic fields in [41] when $\mathbf{k} = \mathbb{Q}$ and $t \in \mathbb{Z} - \{0, -2\}$.

Very early on, Phil Carmody drew the author’s attention to examples of $T(m, A, x)$, $m, A \in \mathbb{Z}$, $m \neq \pm 2$, with repeated factors in $\mathbb{Z}[x]$. It was this observation that originally led us to consider the case $w^2 = 0$. As when $s^2 = 0$, $T(m, A, x)$ is again the square of a quartic in $\mathbf{k}[x]$.

If $m = t^2 + 2$, then $4(m - 2) = 4t^2$, and $w = 0$ when $A = -t^2 - 4 \pm 2t$. To choose the \pm sign, we substitute into the formulas in Lemma 4.4(d); we find that $Q_1 = Q_2 = \pm t$, and

$$(4.15) \quad \text{disc}(q_1(x)) = \text{disc}(q_2(x)) = t(t \mp 2)(u + 1), \quad \text{whence}$$

$$(4.16) \quad \text{disc}(q_1(x)) \text{disc}(q_3(x)) = t^2(t \mp 2)^2(t^2 + 4).$$

Taking $A = -t^2 - 4 - 2t$ gives $\text{disc}(q_1(x)) \text{disc}(q_3(x)) = t^2(t + 2)^2(t^2 + 4)$, the same form as the product of the discriminants of the quadratic factors of the quartics in [41]. Then $T(t^2 + 2, -t^2 - 2t - 4, x) = P_t^2(x)$, where

$$(4.17) \quad P_t(x) = x^4 + (t^2 + 2t + 4)x^3 + (t^3 + 3t^2 + 4t + 6)x^2 + (t^3 + t^2 + 2t + 4)x + 1,$$

the $P(x)$ in Proposition 1.2(c). By Theorem 4.11, $P_t(x)$ is irreducible in $\mathbf{k}[x]$ when $t \in \mathbf{k}$, $t^2 + 4 \notin \mathbf{k}^2$, and $t \neq -2$.

Next, we relate the $P_t(x)$ to the cyclic quartics

$$(4.18) \quad f_t(x) = x^4 - t^2x^3 - (t^3 + 2t^2 + 4t + 2)x^2 - t^2x + 1$$

in [41]. Clearly $f_t(x)$ is a reciprocal polynomial; if $f_t(\rho) = 0$ then the element of order 2 in the Galois group maps ρ to $1/\rho$. Similarly, the element σ^2 of order 2 in the Galois group of $P_t(x)$ maps a zero r to u/r . With $m = t^2 + 2$, we have $u^2 - (t^2 + 2)u + 1 = 0$, and $u = v^2$ where $v^2 - tv - 1 = 0$. Then $\sigma^2(r/v) = 1/(r/v)$, so r/v is a zero of a reciprocal polynomial. By formulating u and σ as rational expressions (mod $P_t(x)$), and using Pari-GP to bludgeon the algebra into submission, we find that

$$(4.19) \quad u \equiv -x^3 - (t^2 + 2t + 3)x^2 - (t^3 + 2t^2 + 3t + 3)x - 1 \pmod{P_t(x)}.$$

Taking $v = (u - 1)/t$, we find that r/v is in fact a zero of $f_t(x)$.

PROPOSITION 4.17. *If $t \in \mathbf{k} - \{0, -2\}$, $P_t(x)$ and $f_t(x)$ define the same extension of \mathbf{k} .*

Proof. Equation (4.19) and $v = (u - 1)/t$ give $f_t(x/v) \equiv 0 \pmod{P_t(x)}$, where

$$x/v \equiv (x^3 + (t^2 + t + 3)x^2 + (t^2 + t + 3)x + 1)/t^2 \pmod{P_t(x)}, \quad \text{if } t \neq 0.$$

Reformulating $v \pmod{f_t(x)}$, we have $P_t(xv) \equiv 0 \pmod{f_t(x)}$, where

$$xv \equiv (-x^2 + (t^2 + t)x - 1)/(t + 2) \pmod{f_t(x)}, \quad \text{if } t \neq -2.$$

Both transformations are defined when $t \in \mathbf{k} - \{0, -2\}$, and the result follows. □

REMARKS. The related octic $T(t^2 + 2, 2t, x)$ has the repeated factor $(x^2 - tx - 1)^2$. The cofactor is a quartic which defines the same extension of \mathbf{k} as $P_t(x)$.

5. $T(m, A, x)$ and number field extensions

We now apply the preceding results when \mathbf{k} is a number field and $m, A \in \mathcal{O}_{\mathbf{k}}$, when the zeroes of $T(m, A, x)$ are units. We continue to assume that u is defined via Equation (4.7b).

If $m^2 - 4 \notin \mathbf{k}^2$ and $w^2y^2 \neq 0$, the zeroes of $T(m, A, x)$ are all simple by Proposition 4.10. We have the following result.

THEOREM 5.1 (Real and complex zeroes). *Let \mathbf{k} be a number field, $\mathbf{k} \subset \mathbb{R}$, $m, A \in \mathcal{O}_{\mathbf{k}}$, $m^2 - 4 \notin \mathbf{k}^2$. Then the number of real zeroes of $T(m, A, x)$ is*

- (a) *None, if $s^2 < 0$ or $w^2 < 0$;*
- (b) *Four, if $s^2 > 0$, $w^2 > 0$, and $y^2 < 0$;*
- (c) *Eight, if $m < -2$;*
- (d) *Eight, if $m > 2$, $w^2 > 0$, $y^2 > 0$, and*

$$|(A + 4)^2 + mA(A + 4) + A^2| > 16; \quad \text{and}$$

- (e) *None, if $m > 2$, $w^2 > 0$, $y^2 > 0$, and*

$$|(A + 4)^2 + mA(A + 4) + A^2| < 16,$$

i.e. if $2 < m < 6$ and $-m - 2 + 2\sqrt{m - 2} < A < -2\sqrt{m - 2}$.

Proof. For (a), $\mathbf{k}(s, w) \subset \mathbf{k}(r)$ by Lemma 4.8, so $\mathbf{k}(r) \not\subseteq \mathbb{R}$ for each zero r .

