Illinois Journal of Mathematics
Volume 54, Number 2, Summer 2010, Pages 485–499
S 0019-2082

COHOMOLOGY OF DECOMPOSITION AND THE MULTIPLICITY THEOREM WITH APPLICATIONS TO DYNAMICAL SYSTEMS

LEON A. LUXEMBURG

ABSTRACT. The article obtains some lower bounds for the sectional category of a map based on the cohomology of the base space and the total space. We also obtain geometric results on the multiplicity of maps and show applications to equilibria on the boundary of stability regions (basins of attraction) of dynamical systems on differentiable manifolds. We consider a number of generalizations of Lusternik–Schnirelmann's theorem which states that if a covering of an n dimensional sphere consists of $n+1$ closed sets, then at least one of the sets contains antipodal points. An elementary proof is given for a generalization of this result to packings of Euclidean spaces.

1. Introduction

In this article, we prove some geometric results on multiplicity of maps of CW complexes which are unions of a finite number of closed sets. A stronger result for Euclidean spaces is obtained via an elementary proof (not requiring algebraic topology) for packings of Euclidean spaces. We also consider applications to flows of dynamical systems. Some of the results of this article connect and trace back to Lusternik–Schnirelmann's theorem [2] stating that if a sphere S^n is a union of $n+1$ closed subsets then at least one of them contains a pair of antipodal points. We generalize it for coverings of H-spaces and of products of real projective spaces.

The sectional category of a map $f : X \to Y$, denoted $\mathrm{secat}(f)$, is one less than the number of sets in the smallest open cover of Y such that f admits a

Received June 1, 2008; received in final form February 8, 2010.

The author wants to offer thanks to the referee of this article for a number of very valuable suggestions.

2010 *Mathematics Subject Classification.* Primary 55M30. Secondary 37D15, 55R10.

cross-section over each member of the cover, see, for example, [4]–[6]. By the Lusternik–Schnirelmann category $\mathrm{cat}(Y)$ of a space Y, we mean the smallest number of open subsets of Y covering Y and contractible in Y.

One of the objectives of this article is to introduce easily computable lower bounds for the sectional category in terms of the cohomology of the base space Y and the total space X and to show some applications of our method to giving lower bounds on the number of equilibria on the boundaries of stability of flows of dynamical systems. The Morse–Smale theory cannot be used for covering spaces of compact manifold because these covering spaces are usually noncompact. However, most practical problem occur in R^n which is a universal cover for the torus. This calls for alternative methods presented in this paper. If the map $f : X \to Y$ is a fibration, then every subset contractible in Y admits a cross-section, therefore, obviously, $\mathrm{secat}(f) + 1 \leq \mathrm{cat}(Y)$. This shows that giving a lower bound on sectional category means providing a lower bound for the Lusternik–Schnirelmann (LS) category, but not vice versa. Theorem 2.1 provides lower bounds for the sectional category and thus also gives the lower bounds for the LS category. Lower bounds for the LS category of a space based on its cohomology were obtained previously in a large number of papers, see, for example, [5, 8, 9].

The sectional category and its relations to the Lusternik–Schnirelmann category was studied in [4, 5] and [6]. In his seminal article [10], Schwartz introduced the concept of a sectional category and described a method of providing lower bounds for it in terms of the nilpotency of the kernel $p^* : H^*(B, Q) \to H^*(E, Q)$ for a fibration $p : E \to B$.

However, the lower bound given by Theorem 2.1 is much easier to compute because it is formulated in terms of cohomology of the base space and the total space only. Additionally, the nilpotency based lower bound would give no results for the products of real projective spaces of dimension > 1 since all the cohomology with rational coefficients Q will be zero in this case. However, our approach allows us to get some interesting results in this particular case as well, see Theorem 2.4 below. See [11] for another interesting approach to obtaining lower bounds for a sectional category which is still more difficult computationally.

2. Statements of main theorems

The proofs of Theorems 2.1–2.4 stated in this section will be given in Section 4. The spaces considered in this article are assumed Hausdorff and maps are assumed continuous. We use singular homology and cohomology throughout the article.

Notation 2.1. Let $p > 0$ be an integer and let $h_1 \leq h_2 \leq \cdots \leq h_p$, be a nondecreasing sequence of positive integers, then by $F[h_1, \ldots, h_p, x_1, \ldots, x_p]$ we denote the graded algebra of truncated polynomials of variables $x_1, \ldots, x_p$

of degree $\leq h_i$ in variable x_i over a field F with the usual skew commutativity assumption $xy = (-1)^{\deg(x)\deg(y)}yx$.

Alternatively, one can describe $F[h_1, \dots, h_p, x_1, \dots, x_p]$ as a tensor product of rings $F(x_i, h_i)$, where $F(x_i, h_i)$ is a factor ring of the ring $F[x_i]$ of polynomials of one variable x_i over an ideal generated by the element $x_i^{h_i+1}$.

We set formally $h_0 = 0$ and let now $k > 0$ be an integer such that for some integer i, $1 \leq i \leq p-1$, we have

$$h_0 + \cdots + h_i < k \leq h_0 + h_1 + \cdots + h_i + h_{i+1}.$$

Then we define $\ell(h_1, \dots, h_p, k)$ by the equality $\ell(h_1, \dots, h_p, k) = i$.

