Illinois Journal of Mathematics
Volume 54, Number 2, Summer 2010, Pages 397–448
S 0019-2082

A CLASSIFICATION OF ARTIN–SCHREIER DEFECT EXTENSIONS AND CHARACTERIZATIONS OF DEFECTLESS FIELDS

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Abstract. We classify Artin–Schreier extensions of valued fields with nontrivial defect according to whether they are connected with purely inseparable extensions with nontrivial defect, or not. We use this classification to show that in positive characteristic, a valued field is algebraically complete if and only if it has no proper immediate algebraic extension and every finite purely inseparable extension is defectless. This result is an important tool for the construction of algebraically complete fields. We also consider extremal fields (=fields for which the values of the elements in the images of arbitrary polynomials always assume a maximum). We characterize inseparably defectless, algebraically maximal and separable-algebraically maximal fields in terms of extremality, restricted to certain classes of polynomials. We give a second characterization of algebraically complete fields, in terms of their completion. Finally, a variety of examples for Artin–Schreier extensions of valued fields with nontrivial defect is presented.

1. Introduction

In this paper, we consider fields K equipped with (Krull) valuations v. The value group of (K,v) will be denoted by vK, and its residue field by Kv. The value of an element a is denoted by va, and its residue by av. We will frequently drop the valuation v and talk of K as a valued field if the context is clear. By $(L|K,v)$, we mean an extension of valued fields where v is a

Received December 17, 2004; received in final form January 7, 2011.
I was partially supported by a Canadian NSERC grant and by sabbatical grants from the University of Saskatchewan.

valuation on L and its subfield K is endowed with the restriction of v. The extension $(L|K,v)$ is called *immediate* if $(vL:vK)=1$ and $[Lv:Kv]=1$.

Assume that $(L|K,v)$ is a finite extension and the extension of v from K to L is unique. Then the Lemma of Ostrowski tells us that

$$[L:K]=(vL:vK)\cdot[Lv:Kv]\cdot p^{\nu} \quad \text{with } \nu\geq 0, \tag{1.1}$$

where p is the *characteristic exponent* of Kv, that is, $p=\mathrm{char}Kv$ if this is positive, and $p=1$ otherwise. The factor $\mathrm{d}(L|K):=p^{\nu}$ is called the *defect* of the extension $(L|K,v)$. If $\nu>0$, then we talk of a *nontrivial* defect and call $(L|K,v)$ a *defect extension*. Otherwise, we call $(L|K,v)$ a *defectless extension*. If $[L:K]=p$, then $(L|K,v)$ is a defect extension if and only if it is immediate. A possibly infinite algebraic extension is called defectless if all of its finite subextensions are defectless; in view of Lemma 2.1 in Section 2.1, this agrees with the definition for finite extensions.

Every finite extension L of a valued field (K,v) satisfies the *fundamental inequality* (cf. [5], [31]):

$$n\geq\sum_{i=1}^{\mathrm{g}}\mathrm{e}_i\mathrm{f}_i, \tag{1.2}$$

where $n=[L:K]$ is the degree of the extension, $v_1,\dots,v_{\mathrm{g}}$ are the distinct extensions of v from K to L, $\mathrm{e}_i=(v_iL:vK)$ are the respective *ramification indices* and $\mathrm{f}_i=[Lv_i:Kv]$ are the respective *inertia degrees*. If $\mathrm{g}=1$ for every finite extension $L|K$, then (K,v) is called *henselian*. This holds if and only if (K,v) satisfies *Hensel's Lemma*, that is, if f is a polynomial with coefficients in the valuation ring $\mathcal{O}$ of (K,v) and there is $b\in\mathcal{O}$ such that $vf(b)>0$ and $vf'(b)=0$, then there is $a\in\mathcal{O}$ such that $f(a)=0$ and $v(b-a)>0$ (see [6], Section 4.1, or [24] for details).

Every valued field (K,v) has a minimal separable-algebraic extension which is henselian; it is unique up to isomorphism over K. We call it the *henselization of* (K,v) and denote it by $(K,v)^h$. It is an immediate extension of (K,v).