For (b), $\mathbf{k}(s, w) \subset \mathbb{R}$, so $q_i(x) \in \mathbb{R}[x]$. Now, $\text{disc}(q_3(x))$ has the same sign as $\text{disc}(q_1(x))$ by Lemma 4.6(b), while $\text{disc}(q_2(x))$ and $\text{disc}(q_4(x))$ have the opposite sign by Equations (4.11a) and (4.11c).

For (c), if $m < -2$ then $E = \mathbf{k}(s, w, y) \subset \mathbb{R}$, so the $\text{disc}(q_i(x))$ all have the same sign. Here $u < 0$, so by Lemma 4.4(d), $\text{disc}(q_1(x)) > 0$.

If $m > 2$, $w^2 > 0$, and $y^2 > 0$, then again the $\text{disc}(q_i(x))$ all have the same sign, but here $u > 0$. Again using Lemma 4.4(d), $\text{disc}(q_1(x))$ and $\text{disc}(q_2(x))$ are both > 0 or both < 0 , according as whether $|\alpha\beta| > 4u$ or $|\alpha\beta| < 4u$. By Lemma 4.4(a), $\alpha\beta = A + 4 + Au$. Multiplying by the corresponding conditions for $q_3(x)$ and $q_4(x)$ gives (d) and (e). □

REMARKS. Suppose $m, A \in \mathbb{R}$. If $m < 2$ (in particular, when $s^2 < 0$), then $w^2 > 0$ and $y^2 > 0$. Since $\text{Max}(|A|, |-m - 2 - A|) \geq |(m + 2)/2|$, we have $\text{Max}(w^2, y^2) \geq (m - 6)^2/4 \geq 0$, so w^2 and y^2 cannot both be negative.

The cases $w^2 < 0$ and $y^2 < 0$ correspond to related octics. Thus, if $m > 2$ and A is such that $w^2y^2 < 0$, one of the two related octics has signature $(0, 4)$ and the other has signature $(4, 2)$.

Suppose $T(m, A, x)$ is irreducible with $G \cong {}_8T_{11}$. Then the fixed field of complex conjugation is nonnormal over \mathbf{k} in cases (a) and (b). In case (e), the fixed field of complex conjugation is the elementary Abelian extension E/\mathbf{k} .

We use the Washington’s cyclic quartic case $T(t^2 + 2, -t^2 - 2t - 4, x) = P_t^2(x)$ of Section 4.7 to deal with repeated zeroes.

THEOREM 5.2. *Let $\mathbf{k} \subset \mathbb{R}$, $t \in \mathcal{O}_{\mathbf{k}}$, and $T(t^2 + 2, -t^2 - 2t - 4, x) = P_t^2(x)$ as in Section 4.7. Then $P_t(x)$ has 4 real zeroes if $|t + 1| > 1$, and none if $|t + 1| < 1$.*

Proof. In Section 4.7, the choice $A = -t^2 - 2t - 4$ makes $\text{disc}(q_1(x)) = t(t + 2)(u + 1)$ in Equation (4.15). Also in Section 4.7, $u = v^2$ where $v^2 - tv - 1 = 0$. Thus, $u + 1 \geq 1$, so $\text{disc}(q_1(x))$ has the same sign as $t(t + 2) = (t + 1)^2 - 1$. □

The following result depends only on the fact that the zeroes of $T(m, A, x)$ are units when m and A are algebraic integers. We let \sim indicate associates.

PROPOSITION 5.3 (Exceptional sequences). *Let \mathbf{k} be a number field, $m, A \in \mathcal{O}_{\mathbf{k}}$, $m^2 - 4 \notin \mathbf{k}^2$, and $T(m, A, r) = 0$. Then*

- (a) $r + 1 \sim r + u$ in $\mathbf{k}(r)$.
- (b) *If $m - 2 \in \mathcal{O}_{\mathbf{k}}^\times$, then $r, r + 1, r + u$ is an exceptional sequence of three units.*

Proof. For part (a), $\sigma(r) = (-r - 1)/(r + u)$ is a zero of $T(m, A, x)$, hence is a unit. For part (b), use $P_s(-1) = \bar{P}_s(-1) = m - 2$ and $(u - 1)^2 = (m - 2)u$. □

REMARK. The three units in (b) may be incorporated in various normalized four-term exceptional sequences as in [20]. One such is $0, 1, u, -r$.

We obtain additional units and associates when m, A , and $w \in \mathcal{O}_{\mathbf{k}}$. The following result does *not* require that $[E : \mathbf{k}] = 4$.

PROPOSITION 5.4 (“Constellations” of units and associates). *Let \mathbf{k} be a number field, $m, d \in \mathcal{O}_{\mathbf{k}}$, $d \mid m - 2$, and A, w and $P_w(x)$ as in Equations (4.13) and (4.14). If $P_w(r) = 0$, then*

- (a) $r + 1 \sim r + u \sim r + 1 + d$, and $r \sim r + 1 + d + u \sim 1$.
- (b) *If $d \in \mathcal{O}_{\mathbf{k}}^\times$, then all the quantities in (a) are units.*

Proof. In this case, $q_1(x) = x(x + 1 + d) + u(x + 1) = x(x + 1 + d + u) + u$, giving (a). For (b), note that in Equation (4.14), $P_w(-1) = d^2$. Thus, if $d \in \mathcal{O}_{\mathbf{k}}^\times$, $r + 1 \sim 1$, and the result follows. □

6. Extensions of $\mathbf{k} = \mathbb{Q}$

We let $\mathbf{k} = \mathbb{Q}$ and $m, A \in \mathbb{Z}$. For $|m|, |A + (m + 2)/2| \leq 10^4$, a simple numerical sweep found that $[E : \mathbb{Q}] = 8$ for over 95% of pairs (m, A) . So it appears that for $m, A \in \mathbb{Z}$, $T(m, A, x)$ “typically” produces nonnormal octic fields whose normal closures over \mathbb{Q} have Galois group $G \cong {}_8T_{11}$.

6.1. Real and complex fields. The only $(m, A) \in \mathbb{Z} \times \mathbb{Z}$ to which Theorem 5.1(e) applies, is $(4, -3)$. Here, $A = -(m + 2)/2$, the “degenerate” case $d = -2$ of Equation (4.14); we have $G \cong V_4$ and $L = \mathbb{Q}(\zeta_{12})$. The only $t \in \mathbb{Z}$ for which $|t + 1| < 1$ as in Theorem 5.2 is $t = -1$. In this case, $L = \mathbb{Q}(\zeta_5)$.