We note that if $h_i = 1$ for all i, $1 \leq i \leq p$ then $\ell(h_1, \dots, h_p, k) = k-1$.

Also, $\ell(h_1, \dots, h_p, k) < p$ since $k \leq h_0 + h_1 + \cdots + h_p$, therefore,

$$p - \ell(h_1, \dots, h_p, k) > 0.$$

If $f : X \to Y$ is an arbitrary map between two sets, then we will say that f has multiplicity n if n is the smallest integer such that the inverse image $f^{-1}(y)$ of every point $y \in Y$ consists of no more than n points.

THEOREM 2.1 (Multiplicity theorem). *Let M be a normal space and let F be any field. Assume that the cohomology ring $H^*(M, F)$ contains a graded subalgebra $F[h_1, \dots, h_p, x_1, \dots, x_p]$ of truncated polynomials of variables x_1, $\dots$, x_p, where for all i, $1 \leq i \leq p$ we have $x_i \in H^1(M, F)$. Let k be a positive integer, $k \leq h_1 + \cdots + h_p$ and let $\rho : \widetilde{M} \to M$ be a map such that*

$$\operatorname{rank}(H^1(\widetilde{M}, F)) < p - \ell(h_1, \dots, h_p, k).$$

Then, the following is true:

$$\text{(2.1)}\quad \begin{minipage}{0.8\textwidth}\textit{For any k closed sets $H_i \subset M$, where $1 \leq i \leq k$, satisfying the condition $\bigcup_{i=1}^k H_i = M$ and any compact sets $L_i \subset \widetilde{M}$ such that $\rho(L_i) = H_i$ for all $i, 1 \leq i \leq k$, there exists a pair of points x, y in some L_i such that $\rho(x) = \rho(y)$, i.e. ρ has multiplicity greater than 1 on at least one set L_i. In particular,} $\operatorname{secat}(\rho) \geq k$.\end{minipage}$$

REMARK 2.1. Since the definition of a sectional category of a map involves cross sections over open sets and Theorem 2.1 talks about closed sets, some clarification about why (2.1) implies that $\operatorname{secat}(\rho) \geq k$, is in order. Let $\mathcal{U} = \{U_i : 1 \leq i \leq k\}$ be a finite open cover of a normal space space M such that each U_i admits a cross-section with respect to a map $\rho : \widetilde{M} \to M$. It is a well known simple fact from the point set topology that there exists a closed cover of M, $\mathcal{V} = \{C_i : 1 \leq i \leq k\}$ such that $C_i \subset U_i$. Then, the restrictions of cross-sections from U_i onto C_i define cross-sections on C_i. Therefore, if for some integer k and any closed cover $\mathcal{V} = \{C_i : 1 \leq i \leq k\}$ of M there do not exist cross-sections over each set C_i, then $\operatorname{secat}(\rho) \geq k$.

Additionally, we can show that under some conditions, the converse is also true. To be more precise, let us assume that $\rho : \widetilde{M} \to M$ is a fibre bundle map with the fibre which is ANR and such that M is paracompact. Let us also consider a closed cover $\mathcal{V} = \{C_i : 1 \le i \le k\}$ of M such that each set C_i admits a cross-section ρ_i with respect to ρ. Then there are open sets U_i, $C_i \subset U_i$ such that for each $i, 1 \le i \le k$, there is an extension $\zeta_i : U_i$ to $\widetilde{M}$ of cross-section ρ_i from C_i onto U_i. So, in this case, it does not matter whether we define the sectional category for closed or open sets. The proof is not difficult, however, we are not going to prove the latter fact here because it is not essential for the results claimed in this article.

THEOREM 2.2. *Let $\pi : R^n \to T^n$ be a factor map from the n dimensional Euclidean space onto a torus mapping an equivalence class of all vectors with coordinates differing by an integer into a corresponding point on a torus T^n. Let T^n be a union of k closed sets H_i, satisfying the condition*

$$\bigcup_{i=1}^{k} H_i = T^n, \quad 1 \le k \le n+1$$

and let compact sets $L_i \subset R^n$ where $1 \le i \le k$ be such that $\pi(L_i) = H_i$. Then the sum $\sum_{i=1}^{k} \mu_i$, where μ_i is a multiplicity of the restriction of map π on L_i, is at least $n+1$. In particular, $\operatorname{secat}(\pi) \ge n$.

THEOREM 2.3. *Let M be an H-space and is a finite CW complex and let F be a field of characteristic* 0. *Let k be an integer such that $1 \le k \le \beta_1(M,F)$ where*

$$\beta_1(M,F) = \operatorname{rank}(H_1(M,F)) = \operatorname{rank}(H^1(M,F)).$$

Let $\rho : \widetilde{M} \to M$ be a map such that

$$\operatorname{rank}(H^1(\widetilde{M},F)) = \beta_1(\widetilde{M},F) < \beta_1(M,F) - k + 1.$$

Then the condition (2.1) *is satisfied, in particular,* $\operatorname{secat}(\rho) \ge k$.