We call a (not necessarily henselian) valued field (K,v) a *defectless field*, *separably defectless field* or *inseparably defectless field* if equality holds in the fundamental inequality (1.2) for every finite, finite separable or finite purely inseparable extension L of K. One can trace this back to the case of unique extensions of the valuation, respectively; for the proof of the following theorem, see [24] (a partial proof was already given in [5]).

THEOREM 1.1. *A valued field (K,v) is a defectless field if and only if its henselization $(K,v)^h$ is (that is, if and only if every finite extension of $(K,v)^h$ is a defectless extension). The same holds for "separably defectless field" and "inseparably defectless field" in the place of "defectless field".*

A valued field is called *algebraically complete* if it is henselian and defectless.

For various reasons (e.g., local uniformization in positive characteristic [13], [14], the study of ramification [3], model theory of valued fields [21]), it is necessary to study the structure of defect extensions. A ramification theoretic method that was used frequently by S. Abhyankar and that is also employed in [19] is to consider the part of an extension $(L|K,v)$ that "lies above" its ramification field. We can reformulate this in the following way. We let K^r denote the *absolute ramification field* of K, i.e., the ramification field of the extension $K^{\mathrm{sep}}|K$ with respect to a fixed extension of v to the separable-algebraic closure K^{sep} of K. Then we consider the extension $L.K^r|K^r$. If K is henselian, then this extension has the same defect as $L|K$ (cf. Proposition 2.8 below). On the other hand, the Galois group of $K^{\mathrm{sep}}|K^r$ is a pro-p-group (cf. [5] or [27]). Consequently, $L.K^r|K^r$ is a tower of normal extensions $L_1|L_2$ of degree p (cf. Lemma 2.9 in Section 2.1). If our fields have characteristic p and if $L_1|L_2$ is separable, then it is an *Artin–Schreier extension*, that is, it is generated by a root ϑ of a polynomial of the form $X^p - X - a$ with $a \in L_2$ (see, e.g., [26]); in this case, ϑ is called an *Artin–Schreier generator* of $L_1|L_2$. Such extensions are always normal and hence Galois since the other roots of $X^p - X - a$ are $\vartheta+1,\ldots,\vartheta+p-1$. This follows from the fact that $0,1,\ldots,p-1$ are all roots of the Artin–Schreier polynomial $\wp(X) = X^p - X$ and that this polynomial is additive. A polynomial $f \in K[X]$ is called *additive* if $f(b+c) = f(b)+f(c)$ for all b,c in every extension field of K (cf. [18], [26]).

Also in the mixed characteristic case where $\mathrm{char} L_1 = 0$ and $\mathrm{char} L_1 v = p$, we will call $L_1|L_2$ an Artin–Schreier extension with Artin–Schreier generator ϑ if $[L_1 : L_2] = p$, $L_1 = L_2(\vartheta)$ and $\vartheta^p - \vartheta \in L_2$.

Because of the representative role of Artin–Schreier extensions that we just pointed out, it is interesting to know more about their structure, in particular when they have nontrivial defect. In this paper, we will classify Artin–Schreier defect extensions according to the question whether they are in some sense similar to immediate purely inseparable extensions. Then we study the relation between the two different types of extensions in our classification. In Section 4.6, we will give several examples of Artin–Schreier defect extensions of both types.

The defect is a bad phenomenon as it destroys the tight connection between valued fields and their invariants, value group and residue field. Therefore, it is desirable to work with defectless fields. As the notion of "defectless field" plays an important role in several applications, it is helpful to have equivalent characterizations. For instance, when it comes to constructing defectless fields, one would like to have criteria that could (more or less) easily be checked. Our results on Artin–Schreier defect extensions will enable us to break down the property "defectless field" into weaker maximality properties of valued fields.