Apart from these cases, assuming $m \neq \pm 2$, by Theorem 5.1 the splitting field of $T(m, A, x)$ is totally real for $m, A \in \mathbb{Z}$ unless $s^2 < 0$ ($m = -1, 0$, or 1); or, $m > 2$ and $w^2 < 0$ or $y^2 < 0$. Now $y^2 < 0$ requires that $A^2 < 4(m - 2)$, so nonreal fields are relatively rare. When $[E : \mathbb{Q}] < 8$, they are even less common. We have the following result.

THEOREM 6.1. *Let $m, A \in \mathbb{Z}$, $s^2w^2y^2 \neq 0$, L the splitting field of $T(m, A, x)$.*

- (a) *If $yw \in \mathbb{Z}$, then L is totally real unless $(m, A) = (1, -4), (1, 1)$, or $(4, -3)$.*
- (b) *If $swy \in \mathbb{Z}$, then L is totally real.*
- (c) *If $m \in \mathbb{Z}$, $d \in \mathbb{Z}$, $d \mid m - 2$, $d^2 \leq |m - 2|$, and A and $P_w(x)$ are as in Equations (4.13) and (4.14), then L is totally real unless $(m, d) = (-1, \pm 1), (0, \pm 1), (1, \pm 1), (4, -1)$, or $(m, -1)$ with $m > 4$.*
- (d) *If $sw \in \mathbb{Z}$, then L is totally real unless $(m, A) = (7, -4)$ or $(7, 1)$.*

Proof. For (a) and (b), one simply checks the criteria of Theorem 5.1, keeping in mind that w^2 and y^2 cannot both be negative. For (c), we have $w^2 > 0$, and if $4 \leq d^2 \leq |m - 2|$, $|(m - 2)/d + d| \leq |(m + 2)/2|$, so $y^2 \geq (m - 6)^2/4$.

For (d), we have the “polynomial square root” identity

$$s^2w^2 = (m^2 + Am + 2A + 4)^2 - 4A(A + 4)m - 8(A^2 + 4A + 8).$$

If s^2w^2 is a perfect square, $s^2w^2 = (m^2 + Am + 2A + 4 - 2k)^2$ for some $k \in \mathbb{Z}$. Now here, L is totally real unless $m > 2$ and $A^2 < 4(m - 2)$. This makes the “remainder” $-4A(A + 4)m - 8(A^2 + 4A + 8)$ so small, we only need to consider the cases $-4 < A < 0$ and $0 \leq k \leq 4$. The only ones where $sw \in \mathbb{Z}$ and $s^2w^2y^2 \neq 0$ have $k = 1$, namely $(m, A) = (7, -4)$ and $(7, 1)$. □

The only cases where L/\mathbb{Q} is Abelian and nonreal are $(m, A) = (3, -3)$ or $(3, -2)$ ($L = \mathbb{Q}(\zeta_5)$); $(4, -3)$ ($L = \mathbb{Q}(\zeta_{12})$); $(1, -4)$ or $(1, 1)$ ($L = \mathbb{Q}(\zeta_{15})$); $(7, -4)$ or $(7, -5)$ ($L = \mathbb{Q}(\zeta_{20})$); and $(7, 1)$ or $(7, -10)$ ($L \subset \mathbb{Q}(\zeta_{95}), [L : \mathbb{Q}] = 8$). The pairs $(m, A) = (7, -4)$ and $(7, 1)$ give “Murphy’s twins.” In both cases, the splitting field of one is $\mathbb{Q}(\zeta_5)$, while that of the other is the real subfield of L . The $C_4 \times C_2$ cyclotomic field $\mathbb{Q}(\zeta_{16})$ is “left out,” although its real subfield is the splitting field of $P_t(x)$ for $t = 2$.

All quaternion fields defined by $T(m, A, x)$ for $m, A \in \mathbb{Z}$ are totally real, by Theorem 6.1(b).

6.2. Special units. We have $\mathbb{Z}^\times = \{-1, 1\}$, so Propositions 5.3(b) and 5.4(b) only apply with $m \in \{1, 3\}$ and $d \in \{-1, 1\}$, respectively.

The 1-parameter family $T(1, A, x)$, $A \in \mathbb{Z}$, produces fields which are totally imaginary by Theorem 5.1(a), and with exceptional sequences of three units by Proposition 5.3(b). We have $[E : \mathbb{Q}] = 8$ except for $A \in \{-3, 0\}$ or $\{-4, 1\}$, when $[E : \mathbb{Q}] = 4$. Equation (4.13) gives $A = -3$ when $d = -1$; the quartic factors $P_w(x) = x^4 + x^3 + 2x^2 + 2x + 1$ and $\bar{P}_w(x) = x^4 + 5x^3 + 8x^2 + 4x + 1$ of $T(1, -3, x)$, both have the minimum quartic discriminant (see [15]) of 117. The related octic, $T(1, 0, x) = x^8 - 3x^6 + 3x^5 + 14x^4 + 15x^3 + 9x^2 + 3x + 1$, is a defining polynomial for the splitting field, the Hilbert Class Field of $\mathbb{Q}(\sqrt{-39})$.

Both related octics $T(1, -4, x)$ and $T(1, 1, x)$ are irreducible in $\mathbb{Z}[x]$, with $G \cong C_4 \times C_2$, by Proposition 4.16(a) with $t = 1$. The splitting field is $\mathbb{Q}(\zeta_{15})$.

The 1-parameter family $T(3, A, x)$, $A \in \mathbb{Z}$, also produces fields with exceptional sequences of three units. But these fields are totally real, except when $A \in \{-6, 1\}$, $\{-5, 0\}$, $\{-4, -1\}$, or $\{-3, -2\}$. Since $4(m - 2) = 4$ for $m = 3$, $w^2 \notin \mathbb{Z}^2$ and $y^2 \notin \mathbb{Z}^2$ unless $w^2y^2 = 0$. It is also not hard to check that with $m = 3$ and $A \in \mathbb{Z}$, $w^2y^2 \notin \mathbb{Z}^2$ and $s^2w^2y^2 = 5w^2y^2 \notin \mathbb{Z}^2$ unless $w^2y^2 = 0$. Thus, $[E : \mathbb{Q}] = 8$ unless either $5w^2 \in \mathbb{Z}^2$ or $5y^2 \in \mathbb{Z}^2$. Taking $5w^2 \in \mathbb{Z}^2$ gives a family of Murphy’s twins (see Equations (6.3a)–(6.4b)). These define a family of cyclic quartic fields which (apart from $\mathbb{Q}(\zeta_5)$) are real, contain $\mathbb{Q}(\sqrt{5})$, and have exceptional sequences of three units by Proposition 5.3(b).