THEOREM 2.4. *Let $M = \prod_{i=1}^{p} RP^{h_i} \times T^n$ be a product of p real projective spaces RP^{h_i} of dimension $h_i > 1$, and an n dimensional torus T^n where p or n can be zero. Let k be an integer, and let*

$$1 \le k \le h_1 + \cdots + h_p + n.$$

Let $\rho : \widetilde{M} \to M$ be a map such that

$$\operatorname{rank}(H^1(\widetilde{M},F)) < p + n - \ell(1,\ldots,1,h_1,\ldots,h_p,k)$$

where 1 *is repeated* n *times. Then* (2.1) *is satisfied, in particular,* $\operatorname{secat}(\rho) \ge k$.

COROLLARY 2.1. *Let $L \subset R^n$ be a compact set such that all of its parallel translations by vectors with integer coordinates fill the whole R^n. Then L contains $n+1$ points $x_1, \ldots, x_{n+1}$ such that for some vector $y \in R^n$ all points $x_1 + y, \ldots, x_{n+1} + y$ have only integer coordinates.*

Proof. This is a restatement of Theorem 2.2 for $k=1$. □

COROLLARY 2.2. *Let $M=T^n$ and let $1\le k\le n$. Let F be a field and let $\rho:\widetilde{M}\to M$ be a map such that*

$$\operatorname{rank}(H^1(\widetilde{M},F))<n-k+1.$$

Then $\operatorname{secat}(\rho)\ge k$.

This corollary immediately follows from Theorem 2.4 above.

REMARK 2.2. The Lusternik–Schnirelmann's result [2] that in any decomposition of a sphere

$$S^n=\bigcup_{i=1}^{n+1}L_i$$

as a union of $n+1$ closed sets L_i, at least one of sets would contain two antipodal points and an independently proved equivalent K. Borsuk's result that the n dimensional sphere as a geometric object cannot be decomposed into a union of $n+1$ sets of smaller diameters, follow from Theorem 2.1. Indeed, suppose none of the sets L_i contains antipodal points, then the union of just any n sets $K=\bigcup_{i=1}^{n}L_i$ satisfies the condition that $\rho(K)=RP^n$ where $\rho:S^n\to RP^n$ is the covering map of S^n onto the projective space RP^n. Now, since $H^*(RP^n)=Z_2(n,x)$ for some generator $x\in H^1(RP^n)$, and S^n is simply connected, by Theorem 2.1 for the case when $m=1$, at least one of the sets L_i, where $1\le i\le n$, contains two points x,y such that $\rho(x)=\rho(y)$ which means they are antipodal. Theorem 2.4 is also another generalization of Lusternik–Schnirelmann's theorem.

Applications to equilibria of flows of dynamical systems on manifolds. A vector field X on a differentiable manifold M is $C(k)$ if it has exactly k stable equilibria $e_1,\dots,e_k$ on M with corresponding open regions of attractions $S(e_1),\dots,S(e_k)$ such that $M=\bigcup_{i=1}^{k}\overline{S(e_i)}$ and every trajectory of the flow corresponding to X converges to some (not necessarily stable) equilibrium point as $t\to\infty$. (Here, by a stable equilibrium we mean an equilibrium point of the vector filed X such that the corresponding Jacobian has all eigenvalues in the open left half plane, and by the region of attraction or stability region of a stable equilibrium we mean its stable manifold, that is, the union of all trajectories converging to this equilibrium.) Let $\rho:\widetilde{M}\to M$ be a covering map from a covering space $\widetilde{M}$ onto M with a vector field X defined on M and let $X(\rho)$ be the induced vector field on the manifold $\widetilde{M}$, that is, the uniquely defined vector field such that $\rho_*(X(\rho))=X$, where ρ_* is the induced vector bundle map.

THEOREM 2.5. *Let M be a compact manifold. Assume that the cohomology ring $H^*(M,F)$ for some algebraic field F contains a graded subalgebra $F[h_1,\dots,h_p,x_1,\dots,x_p]$ of truncated polynomials of variables $x_1,\dots,x_p,x_i\in$*

$H^1(M,F)$ for all i such that $1 \le i \le p$. Let k be an integer such that $1 \le k \le h_1 + \cdots + h_p$ and let $\rho : \widetilde{M} \to M$ be a covering map satisfying

$$\text{rank}(H^1(\widetilde{M},F)) < p - \ell(h_1,\ldots,h_p,k). \tag{2.2}$$

Let also X be a $C(k)$ vector field on M and $X(\rho)$ be the induced vector field on $\widetilde{M}$. Let $e_1,\ldots,e_k$ be all stable equilibria on M with corresponding open regions of attractions $S(e_1),\ldots,S(e_k)$ so that

$$M = \bigcup_{i=1}^{k} \overline{S(e_i)}.$$

Let equilibria $d_1,\ldots,d_k$ in $\widetilde{M}$ satisfy $\rho(d_i) = e_i$ and let them have compact regions of attractions $S(d_1),\ldots,S(d_k)$ respectively. Then there exist at least 2 equilibria x and y on the boundary of at least one of stability regions $S(d_i)$ such that $\rho(x) = \rho(y)$.