A valued field (K,v) is called *algebraically maximal* if it has no proper immediate algebraic extensions, and *separable-algebraically maximal* if it has no proper immediate separable-algebraic extensions. Note that a separable-algebraically maximal valued field is henselian, because the henselization is an immediate separable-algebraic extension. In Section 4.5, we will prove the following useful characterization of the property "defectless field":

Theorem 1.2. *A valued field of positive characteristic is henselian and defectless if and only if it is separable-algebraically maximal and inseparably defectless.*

This characterization has been applied in [16] to construct a valued field extension of the field $\mathbb{F}_p((t))$ of formal Laurent series over the field with p elements which is henselian defectless with value group a $\mathbb{Z}$-group and residue field $\mathbb{F}_p$ but does not satisfy a certain elementary sentence (involving additive polynomials) that holds in $\mathbb{F}_p((t))$. This example shows that the axiom system "henselian defectless valued field of characteristic p with value group a $\mathbb{Z}$-group and residue field $\mathbb{F}_p$" is not complete. Whenever one wants to construct a henselian defectless field, the problem is to get it to be defectless. It is easy to make it henselian (just go to the henselization) or even algebraically maximal (just go to a maximal immediate algebraic extension). But the latter does not imply that the field is defectless, as an example given by F. Delon [4] shows (see also [20]). However, in the case of finite p-degree, Delon also gave a handy characterization of inseparably defectless valued fields, see Theorem 3.5. (Recall that d is called the *p-degree*, or *Ershov invariant*, or *degree of imperfection*, of K if $[K:K^p]=p^d$.) Together with the above theorem, this provides a handy characterization of henselian defectless fields of characteristic p in the case of finite p-degree.

Take a valued field (K,v) with valuation ring $\mathcal{O}$. If f is a polynomial in n variables with coefficients in K, then we will say that (K,v) is *K-extremal with respect to f* if the set

$$v\,\mathrm{im}_K(f) := \{vf(a_1,\dots,a_n) | a_1,\dots,a_n \in K\} \subseteq vK \cup \{\infty\} \tag{1.3}$$

has a maximum, and we will say that (K,v) is *$\mathcal{O}$-extremal with respect to f* if the set

$$v\,\mathrm{im}_{\mathcal{O}}(f) := \{vf(a_1,\dots,a_n) | a_1,\dots,a_n \in \mathcal{O}\} \subseteq vK \cup \{\infty\} \tag{1.4}$$

has a maximum. The former means that

$$\exists Y_1,\dots,Y_n\ \forall X_1,\dots,X_n : vf(X_1,\dots,X_n) \leq vf(Y_1,\dots,Y_n)$$

holds in (K,v). For the latter, one has to build into the sentence the condition that the X_i and Y_j only run over elements of $\mathcal{O}$. It follows that being K-extremal or $\mathcal{O}$-extremal with respect to f is an elementary property in the language of valued fields with parameters from K. Note that in the first case the maximum is ∞ if and only if f admits a zero in K^n; in the second case, this zero has to lie in $\mathcal{O}^n$. A valued field (K,v) is called *extremal* if for all $n\in\mathbb{N}$, it is $\mathcal{O}$-extremal with respect to every polynomial f in n variables with coefficients in K. This property can be expressed by a countable scheme of elementary sentences (quantifying over the coefficients of all possible polynomials of degree at most n in at most n variables). Hence, it is elementary in the language of valued fields.

If we would have chosen K-extremality for the definition of "extremal valued field" (as Yu. Ershov in [9]), then we would have obtained precisely the class of algebraically closed valued fields. Using $\mathcal{O}$-extremality instead yields a much more interesting class of valued fields. See [2] and [10] for details.

The properties "algebraically maximal", "separable-algebraically maximal" and "inseparably defectless" are each equivalent to K- or $\mathcal{O}$-extremality restricted to certain (elementarily definable) classes of polynomials. A polynomial is called a *p-polynomial* if it is of the form $\mathcal{A}(X) + c$ where $\mathcal{A}(X)$ is an additive polynomial and c is a constant. We say that a basis $b_1, \ldots, b_n$ of a valued field extension $(L|K, v)$ is a *valuation basis* if for all choices of $c_1, \ldots, c_n \in K$,

$$v \sum_{i=1}^{n} c_i b_i = \min_i v c_i b_i .$$

In Section 3, we prove the following theorem.

THEOREM 1.3. *A valued field K of positive characteristic is inseparably defectless if and only if it is K-extremal with respect to every p-polynomial of the form*

$$b - \sum_{i=1}^{p^\nu} b_i X_i^p \tag{1.5}$$

with $b, b_1, \ldots, b_{p^\nu} \in K$, $\nu \geq 0$, forming a basis of a finite extension of K^p inside K. If the value group vK of K is divisible or a $\mathbb{Z}$-group, then K is inseparably defectless if and only if it is $\mathcal{O}$-extremal with respect to every p-polynomial (1.5) with $\nu \in \mathbb{N}$ and $b, b_1, \ldots, b_{p^\nu} \in \mathcal{O}$ such that $b_1, \ldots, b_{p^\nu}$ form a valuation basis of a finite defectless extension of K^p and $vb_1, \ldots, vb_{p^\nu}$ are smaller than every positive element of vK.