Using Pari-GP, we checked $T(m, A, x)$ for small values of m and A against the octic fields with small discriminants listed in the tables of [4]. We found that $T(3, -6, x)$, $T(1, -2, x)$, and $T(1, -14, x)$ all define the two conjugate octic fields of minimum discriminant for signature $(0, 4)$ and $G \cong_8 T_{11}$. Each of these polynomials gives exceptional sequences of 3 units in those fields.

If $m, d \in \mathbb{Z}$, $m \neq \pm 2$, $d \mid m - 2$, and $P_w(r) = 0$, $P_w(x)$ as in Equations (4.13) and (4.14), then $P_w(\sigma(r)) \neq 0$, but $\sigma(r) = (-r - 1)/(r + u)$ is a unit in $\mathbb{Q}(r)$. When $d \neq -1$, this allows us to exhibit a system of three independent units for the field $\mathbb{Q}(r)$ when this is a totally real quartic field. The values $d = -1$ and $d = 1$ give the 1-parameter families

$$(6.1a) \quad P_w(x) = x^4 + mx^3 + (m + 1)x^2 + 2x + 1, \quad \text{and}$$

$$(6.1b) \quad P_w(x) = x^4 + (m + 4)x^3 + (3m + 5)x^2 + (2m + 2)x + 1,$$

respectively, to which Proposition 5.4(b) applies. If $P_w(r) = 0$ in Equation (6.1b), then (with u defined via Equation (4.7b)) $r, r + 1, r + 2, r + u$, and $r + 2 + u$ are all units.

6.3. Murphy’s twins. If $m, A \in \mathbb{Z}$ and $sw \in \mathbb{Z}$, the Murphy’s twins $P_{sw}(x)$ and $\bar{P}_{sw}(x)$ define cyclic quartic number fields when the conditions of Theorem 4.11 hold, and (with u and σ defined via Equation (4.7b)) their zeroes are units satisfying (M) with $n = 4$. We give a description of Murphy’s twins cyclic quartic fields having a given quadratic subfield, based on standard results about norms from real quadratic fields. We show that any quadratic

field which is contained in *some* cyclic quartic field, is a subfield of infinitely many Murphy’s twins cyclic quartic fields.

Let $d > 1$ be a squarefree integer, $K = \mathbb{Q}(\sqrt{d})$, and assume that $m \in \mathbb{Z}$, $m^2 - 4 = dN^2$, $N \in \mathbb{Z}$. Then $(m + N\sqrt{d})/2$ is a unit of norm 1 in \mathcal{O}_K , so m and N are (to within sign) generalized Lucas and Fibonacci numbers, respectively, which we describe as follows.

Let $\varepsilon > 1$ be the fundamental unit of K , and $\bar{\varepsilon}$ its conjugate. Let

$$(LF) \quad \varepsilon^n = \frac{L_n + F_n(\varepsilon - \bar{\varepsilon})}{2}, \quad \text{where } L_n, F_n \in \mathbb{Z}.$$

The properties in [11], Theorem 179, where $d = 5$, and L_n and F_n are the Lucas and Fibonacci numbers, easily generalize to any squarefree $d > 1$. We have $\mathbb{Q}(\sqrt{m^2 - 4}) = \mathbb{Q}(\sqrt{d})$ when $m = \pm L_n$, where n is arbitrary if $\mathcal{N}(\varepsilon) = +1$, but n must be *even* if $\mathcal{N}(\varepsilon) = -1$.

Now we want $w^2 = dY^2$ for some $Y \in \mathbb{Z}$. Taking $X = m + A + 2 \in \mathbb{Z}$, we may express this as $(X + Y\sqrt{d})/2 = \pi \in \mathcal{O}_K$, $\mathcal{N}(\pi) = m - 2$. Then $X^2 + (m - 2)^2 = d(N^2 + Y^2)$, so d is the sum of two squares, as expected if K is contained in a cyclic quartic field. We have the following result.

PROPOSITION 6.2. *Let $d > 1$ be squarefree, $d = a^2 + b^2$, $K = \mathbb{Q}(\sqrt{d})$, L_n and F_n as in Equation (LF). There are infinitely many m such that $\mathbb{Q}(s) = K$, for each of which there are infinitely many $A \in \mathbb{Z}$ such that $T(m, A, x) = P_{sw}(x)\bar{P}_{sw}(x)$ in $\mathbb{Z}[x]$.*

Proof. If $\mathcal{N}(\varepsilon) = -1$, then for any $j \in \mathbb{Z}$, we obtain $sw \in \mathbb{Z}$ by taking

$$m = L_{2k}, \quad X + Y\sqrt{d} = 2(\varepsilon^{2k} - 1)\varepsilon^{2j-1}, \quad \text{and} \quad A = -m - 2 \pm X.$$

If $\mathcal{N}(\varepsilon) = +1$, choose n and the \pm sign so that $u = \pm\varepsilon^n \equiv 1 \pmod{a\mathcal{O}_K}$. Then $(u - 1)/a \in \mathcal{O}_K$, so $sw \in \mathbb{Z}$ for any $j \in \mathbb{Z}$, if

$$m = \pm L_n, \quad X + Y\sqrt{d} = 2((u - 1)/a)(b - \sqrt{d})\varepsilon^j, \quad \text{and} \quad A = -m - 2 \pm X.$$

□

REMARK. The equation $\mathcal{N}(\pi) = m - 2$ may well have other solutions than those given in Proposition 6.2.