This is a principal result showing that under certain restrictions on k in terms of the cohomology of the manifold M, the number of equilibria on the boundaries of stability regions strictly increases when we consider the induced vector field on universal coverings, and, more generally, on coverings satisfying (2.2), because at least some equilibria on a boundary of a stability region in $\widetilde{M}$ will be mapped into the same equilibrium in M. The implication of this is that the Morse–Smale theory [7], which gives the lower bounds for equilibria on compact manifolds M for Morse–Smale fields, fails to give precise lower bounds for the induced vector fields on generally noncompact covering manifolds $\widetilde{M}$. This fact is important in applications when we have a periodic vector field on R^n which factors through the universal cover over a torus, $\pi : R^n \to T^n$, see [12]–[13].

Proof of Theorem 2.5. Theorem 2.5 immediately follows from Theorem 2.1. Indeed, according to Theorem 2.1, for $H_i = \overline{S(e_i)}$, there exist two distinct points x_1 and y_1 in $L_i = \overline{S(d_i)}$ for some integer i, where $1 \le i \le k$, such that $\rho(x_1) = \rho(y_1)$. Since X is $C(k)$, for trajectories $x(t), y(t)$ passing through these points respectively, we will have

$$\lim_{t\to\infty} x(t) = x, \qquad \lim_{t\to\infty} y(t) = y.$$

It is easy to see that x, y are distinct equilibria which lie on the boundary of $S(d_i)$ and $\rho(x) = \rho(y)$. □

Let us consider another example. Let Y be a periodic vector field on R^n with exactly one stable equilibrium point in each period and let us assume that every trajectory of Y converges to an equilibrium point, that is, we will assume that $\bigcup_{i=1}^{\infty} \overline{S(d_i)} = R^n$, where $S(d_i)$ is the region of stability of the stable equilibrium d_i and all equilibria d_i are obtained by translation of some equilibrium point d_1 via a vector with integer coordinates. (This means that $Y = X(\pi)$ for some $C(1)$ vector field X on the torus T^n.) Assume also that sets $S(d_i)$ are bounded.

Then from Corollary 2.1 it follows that the factor map $\pi : R^n \to T^n$ has multiplicity $n+1$ on every $\overline{S(d_i)}$. Just as in the proof of Theorem 2.5 given above, we can now show that there are at least $n+1$ equilibria on $\overline{S(d_i)}$ which are mapped into one point by π. Since π is obviously one to one on $S(d_i)$, these equilibria lie on the boundary of $S(d_i)$. This proves the following corollary.

Corollary 2.3. *Let X be a $C(1)$ vector field on a torus T^n and let $Y = X(\pi)$ be the vector field on R^n induced by the factor map $\pi : R^n \to T^n$. Then there are at least $n+1$ equilibria on the boundary of every region of stability $S(d_i)$ of any stable equilibrium d_i of the vector field $X(\pi)$ which are mapped into the same point by the factor map π.*

3. Truncated polynomial algebra decomposition

This section contains more than a minimum necessary to prove the results of Section 2, because the cohomology of decomposition is of some independent interest.

Theorem 3.1. *Let the cohomology ring $H^*(M,F)$ for some field F contain algebra*

$$F[h_1,\dots,h_p,x_1,\dots,x_p], \quad h_1 \le \dots \le h_p,$$
$$x_i \in H^1(M,F) \text{ for each } i, 1 \le i \le p.$$

Let for some integer $k, 1 \le k \le h_1 + \dots + h_p$, a space $M = \bigcup_{s=1}^k H_s$ be a union of k arbitrary closed sets H_s. Then for the inclusion maps $i_s : H_s \to M$ and the induced homomorphisms $i_s^* : H^1(M,F) \to H^1(H_s,F)$ we have:

$$\text{(i)} \qquad \sum_{s=1}^{k} \operatorname{rank} i_s^*(H^1(M,F)) \ge p,$$

$$\text{(ii)} \qquad \max\{\operatorname{rank} i_s^*(H^1(M,F)) : 1 \le s \le k\} \ge p - \ell(h_1,\dots,h_p,k).$$

Notation 3.1. Let A=$F[h_1,\dots,h_p,x_1,\dots,x_p], h_1 \le \dots \le h_p$ and let A_i be the submodule of A consisting of polynomials of degree i. Then A_1 is a linear vector space over F spanning elements $x_1,\dots,x_p$ and $\dim A_1 = p$. Let

$$x = \sum_{i=1}^{p} \xi_i x_i \in A_1, \quad \xi_i \in F \tag{3.1}$$

be an arbitrary nonzero vector from A_1. Let i be the smallest integer such that $\xi_i \ne 0$, then we set $x_i = L(x)$ and will call x_i the leading element of x. Any set S consisting of some, possibly repeated, elements of $T = \{x_1,\dots,x_p\}$

will be called admissible if each x_i occurs in S no more than h_i times. An element $x_i \in S$ will be called saturated if it occurs in S exactly h_i times and S will be called saturated if its every element is (saturation of S does not imply that every $x_i \in A_1$ also belongs to S). We will introduce a lexicographic order for monomials in A so that $\prod_{i=1}^{p} x_i^{\mu_i} < \prod_{i=1}^{p} x_i^{\nu_i}$ if for the smallest i such that $\mu_i \neq \nu_i$ we have $\mu_i < \nu_i$. Let also

$$k_0 = h_1 + \cdots + h_p. \tag{3.2}$$

LEMMA 3.1. *Let $c_1, \ldots, c_k, k \leq k_0$ be some elements of A_1 as above and let $S = \{x_{i_1}, \ldots, x_{i_k}\}$ be the set of their corresponding leading elements, $x_{i_s} = L(c_s)$. Then, if S is an admissible set, we have*

$$\prod_{j=1}^{k} c_j \neq 0. \tag{3.3}$$

Proof. The product $\prod_{j=1}^{k} c_j$ is a sum of nonzero monomials of degree k and the monomial $\prod_{j=1}^{k} x_{i_j}$ which is the product of the leading elements of $c_j, x_{i_j} = L(c_j)$, has a nonzero coefficient in this sum, is itself nonzero, and is strictly smaller than all other monomials in this sum. Therefore, it is not cancelled and $\prod_{j=1}^{k} c_j \neq 0$. □