In Section 6.1, we prove the following theorem.

THEOREM 1.4. *A valued field K is algebraically maximal if and only if it is $\mathcal{O}$-extremal with respect to every polynomial in one variable with coefficients in K.*

THEOREM 1.5. *A henselian valued field K of positive characteristic is algebraically maximal if and only if it is $\mathcal{O}$-extremal with respect to every p-polynomial in one variable with coefficients in K.*

In Section 6.2, we prove the following.

THEOREM 1.6. *A valued field K is separable-algebraically maximal if and only if it is $\mathcal{O}$-extremal with respect to every separable polynomial in one variable with coefficients in K.*

THEOREM 1.7. *A henselian valued field K of positive characteristic is separable-algebraically maximal if and only if it is $\mathcal{O}$-extremal with respect to every separable p-polynomial in one variable with coefficients in K.*

THEOREM 1.8. *In Theorems 1.4, 1.5, 1.6 and 1.7, "$\mathcal{O}$-extremal" can be replaced by "K-extremal".*

Theorems 1.4 and 1.6 in the "K-extremal" version were presented by Delon in [4], but the proofs had gaps in both directions.

In [2], we prove the following theorem.

THEOREM 1.9. *Every extremal field is henselian and defectless. Every finite extension of an extremal field is again extremal.*

These results were proved by Yu. Ershov in [9] for "K-extremal" in the place of "$\mathcal{O}$-extremal". But in this case, they are trivial consequences of the fact that every K-extremal valued field is algebraically closed (cf. [2]).

We also obtain that the properties "algebraically maximal", "separable-algebraically maximal" and "inseparably defectless" are elementary in the language of valued fields, see Corollary 6.5, Corollary 6.9 and Corollary 3.4. By Theorem 1.2, this fact provides an easy proof of the following result, which was proved by Ershov [8], and independently by Delon [4], by different methods.

THEOREM 1.10. *The property "henselian and defectless valued field of characteristic $p > 0$" is elementary in the language of valued fields.*

In Section 5, we will give another characterization of henselian defectless fields, in terms of their completion (Theorem 5.1). We will also show that a henselian field of positive characteristic is separably defectless if and only if its completion is defectless (Theorem 5.2).

A field of positive characteristic is called *Artin–Schreier closed* if it admits no nontrivial Artin–Schreier extensions. In Section 4.3, we will prove the following theorem.

THEOREM 1.11. *Every Artin–Schreier closed nontrivially valued field lies dense in its perfect hull, and its completion is perfect. In particular, every separable-algebraically closed nontrivially valued field lies dense in its algebraic closure.*

The second part of this theorem is well known (cf. [30], Theorem 30.28).

Several of the results of this paper, and in particular Theorem 1.2, have been inspired by the work presented in Francoise Delon's thesis [4].

2. Preliminaries

For the basic facts of valuation theory, we refer the reader to [5], [28], [30] and [31]. For ramification theory, we recommend [5] and [27]. In parts of this paper, we will assume some familiarity with the theory of pseudo Cauchy sequences as presented in the first half of [12]. Note that our "pseudo Cauchy sequence" is what Kaplansky calls "pseudo convergent set".

The algebraic closure of a field K will be denoted by $\tilde{K}$. Note that if v is a valuation on K, then any possible extension of v to $\tilde{K}$ has residue field $\widetilde{Kv}$ and value group $\widetilde{vK}$, the divisible hull of vK (isomorphic to $\mathbb{Q}\otimes vK$).

2.1. Defectless extensions, defect and immediate extensions. The defect is multiplicative in the following sense. Let $L|K$ and $M|L$ be finite extensions. Assume that the extension of v from K to M is unique. Then the defect satisfies the following product formula

$$\mathrm{d}(M:K)=\mathrm{d}(M:L)\cdot\mathrm{d}(L:K) \tag{2.1}$$

which is a consequence of the multiplicativity of the degree of field extensions and of ramification index and inertia degree. This formula implies the following.