Thus, there are Murphy’s twins cyclic quartic fields containing $\mathbb{Q}(\sqrt{d})$ whenever $d > 1$ is squarefree and the sum of two squares. But suppose d is even as well; and further, that $K = \mathbb{Q}(\sqrt{d})$ has $\mathcal{N}(\varepsilon) = +1$. Then, K is *not* a subfield of any of Washington’s cyclic quartic fields (whose quadratic subfields always have $\mathcal{N}(\varepsilon) = -1$). It is also easy to show that K is not a subfield of any “simplest” quartic field. The smallest such d is 34.

The only nonreal Murphy’s twins cyclic quartic field is $\mathbb{Q}(\zeta_5)$, which occurs when $(m, A) = (3, -3), (3, -2), (7, -4)$, and $(7, 1)$.

We can make $P_{(sw)'}(x) = x^4 P_{sw}(1/x)$ as per Proposition 2.2, with parameter $m' = m$, the same square root s of $m^2 - 4$, and $(sw)' = s \cdot w'$, specifying A' and w' by

$$(6.2a) \quad \frac{m + 2 + A' + w'}{2} = \left(\frac{m + 2 + A - w}{2} \right) \left(\frac{m + s}{2} \right),$$

that is

$$(6.2b) \quad A' = \frac{m^2 + mA - 4 - sw}{2} \quad \text{and} \quad w' = \frac{(m + A + 2)s - mw}{2}.$$

Of course, $(m, A', -s, -w')$ gives the same $P_{(sw)'}(x)$ as (m, A', s, w') . Note, however, that if $sw \neq 0$, $P_{sw}(x)$ and $\bar{P}_{sw}(x)$ produce different values of A' . For the $P_t(x)$ in Section 4.7, $A' = -t^3 - t^2 - 2t - 4$ and $(sw)' = -t^5 - 4t^3$ give $P_{(sw)'}(x) = x^4 P_t(1/x)$.

The case $m = 3$ is particularly simple. Here, $d = 5$, $\varepsilon = \tau$, the “golden ratio,” and $(X + Y\sqrt{5})/2$ has norm $m - 2 = 1$. With $X = L_{2j}$, $A = -5 + L_{2j}$ and $sw = 5F_{2j}$ (the original Lucas and Fibonacci numbers), we obtain

$$(6.3a) \quad P_{sw}(x) = x^4 + (5 - L_{2j})x^3 + (9 - 5F_{2j-1})x^2 + (5 - L_{2j-2})x + 1 \quad \text{and}$$

$$(6.3b) \quad \bar{P}_{sw}(x) = x^4 + (5 - L_{2j})x^3 + (9 - 5F_{2j+1})x^2 + (5 - L_{2j+2})x + 1.$$

With $A = -5 - L_{2j}$ and $sw = 5F_{2j}$, we obtain

$$(6.4a) \quad P_{sw}(x) = x^4 + (5 + L_{2j})x^3 + (9 + 5F_{2j+1})x^2 + (5 + L_{2j+2})x + 1 \quad \text{and}$$

$$(6.4b) \quad \bar{P}_{sw}(x) = x^4 + (5 + L_{2j})x^3 + (9 + 5F_{2j-1})x^2 + (5 + L_{2j-2})x + 1.$$

As suggested by Equation (6.2a), in both families $x^4 P_{sw}(1/x)$ is obtained by taking $\bar{P}_{sw}(x)$ and shifting the index j .

If $P(r) = 0$ for one of the cyclic indexes in Equations (6.3a)–(6.4b) and u is defined via Equation (4.7b), $r, r + 1, r + u$ is an exceptional sequence of 3 units by Proposition 5.3(b). If $T(3, \pm L_{2j}, r) = 0$, then $r, r + 1, r + u$ is also an exceptional sequence of 3 units, which are generally of degree 8 over \mathbb{Q} .

It is easy to show that for a given $m \in \mathbb{Z}$, there are only finitely many $A \in \mathbb{Z}$ for which either $y^2 \in \mathbb{Z}^2$ or $y^2 w^2 \in \mathbb{Z}^2$. Consequently, for a given m there can be only finitely many A yielding Murphy’s twins which do *not* have distinct splitting fields. The octics with $(m, A) = (-3, -4)$, $(-7, 8)$, and $(-66, 13)$, are the only examples we know where $s^2 w^2 y^2 \neq 0$ and the “twins” define the *same* C_4 extension of \mathbb{Q} .

6.4. The cases $[L : \mathbb{Q}] = 8$ when $wy \in \mathbb{Z}$, $swy \in \mathbb{Z}$, and $w \in \mathbb{Z}$. We apply Proposition 4.16 to the cases $wy \in \mathbb{Z}$ and $swy \in \mathbb{Z}$ by taking $t \in \mathbb{Z}^+$. In (a), we have $[E : \mathbb{Q}] = 4$ when $t \neq 2$; and in (b), $[E : \mathbb{Q}] = 4$ when $t > 1$. Clearly $t^2 - 4$ and $t^2 + 4$ are both squarefree for a positive proportion of $t \in \mathbb{Z}^+$; likewise for $t^2 + 4$ and $t^2 + 8$, so these families of $C_4 \times C_2$ and Q_8 number fields are infinite. The family of quaternion fields mentioned in the Remarks after Proposition 4.16 may also be shown to be infinite.

When $m, d \in \mathbb{Z}$, $m \neq \pm 2$, and $d \mid m - 2$ in Equations (4.13) and (4.14), it is not hard to show that $P_w(x)$ is irreducible in $\mathbb{Z}[x]$ unless $d = -2$ and $m = 6$. Apart from the special cases described after Equations (4.13) and (4.14), the only instances we know where $P_w(x) \in \mathbb{Z}[x]$ has $G \not\cong D_4$ are $(m, A) = (-3, 5)$, $(-7, -3)$ and $(-66, 51)$. In these instances, $s^2y^2 \in \mathbb{Z}^2$, and $G \cong C_4$. The related octics are the examples mentioned at the end of Section 6.3.

6.5. Regulator formulas. Let $\mathbf{k} = \mathbb{Q}$, and $m, A \in \mathbb{Z}$. If $T(m, A, r) = 0$ and $F = \mathbb{Q}(r)$, then \mathcal{O}_F^\times has rank 3 if F has signature $(0, 4)$ or $(4, 0)$. We show that in most such cases, the system $\langle \zeta, \varepsilon, r, \sigma(r) \rangle$ has rank 3, where $\langle \zeta \rangle$ is the torsion units in F , ε is the fundamental unit of a real quadratic subfield of F , $\sigma(r)$ is as in Equation (C4) with u as per Equation (4.7b). The regulator formulas are similar to those in [17] and [41].