LEMMA 3.2. *Let $V_1, \ldots, V_l, V_{l+1}, l \geq 1$ be linear subspaces of A_1. Suppose for every i, $1 \leq i \leq l$, an element $c_i \in V_i$ is chosen so that the set of leading elements $S = \{L(c_i) : 1 \leq i \leq l\}$ is an admissible set. Then if for some integer $j, 0 \leq j < p$ we have*

$$l + 1 \leq h_1 + \cdots + h_{j+1}, \tag{3.4}$$

$$\dim V_{l+1} \geq j + 1 \tag{3.5}$$

there exists an element $c_{l+1} \in V_{l+1}$ such that the set $\{L(c_{l+1})\} \cup S$ is also admissible and, therefore, by Lemma 3.1,

$$\prod_{j=1}^{l+1} c_j \neq 0.$$

Proof. Let $Y = \{x_{i_1}, \ldots, x_{i_r}\}$ be the set of all elements in S which are saturated (without repetition), then since cardinality$(S) = l$, (3.4) and the sequence $h_1 \leq \cdots \leq h_p$ is nondecreasing, it follows that

$$r \leq j. \tag{3.6}$$

Let

$$T = \{x_1, \ldots, x_p\} \tag{3.7}$$

and let Q be the vector subspace of A_1 spanned over all elements in the set $T \backslash Y$, then

$$\dim Q \geq p - r$$

and (3.5) and (3.6) imply that $\dim V_{l+1} \geq r+1$ and, therefore, there exists a nonzero element c_{l+1} such that

$$c_{l+1} \in V_{l+1} \cap Q. \tag{3.8}$$

Clearly, its leading coefficient is not in Y and, therefore, the set $\{L(c_{l+1})\} \cup S$ is admissible. $\square$

LEMMA 3.3. *Let $V_1, \ldots, V_k$ be linear subspaces of $A_1, 1 \leq k \leq k_0$. Then if*

$$\sum_{i=1}^{k} \dim V_i > (k-1)p \tag{3.9}$$

then there exist elements $c_i \in V_i$ for each i satisfying $1 \leq i \leq k$ such that the product

$$\prod_{i=1}^{k} c_i \neq 0.$$

Proof. We will select elements $c_i \in V_i$ with the property that

(3.10) the leading elements $L(c_i)$ for $i \leq l$ form an admissible set

by induction on l. Let spaces $V_1, \ldots, V_k$ be numbered in the order of nondecreasing dimensions,

$$\dim V_1 \leq \cdots \leq \dim V_k. \tag{3.11}$$

Note, that $\dim V_1 > 0$, otherwise, we would have (contrary to (3.9)) that

$$\sum_{i=1}^{k} \dim V_i \leq (k-1)p$$

due to the fact that for all i, $\dim V_i \leq \dim A_1 = p$. Let $c_1 \in V_1$ be an arbitrary nonzero element. Assume inductively, that elements $c_i \in V_i$, $1 \leq i \leq l$ are selected so that the set $\Theta = \{L(c_i) : 1 \leq i \leq l\}$ is admissible. Let $\dim V_{l+1} = p$. Then, since $l+1 \leq k \leq k_0 = h_1 + \cdots + h_p$, for $j = p-1$, the conditions (3.4) and (3.5) of Lemma 3.2 are satisfied and, therefore, by the same lemma, an element $c_{l+1} \in V_{l+1}$ can be chosen so that the set $\Theta \cup \{L(c_{l+1})\}$ is admissible, which completes the inductive step. Now assume that

$$\dim V_{l+1} < p. \tag{3.12}$$

This, and (3.11), (3.9) imply that

$$\begin{aligned}(k-1)p < \sum_{i=1}^{k} \dim V_i &= \sum_{i=1}^{l+1} \dim V_i + \sum_{i=l+2}^{k} \dim V_i \\ &\leq (l+1)(p-1) + p(k-l-1) = p(k-1) + (p-l-1).\end{aligned}$$

Therefore,

$$l+1 < p, \tag{3.13}$$

(3.12) and (3.11) imply that $\dim V_1 \leq \cdots \leq \dim V_{l+1} \leq p-1$. Therefore,

$$\dim V_{l+1} \geq l+1 \tag{3.14}$$

because, otherwise, from (3.11), (3.13) it follows that

$$\dim V_1 \leq \cdots \leq \dim V_{l+1} \leq l$$

and

$$\begin{aligned}\sum_{i=1}^{k} \dim V_i &= \sum_{i=1}^{l+1} \dim V_i + \sum_{i=l+2}^{k} \dim V_i \leq l(l+1) + p(k-l-1)\\ &= l^2 + l + pk - pl - p = p(k-1) + l(l+1-p) < p(k-1)\end{aligned}$$

which contradicts (3.9).