Lemma 2.1. *$M|K$ is defectless if and only if $M|L$ and $L|K$ are defectless.*

The following corollary can be derived in the same way by a tedious computation ranging over all possible extensions of the valuation v. It is more elegant, however, to deduce it from the foregoing lemma with the help of Theorem 1.1.

Corollary 2.2. *If (K,v) is a defectless field and L is a finite extension of K, then L is also a defectless field with respect to every extension of v from K to L. Conversely, if there exists a finite extension L of K for which equality holds in the fundamental inequality (1.2) and such that L is a defectless field with respect to every extension of v from K to L, then (K,v) is a defectless field. The same holds for "separably defectless" in the place of "defectless" if $L|K$ is separable, and for "inseparably defectless" if $L|K$ is purely inseparable.*

Recall that an infinite algebraic extension $(L|K,v)$ with unique extension of the valuation is called defectless if every finite subextension is defectless. This definition is compatible with the definition of "defectless" for finite extensions as given in the introduction, because by Lemma 2.1 every subextension of a finite defectless extension is again defectless. We have the following lemma.

Lemma 2.3. *If $(L|K,v)$ and $(L'|L,v)$ are (not necessarily finite) defectless extensions, then also $(L'|K,v)$ is defectless.*

Proof. Let $F|K$ be any finite subextension of $L'|K$. Since $L'|L$ is defectless, so is its finite subextension $F.L|L$. Hence, $[F.L:L]=(v(F.L):vL)[(F.L)v:$

$Lv]$. Pick a set $\alpha_1,\dots,\alpha_k$ of generators of $vF.L$ over vL, and a basis $\zeta_1,\dots,\zeta_\ell$ of $(F.L)v|Lv$. Choose a finite subextension $L_0|K$ of $L|K$ such that

$$[F.L_0 : L_0] = [F.L : L], \quad \alpha_1,\dots,\alpha_k \in v(F.L_0), \zeta_1,\dots,\zeta_\ell \in (F.L_0)v.$$

Then $(v(F.L_0) : vL_0) \geq (v(F.L) : vL)$, $[(F.L_0)v : L_0v] \geq [(F.L)v : Lv]$, and

$$\begin{aligned}[F.L : L] = [F.L_0 : L_0] &\geq (v(F.L_0) : vL_0)[(F.L_0)v : L_0v] \\ &\geq \big(v(F.L) : vL\big)[(F.L)v : Lv] = [F.L : L],\end{aligned}$$

where the first inequality follows from (1.1). Hence, equality must hold everywhere, and we obtain that $F.L_0|L_0$ is defectless. Since $L|K$ is assumed to be defectless, also $L_0|K$ is defectless. By Lemma 2.1, it follows that $F.L_0|K$ is defectless. Again by Lemma 2.1, also the subextension $F|K$ is defectless. This proves that $L'|K$ is defectless. □

Let (K,v) be any valued field. A valuation w on K is a *coarsening* of v if its valuation ring $\mathcal{O}_w$ contains the valuation ring $\mathcal{O}_v$ of v. Note that we do not exclude the case of $w = v$. If H is a convex subgroup of vK, then it gives rise to a coarsening w through the definition $\mathcal{O}_w := \{x \in K | \exists \alpha \in H : \alpha \leq vx\}$. Then v induces a valuation $\overline{w}$ on Kw through the definition $\mathcal{O}_{\overline{w}} := \{xw | x \in \mathcal{O}_v\}$, and there are canonical isomorphisms $wK \simeq vK/H$ and $\overline{w}(Kw) \simeq H$.

If (K,w) is any valued field and if w' is any valuation on the residue field Kw, then $w \circ w'$, called the *composition of* w *and* w', will denote the valuation whose valuation ring is the subring of the valuation ring of w consisting of all elements whose w-residue lies in the valuation ring of w'. (Note that we identify equivalent valuations.) In our above situation, v is the composition of w and $\overline{w}$. While $w \circ w'$ does actually not mean the composition of w and w' as mappings, this notation is used because in fact, up to equivalence the place associated with $w \circ w'$ is indeed the composition of the places associated with w and w'.