In the following three results (which we state without proof), by Theorem 4.11 we can treat σ as a field isomorphism. Checking that the $\ln^2(\cdot)$ values are not both 0, is left as an exercise for the reader.

PROPOSITION 6.3. *Let $m \in \{-1, 0, 1\}$, $(m, A) \neq (-1, -3), (-1, 1), (0, -3), (0, -1)$, or $(1, -3)$. Let $\varepsilon > 1$ be the fundamental unit of $\mathbb{Q}(w)$ and $\langle \zeta \rangle$ the torsion units in $\mathbb{Q}(r)$. Then*

$$\text{Reg}\langle \zeta, \varepsilon, r, \sigma(r) \rangle = 16 \ln(\varepsilon)(\ln^2|r| + \ln^2|\sigma(r)|) \neq 0.$$

PROPOSITION 6.4. *Let $m > 2$ and $(m + 2 + A)^2 < 4(m - 2)$. Let $\varepsilon > 1$ be the fundamental unit of $\mathbb{Q}(s)$ and $\langle \zeta \rangle$ the torsion units in $\mathbb{Q}(r)$. Then*

$$\text{Reg}\langle \zeta, \varepsilon, r, \sigma(r) \rangle = 4 \ln(\varepsilon)(\ln^2|r^2/u| + \ln^2|\sigma(r)^2u|) \neq 0.$$

PROPOSITION 6.5. *Let $|m| > 2$. Assume $T(m, A, x) = P_{sw}(x)\bar{P}_{sw}(x)$ in $\mathbb{Z}[x]$, $(m, A) \neq (3, -2)$ or $(3, -3)$. Let $\varepsilon > 1$ be the fundamental unit of $\mathbb{Q}(s)$. If $T(m, A, r) = 0$, $[\mathbb{Q}(r) : \mathbb{Q}] = 4$, and $\mathbb{Q}(r)$ is real, then*

$$\text{Reg}\langle -1, \varepsilon, r, \sigma(r) \rangle = \frac{1}{2} \ln(\varepsilon)(\ln^2|r^2/u| + \ln^2|\sigma(r)^2u|) \neq 0.$$

For $P_w(x)$ as in Equations (4.13) and (4.14), $\sigma(r)$ is a unit in $\mathbb{Q}(r)$, but it is not an algebraic conjugate of r . In this case we have the following result.

PROPOSITION 6.6. *Let $m, d \in \mathbb{Z}$, $d \mid m - 2$, $d \neq -1$, $d^2 \leq |m - 2|$. Let $P_w(x)$ be as in Equations (4.13) and (b). Assume $P_w(x)$ is irreducible with signature $(4, 0)$, and $\varepsilon > 1$ is the fundamental unit of $\mathbb{Q}(s)$. Then up to a factor of $(1 + O(|d/m - 2|)) = (1 + O(1/\sqrt{|m - 2|}))$,*

$$\begin{aligned} &\text{Reg}\langle -1, \varepsilon, r, \sigma(r) \rangle \\ &\approx 2 \ln(\varepsilon) |\ln^2|m - 2| + \ln|(d + 1)^2/d| \ln|m - 2| - \ln|d| \ln|d + 1|. \end{aligned}$$

Proof. With $R = 1/(m - 2)$, we can express the zeroes of $x^2 - mx + 1$ or $x^2 - (1/R + 2)x + 1$ as formal power series in R , $u_1 = 1/R + 2 - R + 2R^2 + \dots$ and $u_2 = R - 2R^2 + \dots$, which converge for $|m - 2| > 4$.

Taking $q_1(x) = x^2 + (1 + d + u_1)x + u_1$, with a zero $r_1 \approx -1$, we obtain a formal series for r_1 with terms $p_k(d)R^k$, where $p_k(d) \in \mathbb{Z}[d]$ has degree k . This gives a series for $\sigma(r_1) = -(r_1 + 1)/(r_1 + u_1)$. Now $P_w(x) = q_1(x)q_4(x)$, where $q_4(x) = x^2 + (d + 1 + u_2)x + u_2$, which has a zero $r_3 \approx -d - 1$. (The coefficient of R^k in the series for r_3 has a power of $d + 1$ in the denominator.) Using $\varepsilon, r_1, \sigma(r_1); \varepsilon' = \pm\varepsilon^{-1}, r_3, \sigma(r_3) = -(r_3 + 1)/(r_3 + u_2)$; and $\varepsilon, r_2 = u_1/r_1, \sigma(r_2) = u_2/\sigma(r_1)$ to form the regulator determinant then gives the result. \square

REMARKS. When $d = -1$ the quartic fields defined by the $P_w(x)$ in Equation (6.1a) are totally real when $m < -2$, but $r + 1, r + u \in \langle -1, \varepsilon, r \rangle$, so $\langle -1, \varepsilon, r, \sigma(r) \rangle$ has rank 2 at most. When $m = 4 - t - t^2, t \in \mathbb{Z}, t > 2$, we have $G = V_4$. In this case, the fundamental units of the 3 quadratic subfields of the splitting field give 3 independent units.

Now $\sigma(r) = (-r - 1)/(r + u) = -(r + 1)^2/rq_1(-1)$. Thus, $r + 1$ and $r + u$ are both units precisely when $q_1(-1) = (m + A + 2 + w)/2$ is a unit. With $\mathbf{k} = \mathbb{Q}$ and $m, A \in \mathbb{Z}$, we then find that $q_1(-1) \in \langle \zeta, \varepsilon \rangle$, so

$$(6.5) \quad \begin{aligned} & [\langle \zeta, \varepsilon, r, r + 1 \rangle : \langle \zeta, \varepsilon, r, \sigma(r) \rangle] = 2, \\ & \text{when } q_1(-1) = \pm 1, \text{ or } m - 2 = \pm 1. \end{aligned}$$

This gives a “one-half” regulator in Proposition 6.3 when $m = 1$; in Proposition 6.4 when $m = 3$ and $A = -6, -5, \text{ or } -4$; for the quartics in Equations (6.3a)–(6.4b); and for the quartics in Equation (6.1b).