Let now $\mathcal{V}$ be a vector space spanned on elements $T \backslash \Theta$ (where T is defined by (3.7)), then, since Θ consists of no more than l different elements, $\dim \mathcal{V} \geq p-l$. This, and (3.14) imply that there exists a nonzero element c_{l+1} such that $c_{l+1} \in \mathcal{V} \cap V_{l+1}$. From the definition of the space $\mathcal{V}$, it follows that the leading element of c_{l+1} is different from the elements of the set Θ. This immediately implies that the set $\Theta \cup \{c_{l+1}\}$ is admissible, which completes the inductive step in the selection process satisfying (3.10). Now, from Lemma 3.1 it follows that the product $\prod_{i=1}^{k} c_i$ is nonzero. ☐

LEMMA 3.4. *Let $V_1, \dots, V_k$ be linear subspaces of $A_1, 1 \leq k \leq k_0$. If $\dim V_j \geq i+1 = \ell(h_1, \dots, h_p, k)+1$ (see Notation 3.1), for some integer i satisfying $0 \leq i \leq p$ and all integers j satisfying $1 \leq j \leq k$ so that*

$$h_0 + \cdots + h_i < k \leq h_0 + \cdots + h_i + h_{i+1}$$

then there exist elements $c_s \in V_s$ for all s where $1 \leq s \leq k$ such that their leading coefficients $L(c_s)$ form an admissible set and their product $\prod_{s=1}^{k} c_s$ is nonzero.

Proof. We choose elements c_s by induction. $c_1 \in V_1$ is chosen as an arbitrary nonzero element. The inductive step is now afforded by Lemma 3.2. ☐

LEMMA 3.5. *Let $M = \bigcup_{s=1}^{k} H_s$ be a union of arbitrary closed sets H_s. Consider for each positive integer $s, 1 \leq s \leq k$, the following exact cohomology sequence, where the cohomology groups are taken with coefficients in an arbitrary commutative ring with unity. The coefficient ring is omitted from notations for brevity.*

$$\xleftarrow{j^*} H^2(M, H_s) \xleftarrow{\delta} H^1(H_s) \xleftarrow{i_s^*} H^1(M) \xleftarrow{j_s^*} H^1(M, H_s) \xleftarrow{\delta}. \tag{3.15}$$

Let

$$V_s = \operatorname{Ker}(i_s^*) = \operatorname{Im}(j_s^*) \tag{3.16}$$

and let $c_s \in V_s$ be arbitrary elements, $1 \leq s \leq k$. Then the cup product $\prod_{s=1}^{k} c_s \in H^k(M)$ is zero.

Proof. Our statement immediately follows from the following commutative diagram

$$\begin{array}{ccc} \bigotimes_{s=1}^{k} H^1(M,H_s) & \xrightarrow{\cup} & H^k\left(M, \bigcup_{s=1}^{k} H_s\right) = H^k(M,M) = 0 \\ \downarrow \bigotimes_{s=1}^{k} j_s^* & & \downarrow j^* \\ \bigotimes_{s=1}^{k} H^1(M) & \xrightarrow{\cup} & H^k(M) \end{array}$$

where $\cup$ is the cup product. □

Proof of Theorem 3.1. (i) Let $V_s = \mathrm{Ker}(i_s^*) = \mathrm{Im}(j_s^*)$ where the notations are taken from (3.15) with coefficients in F. Then from the exactness of the sequence (3.15) it follows that

$$\operatorname{rank} i_s^*(H^1(M,F)) = \operatorname{rank}(H^1(M,F)) - \dim V_s = p - \dim V_s. \tag{3.17}$$

Therefore,

$$\sum_{s=1}^{k} \operatorname{rank} i_s^*(H^1(M,F)) = \sum_{s=1}^{k} (p - \dim V_s) = pk - \sum_{s=1}^{k} \dim V_s. \tag{3.18}$$

If

$$\sum_{s=1}^{k} \dim V_s > p(k-1) \tag{3.19}$$

then by Lemma 3.3 there exist elements $c_i \in V_i, 1 \le i \le k$ such that the product $\prod_{i=1}^{k} c_i \neq 0$ which contradicts Lemma 3.5. This implies that (3.19) is false and, therefore, (3.18) implies (i).

Let us prove (ii). Suppose inequality (ii) is false. Then for every integer s satisfying $1 \le s \le k$ we have

$$\operatorname{rank} i_s^*(H^1(M,F)) = p - \dim V_s < p - \ell(h_1, \ldots, h_p, k)$$

which implies that $\dim V_s \ge \ell(h_1, \ldots, h_p, k) + 1$. Therefore, by Lemma 3.4 there exist elements $c_s \in V_s$ for all $s, 1 \le s \le k$ such that their product $\prod_{s=1}^{k} c_s$ is nonzero which again contradicts Lemma 3.5. □