LEMMA 2.4. *Take a valued field* (K,v), *a finite extension* $(L|K,v)$ *and a coarsening* w *of* v *on* L. *If* (K,v) *is henselian, then so is* (K,w). *If* $(L|K,v)$ *is defectless, then so is* $(L|K,w)$.

Proof. If there are two distinct extensions w_1 and w_2 of w from K to $\tilde{K}$, and if we take any extension of $\overline{w}$ to the algebraic closure $\widetilde{Kw} = \tilde{K}w_1 = \tilde{K}w_2$ of Kw, then the compositions $w_1 \circ \overline{w}$ and $w_2 \circ \overline{w}$ will be distinct extensions of v to $\tilde{K}$. This shows that (K,v) cannot be henselian if (K,w) isn't.

Now assume that $(L|K,v)$ is defectless, that is, $[L : K] = (vL : vK) \cdot [Lv : Kv]$. We have that $(vL : vK) = (wL : wK)(\overline{w}(Lw) : \overline{w}(Kw))$, $(Lw)\overline{w} = Lv$ and $(Kw)\overline{w} = Kv$. Therefore,

$$\begin{aligned}[L : K] &\geq (wL : wK)[Lw : Kw] \\ &\geq (wL : wK)\big(\overline{w}(Lw) : \overline{w}(Kw)\big)[(Lw)\overline{w} : (Kw)\overline{w}] \\ &= (vL : vK)[Lv : Kv] = [L : K].\end{aligned}$$

This shows that equality holds everywhere, which proves that $(L|K,w)$ is defectless. □

In the next lemma, the relation between immediate and defectless extensions is studied.

LEMMA 2.5. *Take an arbitrary immediate extension $(F|K,v)$ and a finite defectless extension $(L|K,v)$. Then the extension of v from K to $F.L$ is unique, $(F.L|F,v)$ is defectless, and $(F.L|L,v)$ is immediate. Moreover,*

$$[F.L:F]=[L:K],$$

i.e., F is linearly disjoint from L over K.

Proof. $v(F.L)$ contains vL and $(F.L)v$ contains Lv. On the other hand, we have $vF=vK$ and $Fv=Kv$ by hypothesis. Therefore,

$$\begin{aligned}[F.L:F] &\geq \big(v(F.L):vF\big)\cdot[(F.L)v:Fv]\\ &\geq (vL:vK)\cdot[Lv:Kv]=[L:K]\geq[F.L:F],\end{aligned}$$

hence equality holds everywhere. This shows that $[F.L:F]=[L:K]$ and that $F.L|F$ is defectless. Furthermore, it follows that $v(F.L)=vL$ and $(F.L)v=Lv$, i.e., $F.L|L$ is immediate. □

Let us observe that for an inseparably defectless field (K,v), every immediate extension is separable. Indeed, it follows from Lemma 2.5 that every immediate extension of (K,v) is linearly disjoint from the defectless extension $(K^{1/p^\infty}|K,v)$. In the literature, one can find the expression "excellent" for those fields for which all immediate extensions are separable (cf. [4], Définition 1.41). But there are also other properties of certain valuation rings for which this expression is used.

Similarly, every immediate extension of a henselian defectless field is linearly disjoint from the algebraic closure of K. We obtain the following corollary.

COROLLARY 2.6. *If K is an inseparably defectless field then every immediate extension is separable. If K is a henselian defectless field then every immediate extension is regular.*

Let (K,v) be a henselian field and p the characteristic exponent of its residue field Kv. An algebraic extension $(L|K,v)$ is called a *tame extension* if for every finite subextension $(L_0|K,v)$, the following conditions hold:

(1) p is prime to $(vL_0:vK)$,
(2) $L_0v|Kv$ is separable,
(3) $(L_0|K,v)$ is defectless.

On the other hand, an algebraic extension $(L|K,v)$ is called a *purely wild extension* if

(1) vL/vK is a p-group,

(2) $Lv|Kv$ is purely inseparable.

If $p=1$, that is, $\mathrm{char}Kv=0$, then every algebraic extension of a henselian field is tame, and only the trivial extension is purely wild.