The regulators in Propositions 6.3–6.6 can be extremely small. We estimate the “one-half” regulator $R_1 = \text{Reg} \langle \zeta, \varepsilon, r, r + 1 \rangle$ for infinite families with $\varepsilon = \tau$, the “golden ratio;” this makes the factor $\ln(\varepsilon)$ as small as possible.

When $m = 1, A \in \mathbb{Z}, A \neq -3$, and $T(1, A, r) = 0, F = \mathbb{Q}(r)$ is a totally imaginary octic field. Here $y^2 = A^2 + 4$. By Equation (4.11c), if $A^2 + 4$ is squarefree, $\Delta(F/\mathbb{Q}) = (\Delta(\mathbb{Q}(s, w)/\mathbb{Q}))^2(A^2 + 4)^2$. Taking $A = L_{2j-1} - 3$ makes $\varepsilon = \tau$. Then, refining $r \approx -1$ gives, assuming $(L_{2j-1} - 3)^2 + 4$ is squarefree,

$$(6.6) \quad R_1 = \frac{1}{2} \ln(\tau) \ln^2 \left(\frac{1}{15^4} \Delta(F/\mathbb{Q}) \right) (1 + O(1/|A|)).$$

Let F be a real cyclic quartic field given by Equation (6.3a). Using Lemma 4.4(d), $\text{disc}(q_1(x)) \text{disc}(q_3(x)) = 5(F_{2j-1} - 2)^2$. Refining $r \approx -1$, and using Proposition 6.5, we find that if $F_{2j-1} - 2$ is squarefree and prime to 10, then

$$(6.7) \quad R_1 = \frac{1}{4} \ln(\tau) \ln^2 \left(\frac{1}{5^2} \Delta(F/\mathbb{Q}) \right) (1 + O(1/j^2)).$$

Replacing $F_{2j-1} - 2$ with $F_{2j-1} + 2$, and assuming this is squarefree and prime to 10, we again obtain Equation (6.7) for the real cyclic quartic fields given by Equation (6.4b).

Using Proposition 6.6 and Equation (4.11c), we obtain an estimate asymptotically equal to that in Equation (6.7) for the nonnormal quartic fields defined by Equation (6.1b), taking $m = L_{2j}$, a Lucas number of even index, if $4L_{2j}^2 + 9$ is squarefree.

The regulator estimates in Equations (6.6) and (6.7) are comparable to the lower bound in [35], for totally imaginary octic fields with a real quadratic but not a real quartic subfield, or for real quartic fields with a quadratic subfield.

We do not know that the squarefreeness conditions are satisfied infinitely often, but we do not know any reason to assume otherwise. The numbers $F_{2j-1} \pm 2$ are always the product of a Fibonacci number and a Lucas number whose indexes differ by 3, but the squarefreeness question for these is also open.

In [41], Washington obtains a lower bound for the regulators of the cyclic quartic fields defined by $f_t(x)$ which proves the system of 3 units he gives is fundamental when t , $t + 4$, and $t^2 + 4$ are squarefree, except when $t = 1$. The system in Proposition 6.5 for $P_t(x)$ has the same regulator. In the case $t = 1$, the system $\langle -1, \tau, r, r + 1 \rangle$ for Equations (6.4a) and (6.4b) with $j = 0$ is fundamental.

7. Concluding remarks

The condition (M) provides a conceptual unification of all families of number fields previously dubbed “simplest.” It also enabled us to rederive both the “simplest” quartic fields and Washington’s cyclic quartic fields by elementary methods, and to place both families together in a larger context.

For purely algebraic purposes, the substitution $A \leftarrow -(m + 2)/2 + \mathbf{v}$ used in the proof of Theorem 4.13 may be useful.

However, more sophisticated methods than used here would surely be required to construct algebraic maps σ satisfying (M) for $n > 4$. More powerful techniques would probably also refine and elaborate some of our results considerably, particularly those in Section 6.5. Those wanting to investigate class number or unit index questions might want to read [8], [9], [17], [26], [12] and [13] first.

We note that (M) and its σ -conjugates can be viewed as a system of n constraints which may limit the relative sizes of $r, \sigma(r), \dots, \sigma^{n-1}(r)$. This might help account for the previously-noted feature of small regulators in “simplest” cyclic number fields of low degree. In this regard, the factorization of the “circulant” matrix determinant (see, for example, Lemma 5.26 in [39]) might be of interest.

We chose the variable names s and w prior to learning about the “simplest” quartic fields or Washington’s cyclic quartic fields. The fact that the conditions $s = 0$ and $w = 0$ turned out to produce the “simplest” quartic fields and

Washington’s cyclic quartic fields respectively, is a coincidence that absolutely delights the author.

While investigating Equation (C4), we noticed that $f: x \mapsto (-x - 1)/(x + u)$ has compositional order 10 when $u^2 + 3u + 1 = 0$. The compositional powers $f^{(3)}$ and $f^{(7)}$ make (M) a formal identity with $n = 10$. With $K = \mathbb{Q}(\sqrt{5})$,

$$\sum_{k=0}^9 f^{(k)}(x) = 2(A + Bu), \quad A, B \in \mathbb{Z}$$

produces a family of totally real C_{10} extensions of K defined by units whose conjugates satisfy (M), and which have exceptional sequences of *four* units. The normal closure over \mathbb{Q} has “generic” Galois group ${}_{20}T_{53}$.