4. Proofs of Theorems 2.1–2.4

Proof of Theorem 2.1. Assume the contrary, that is, that ρ maps each set L_i bijectively (and, therefore, homeomorphically) onto H_i. From Theorem 3.1, it follows that for some integer $s, 1 \le s \le k$ for the set H_s we have

$$p - \ell(h_1, \ldots, h_p, k) \le \operatorname{rank} i^*(H^1(M,F)) \le \operatorname{rank}(H^1(H_s,F)), \tag{4.1}$$

where $i : H_s \subset M$ is the inclusion map. Since F is a field, there is a natural isomorphism $H^k(S,F) \approx \mathrm{Hom}(H_k(S,F),F)$ for any space S and $k>0$. See [14] for various generalizations. This statement, together with (4.1), implies that the group $i_*(H_1(H_s,F)) \subset H_1(M,F)$ also has rank at least $p-\ell(h_1,\dots,h_p,k)$ where $i_* : H_1(H_s,F) \to H_1(M,F)$ is the homorphism induced by the inclusion map i. Since F is a field, we also have from the condition of our theorem

$$\begin{aligned}\operatorname{rank}(H^1(\widetilde{M},F)) &= \operatorname{rank}(H_1(\widetilde{M},F)) < p-\ell(h_1,\dots,h_p,k)\\ &\le \operatorname{rank} i_*(H_1(H_s,F)).\end{aligned} \tag{4.2}$$

Consider the following commutative diagram

$$\begin{array}{ccc} H_1(L_s,F) & \xrightarrow{j_*} & H_1(\widetilde{M},F)\\ \rho_{1*}\downarrow & & \rho_*\downarrow\\ H_1(H_s,F) & \xrightarrow{i_*} & H_1(M,F)\end{array} \tag{4.3}$$

where the vertical homomorphisms are induced by ρ and the horizontal ones by inclusions. Due to (4.2) and the fact that ρ is a homeomorphism on L_s (and, therefore, ρ_{1*} is an isomorphism), it follows that

$$\operatorname{rank}\bigl(i_*\circ\rho_{1*}(H_1(L_s))\bigr) \ge p-\ell(h_1,\dots,h_p,k) \tag{4.4}$$

however, the commutativity of the diagram above and the condition of the theorem imply that,

$$\operatorname{rank}\bigl(\rho_*\circ j_*(H_1(L_s))\bigr) \le \operatorname{rank}(H_1(\widetilde{M})) < p-\ell(h_1,\dots,h_p,k). \tag{4.5}$$

Since $\rho_*\circ j_*(H_1(L_s)) = i_*\circ\rho_{1*}(H_1(L_s))$, (4.4) contradicts (4.5). This contradiction proves the theorem. □

Proof of Theorem 2.2. We can assume that all sets H_i and, therefore, all L_i are different. Otherwise, we can remove the repeated sets and prove our theorem for the reduced collection of sets which will imply it for the original collection. Let N denote the set of all positive integers and let N_k be the set of all integers s, $1\le s\le k$ and let $S=\{J_m : m\in N\}$ be the set of all nonzero vectors in R^n with integer coordinates. Let also L_i^m denote a set obtained via a parallel translation $L_i^m = P(L_i,J_m)$ of L_i by a vector J_m. From the conditions of the theorem, it follows that

$$\bigcup_{i=1}^{k}\bigcup_{m=1}^{\infty} L_i^m = R^n. \tag{4.6}$$

Since all sets L_i^m are bounded, the covering $\{L_i^m : i\in N_k, m\in N\}$ is locally finite and, therefore, only a finite number of its elements intersects every n dimensional cube $I^n\subset R^n$.

If I^n is large enough so that no set L_i^m intersects both of any two of its opposite faces, then there exist at least $n+1$ sets $L_{i_1}^{m_1},\dots,L_{i_{n+1}}^{m_{n+1}}$ with nonempty intersection,

$$\bigcap_{q=1}^{n+1} L_{i_q}^{m_q} \neq \emptyset \tag{4.7}$$

see [1], Theorem IV.2, p. 43 (Lebesgue's Covering theorem which states that for any finite closed covering of a cube I^n with sets which do not simultaneously intersect any two opposite faces of I^n, there exist at least $n+1$ of the elements of this covering with nonempty intersection). Let us partition the collection of sets $\mathcal{U} = \{L_{i_q}^{m_q} : q \in N_{n+1}\}$ into groups with the same lower indices, that is, into groups of sets such that sets in each group are obtained by parallel translations of the same set L_i via a vector from S. Let there be l groups and let the ith group have r_i elements so that

$$\sum_{i=1}^{l} r_i = n+1. \tag{4.8}$$

It is clear that if for every integer i, such that $1 \leq i \leq l$, the map ρ has multiplicity at least r_i on every set in the ith group then (4.8) implies the theorem. Let us consider a group consisting of $p = r_s$ elements $\{L_s^k : 1 \leq k \leq p\}$ where we simplified the notations to avoid multiple indices (which may require renumbering of the set of vectors S). (4.7) implies that

$$\bigcap_{k=1}^{p} L_s^k \neq \emptyset. \tag{4.9}$$

Let

$$x \in \bigcap_{k=1}^{p} L_s^k. \tag{4.10}$$

Then our notational convention implies that

$$L_s^k = P(L_s, J_k), \qquad L_s^k = P(L_s^1, J_k - J_1). \tag{4.11}$$

Now, (4.10), (4.11) imply that for every $k, 2 \leq k \leq p$ we have

$$y_k = x + J_1 - J_k \in L_s^1. \tag{4.12}$$

Therefore, the coordinates of the points $x, y_2, \ldots, y_p$ differ only by an integer and they all belong to L_s^1. This means that the multiplicity of π on L_s^1 is at least $p = r_s$ and, therefore, the multiplicity μ_s of π on L_s is also at least r_s. This, together with (4.8), (4.11) imply $\sum_{i=1}^{k} \mu_i \geq n+1$. □