By Proposition 4.1 of [25], the absolute ramification field K^r of K (which we defined in the Introduction) is the unique maximal tame algebraic extension of K; it is a normal separable extension of K. By Lemma 4.2 of [25], an algebraic extension $L|K$ is purely wild if and only if it is linearly disjoint from $K^r|K$. From these facts, it follows that $K^r v$ is the separable-algebraic closure of Kv and vK^r is the *p-prime divisible hull* of vK:

$$vK^r = \bigcup_{n\in\mathbb{N}\setminus p\mathbb{N}} \frac{1}{n}\mathbb{Z} \quad \text{and} \quad K^r v = (Kv)^{\mathrm{sep}}. \tag{2.2}$$

We will now consider the behaviour of the defect when a finite extension $L|K$ of a henselian field K is shifted up through a tame extension $N|K$. We need the following information on K^r.

LEMMA 2.7. *Let (K,v) be an arbitrary valued field and p the characteristic exponent of Kv. Take an algebraic extension $L|K$. Then $L^r = L.K^r$; hence if $L\subset K^r$, then $L^r = K^r$. The separable-algebraic closure K^{sep} is a p-extension of K^r.*

Proof. For separable extensions, the first assertion follows from [5], p. 166, (20.15)(b) (where we put $N=K^{\mathrm{sep}}$ since we define K^r to be the ramification field of the separable extension $K^{\mathrm{sep}}|K$). Since every algebraic extension can be viewed as a purely inseparable extension of a separable extension, it remains to show the first assertion for a purely inseparable extension $L|K$. Here, it follows from the fact that $\mathrm{Gal}(K)\cong\mathrm{Gal}(L)$ and that by this isomorphism, the Galois group of an intermediate field K' of $K^{\mathrm{sep}}|K$ is isomorphic to the Galois group of the intermediate field $L.K'$ of $L^{\mathrm{sep}}|L$. The second assertion follows from [5], p. 167, Theorem (20.18). □

The next proposition shows the invariance of the defect under lifting up through tame extensions.

PROPOSITION 2.8. *Let K be a henselian field and N an arbitrary tame algebraic extension of K. If $L|K$ is a finite extension, then*

$$\mathrm{d}(L|K)=\mathrm{d}(L.N|N).$$

In particular, $L|K$ is defectless if and only if $L.N|N$ is defectless. This implies: K is a defectless field if and only if N is a defectless field, and the same holds for "separably defectless" and "inseparably defectless" in the place of "defectless".

Proof. Since $N^r=K^r$, it suffices to prove our lemma for the case of $N=K^r$, because then we obtain

$$\mathrm{d}(L|K)=\mathrm{d}(L.K^r|K^r)=\mathrm{d}(L.N^r|N^r)=\mathrm{d}((L.N).N^r|N^r)=\mathrm{d}(L.N|N)$$

for general N.

We put $L_0 := L \cap K^r$. We have $L.K^r = L^r$ and $L_0^r = K^r$ by Lemma 2.7. Since $K^r|K$ is normal, L is linearly disjoint from $K^r = L_0^r$ over L_0, and $L|L_0$ is thus a purely wild extension.

As a finite subextension of the tame extension $K^r|K$, $L_0|K$ is defectless. Hence, by the multiplicativity of the defect (2.1),

$$\mathrm{d}(L|K) = \mathrm{d}(L|L_0). \tag{2.3}$$

It remains to show $\mathrm{d}(L|L_0) = \mathrm{d}(L.K^r|K^r)$. Since $L|L_0$ is linearly disjoint from $K^r|L_0$, we have

$$[L^r : K^r] = [L.K^r : K^r] = [L : L_0]. \tag{2.4}$$

Since $L|L_0$ is purely wild, vL/vL_0 is a p-group and $Lv|L_0v$ is purely inseparable. On the other hand,

$$vL^r \text{ is the } p\text{-prime divisible hull of } vL \text{ and } L^r v = (Lv)^{\mathrm{sep}},$$

$$vL_0^r \text{ is the } p\text{-prime divisible hull of } vL_0 \text{ and } L_0^r v = (L_0 v)^{\mathrm{sep}}.$$

It follows that

$$(vL^r : vL_0^r) = (vL : vL_0) \quad \text{and} \quad [L^r v : L_0^r v] = [Lv : L_0 v]. \tag{2.5}$$