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REFERENCES

- [1] D. A. Buell and V. Ennola, *On a parameterized family of quadratic and cubic fields*, J. Number Theory **54** (1995), 134–148. MR 1352641
- [2] R. J. Chapman, *Automorphism polynomials in cyclic cubic extensions*, J. Number Theory **61** (1996), 283–291. MR 1423054
- [3] H. Cohen, *A course in computational algebraic number theory*, Springer-Verlag, Berlin, 1993. MR 1228206
- [4] H. Cohen, F. Diaz y Diaz and M. Olivier, *Tables of octic fields with a quartic subfield*, Math. Comp. **68** (1999), 1701–1716. MR 1642813
- [5] H. Cohn, *A device for generating fields of even class number*, Proc. Amer. Math. Soc. **7** (1956), 595–598. MR 0079613

- [6] H. Cohn, *Advanced number theory*, Dover, New York 1980. Originally published in 1962 as *A second course in number theory* by John Wiley and Sons. MR 0133281
- [7] H. Darmon, *Note on a polynomial of Emma Lehmer*, Math. Comp. **56** (1991), 795–800. MR 1068821
- [8] M. N. Gras, *Special units in real cyclic sextic fields*, Math. Comp. **48** (1977), 179–182. MR 0866107
- [9] M. N. Gras, *Table numérique du nombre de classes et des unités des extensions cycliques réelles de degré 4 de \mathbf{Q}* , Publ. Math. Fac. Sci. Bensaçon **2** (1977/8), 1–26, 1–53.
- [10] M. Hall, Jr., *The theory of groups*, Macmillan, New York, 1959. MR 0103215
- [11] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, Oxford University Press, New York, 1979. MR 0568909
- [12] H. Hasse, *Arithmetische Bestimmung von Grundeinheit und Klassenzahl in zyklischen kubischen und biquadratischen Zahlkörpern*, Abh. Deutsch. Akad. Wiss. Berlin. Math.-Nat. Kl. **1948** (1948), no. 2, 95 pp. (1950). Republished in *Mathematische Abhandlungen*, Walter de Gruyter, Berlin, 1975, pp. 285–379. MR 0033863
- [13] H. Hasse, *Über die Klassenzahl abelscher Zahlkörper*, Akademie-Verlag, Berlin, 1952. MR 0049239
- [14] D. Holt, private communications.
- [15] L. K. Hua, *Introduction to number theory*, English ed., Springer-Verlag, Berlin, 1982. MR 0665428
- [16] S. Lang, *Algebra*, revised ed., Addison-Wesley, Reading, MA, 1971. MR 0197234
- [17] A. J. Lazarus, *On the class number and unit index of simplest quartic fields*, Nagoya Math J. **121** (1991), 1–13. MR 1096465
- [18] O. Lecacheux, *Unités d’une famille de corps cycliques réels de degré 6 liés à la courbe modulaire $X_1(13)$* , J. Number Theory **31** (1989), 54–63. MR 0978099
- [19] E. Lehmer, *Connection between Gaussian periods and cyclic units*, Math. Comp. **50** (1988), 535–541. MR 0929551
- [20] H. W. Lenstra, Jr., *Euclidean number fields of large degree*, Invent. Math. **38** (1977), 237–254. MR 0429826
- [21] G. Lettl, A. Pethő and P. Voutier, *Simple families of Thue inequalities*, Trans. Amer. Math. Soc. **351** (1999), 1871–1894. MR 1487624
- [22] G. Lettl, A. Pethő and P. Voutier, *On the arithmetic of simplest sextic fields and related Thue equations*, Number theory (K. Györy, A. Pethő and V. T. Sós, eds.), Walter de Gruyter, Berlin, 1998, pp. 331–348. MR 1628852
- [23] A. Leutbecher, *Euclidean fields having a large Lenstra constant*, Ann. Inst. Fourier Grenoble **35** (1985), 83–106. MR 0786536
- [24] A. Leutbecher and J. Martinet, *Lenstra’s constant and Euclidean number fields*, Astérisque **94** (1982), 87–131. MR 0702368
- [25] A. Leutbecher and G. Niklasch, *On cliques of exceptional units and Lenstra’s construction of Euclidean fields*, Number theory, Proc. Jour. Arith., Ulm, 1987 (H. P. Schlickewei and E. Wirsing, eds.), Lecture Notes in Mathematics, vol. 1380, Springer-Verlag, New York, 1989, pp. 150–178. MR 1009799
- [26] S. R. Louboutin, *Efficient computation of root numbers and class numbers of parametrized families of real abelian number fields*, Math. Comp. **76** (2007), 455–473. MR 2261031
- [27] J. Mináč and T. L. Smith, *A characterization of C -fields via Galois groups*, J. Algebra **137** (1991), 1–11. MR 1090208
- [28] P. Morton, *Characterizing cyclic cubic extensions by automorphism polynomials*, J. Number Theory **49** (1994), 183–208. MR 1305089
- [29] P. Morton, *Arithmetic properties of periodic points of quadratic maps, II*, Acta Arithmetica **87** (1998). MR 1665198

- [30] T. Nagell, *Sur un type particulier d'unités algébriques*, Arkiv f. Matem. **8** (1969), no. 18, 163–164. [MR 0268165](#)
- [31] Y. Rikuna, *On simple families of cyclic polynomials*, Proc. Amer. Math. Soc. **130** (2002), no. 8, 2215–2218. [MR 1896400](#)
- [32] R. Schoof and L. C. Washington, *Quintic polynomials and real cyclotomic fields with large class numbers*, Math. Comp. **50** (1988), no. 182, 543–556. [MR 0929552](#)
- [33] D. Shanks, *The simplest cubic fields*, Math. Comp. **28** (1974), 1137–1152. [MR 0352049](#)
- [34] Y. Y. Shen, *Unit groups and class numbers of real cyclic octic fields*, Trans. Amer. Math. Soc. **326** (1991), 179–209. [MR 1031243](#)
- [35] J. H. Silverman, *An inequality relating the regulator and the discriminant of a number field*, J. Number Theory **19** (1984), 437–442. [MR 0769793](#)
- [36] B. K. Spearman and K. S. Williams, *Normal integral bases for Emma Lehmer's parametric family of cyclic quintics*, J. Théor. Nombres Bordeaux **16** (2004), 215–220. [MR 2145582](#)
- [37] E. Thomas, *Fundamental units for orders in certain cubic number fields*, J. Reine Angew. Math. **310** (1979), 33–55. [MR 0546663](#)
- [38] K. Uchida, *Class numbers of cubic cyclic fields*, J. Math. Soc. Japan **26** (1974), no. 3, 447–453. [MR 0360518](#)
- [39] L. C. Washington, *Introduction to cyclotomic fields*, Graduate Texts in Math., vol. 83, Springer-Verlag, New York, 1982. [MR 0718674](#)
- [40] L. C. Washington, *Class numbers of the simplest cubic fields*, Math. Comp. **48** (1987), no. 177, 371–384. [MR 0866122](#)
- [41] L. C. Washington, *A family of cyclic quartic fields arising from modular curves*, Math. Comp. **57** (1991), 763–775. [MR 1094964](#)

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