Proof of Theorem 2.3. By the Hopf's theorem, Corollary 13, Chapter 5.9 in [3], the cohomology ring over any field F of characteristic 0 of a finite type H-space is isomorphic to that of the finite product of spheres. Therefore, $H^*(M, F)$ contains the exterior algebra of $n = \beta_1(M, F)$ variables which is the same as $F[1, \ldots, 1, x_1, \ldots, x_n]$ (here, 1 is repeated n times). In this case, from

Notation 2.1 it follows that $\ell(h_1,\dots,h_p,k)=k-1$, therefore the inequality $\operatorname{rank}(H^1(\widetilde{M},F))=\beta_1(\widetilde{M},F)<\beta_1(M,F)-k+1$ is equivalent to the one given in Theorem 2.1. Our theorem now follows from Theorem 2.1. □

Remark 4.1. Let $M=\prod_{i=1}^{p} RP^{h_i}\times T^n$ be a product of p real projective spaces RP^{h_i} of dimension $h_i>1$, and an n dimensional torus T^n where p or n can be zero. Then the cohomology algebra $H^*(M,Z_2)$ is

$$Z_2[1,\dots,1,h_1,\dots,h_p,t_1,\dots,t_n,x_1,\dots,x_p],$$

where 1 is repeated n times, and $t_1,\dots,t_n$ are generators corresponding to the cohomology of the torus. In this case, from the Notation 2.1 it follows that for $k\le n+1$,

$$\ell(1,\dots,1,h_1,\dots,h_p,k)=k-1$$

and for $n+1<k\le n+h_0+\dots+h_p$ we have

$$\ell(1,\dots,1,h_1,\dots,h_p,k)=n+i,$$

where $n+h_0+\dots+h_i<k\le n+h_0+\dots+h_i+h_{i+1}$.

Proof of Theorem 2.4. First, notice that the one dimensional sphere is homeomorphic to the one dimensional real projective space, so M could be assumed to be the product of projective spaces only, that is, we can assume that

$$M=\prod_{i=1}^{p} RP^{h_i}.$$

The cohomology ring $H^*(\prod_{i=1}^{p} RP^{h_i},Z_2)$ is isomorphic to $Z_2[h_1,\dots,h_p,x_1,\dots,x_p]$. The needed result now follows from Theorem 2.1 and Remark 4.1. □

References

[1] V. Hurewicz and H. Wallman, *Dimension theory*, Revised edition, Princeton University Press, Princeton, NJ, 1996.

[2] L. Lyusternik and L. Schnirelmann, *Topological methods in variational problems*, Issledowatelskiy Institut Mathematiki i Mechaniki pri M G U, Moscow, 1930 (Russian).

[3] E. Spanier, *Algebraic topology*, McGraw-Hill, NewYork, 1966. MR 0210112

[4] M. Arkowitz and J. Strom, *The sectional category of a map*, Proc. of Royal Soc. Edinburgh, **134A** (2004), 639–650. MR 2079797

[5] I. James, *On category in the sense of Lusternik–Schnirelmann*, Topology **17** (1978), 331–348. MR 0516214

[6] I. Berstein and T. Ganea, *The category of a map and a cohomology class*, Fund. Math. **50** (1962), 265–279. MR 0139168

[7] S. Smale, *Morse inequalities for dynamical systems*, Bull. Amer. Math. Soc. **66** (1960), 43–49. MR 0117745

[8] R. H. Fox, *On the Lusternik–Schnirelmann category*, Annals of Mathematics **42** (1941), 333–370. MR 0004108

[9] S. Eilenberg and T. Ganea, *On the Lusternik–Schnirelmann category of abstract groups*, Annals of Mathematics, 2nd Ser., **65** (1957), 517–518. MR 0085510

[10] A. Schwarz, *The genus of a fiber space*, A.M.S. Trans. **55** (1966), 49–140.

[11] L. F. Suarez, P. Ghienne, T. Kahl and L. Vandembroucq, *Joins of DGA modules and sectional category*, Algebraic & Geometric Topology **6** (2006), 119–144. MR 2199456

[12] L. A. Luxemburg and G. Huang, *Generalized Morse theory and its applications to control and stability analysis*, Circuits Systems Signal Process. **10** (1991), 175–209. MR 1096108

[13] L. A. Luxemburg and G. Huang, *Characterization of equilibrium points on the stability boundary via algebraic topology approach*, Analysis and Control of Nonlinear Systems, (C. I. Byrnes, C. F. Martin and R. E. Saeks, eds.), Amsterdam, North Holland, 1988, pp. 71–76.

[14] M. Huber and W. Meier, *Cohomology theories and infinite CW complexes*, Comm. Math. Helvet. **53** (1978), 239–257. MR 0482742

Leon A. Luxemburg, Texas A&M University at Galveston, Texas, USA

E-mail address: luxembul@tamug.edu; lluxemburg@hotmail.com