From (2.3), (2.4) and (2.5), keeping in mind that $L.K^r = L^r$ and $L_0^r = K^r$, we deduce

$$\begin{aligned}\mathrm{d}(L.K^r|K^r) = \mathrm{d}(L^r|L_0^r) &= \frac{[L^r : L_0^r]}{(vL^r : vL_0^r)[L^r v : L_0^r v]} \\ &= \frac{[L : L_0]}{(vL : vL_0)\cdot[Lv : L_0 v]} = \mathrm{d}(L|L_0) = \mathrm{d}(L|K).\end{aligned}$$

It now remains to show the second assertion of our proposition. Assume that N is a defectless field and let $L|K$ be an arbitrary finite extension. Then by hypothesis, $L.N|N$ is defectless; hence by what we have shown, $L|K$ is defectless. Since $L|K$ was arbitrary, K is shown to be a defectless field. Note that $L.N|N$ is separable if $L|K$ is separable, and $L.N|N$ is purely inseparable if $L|K$ is.

Conversely, assume that K is a defectless field. Since any finite extension $N'|N$ is contained in an extension $L.N|N$ where $L|K$ is a finite and, by hypothesis, defectless extension, we see that by what we have shown, $L.N|N$ and by virtue of Lemma 2.1 also its subextension $N'|N$ are defectless. Note that $L|K$ can be chosen to be separable if $N'|N$ is separable, and to be purely inseparable if $N'|N$ is purely inseparable. This completes the proof of our lemma. ☐

LEMMA 2.9. *Every finite extension of K^r is a tower of normal extensions of degree p. For every finite extension $L|K$, there is already a finite tame extension N of K^h such that $L.N|N$ is such a tower.*

Proof. We know from Lemma 2.7 that $K^{\mathrm{sep}}|K^r$ is a p-extension, that is, $\mathrm{Gal}\, K^{\mathrm{sep}}|K^r$ is a pro-p-group. For pro-p-groups G, the following is well known: for every open subgroup $H \subset G$ there exists a chain of open subgroups $H = H_0 \subset H_1 \subset \cdots \subset H_n = G$ such that $H_{i-1} \lhd H_i$ and $(H_i : H_{i-1}) = p$ for $i = 1, \ldots, n$. Hence by Galois correspondence, every finite separable extension of K^r is a tower of Galois extensions of degree p. Since every finite purely inseparable extension is a tower of purely inseparable extensions of degree p, this proves our first assertion.

We take N to be generated over K^h by all the finitely many elements of K^r that are needed to define the extensions in the tower $L.K^r|K^r$. □

COROLLARY 2.10. *A valued field (K, v) is henselian and defectless if and only if all of its finite defectless extensions are algebraically maximal.*

Proof. Suppose that (K, v) is henselian and defectless and (L, v) is a finite extension. Then (L, v) is henselian, and by Corollary 2.2 it is also defectless.

Suppose now that all finite extensions of (K, v) are algebraically maximal. Then (K, v) itself is algebraically maximal and hence henselian. Take any finite extension $(L|K, v)$; we wish to show that it is defectless. Take a finite tame extension $N|K$ as in the preceding lemma. Let $L_1|L_2$ be any extension of degree p in the tower $L.N|N$ such that $(L_2|N, v)$ is defectless. Since the finite tame extension $(N|K, v)$ is defectless, we know by Lemma 2.1 that the finite extension $(L_2|K, v)$ is defectless. So by our hypothesis, (L_2, v) is algebraically maximal. Thus, $(L_1|L_2, v)$ is not immediate and hence it is defectless. By induction over the extensions in the tower, together with repeated applications of Lemma 2.1, this shows that $(L.N|N, v)$ is defectless. From Proposition 2.8, it now follows that $(L|K, v)$ is defectless. □

2.2. Immediate extensions and pseudo Cauchy sequences.

LEMMA 2.11. *Take an algebraic extension $(K(a)|K, v)$ and let $f \in K[X]$ be the minimal polynomial of a over K. Suppose that $(c_\nu)_{\nu<\lambda}$ is a pseudo Cauchy sequence in K without limit in K, having a as a limit in L. Then $(c_\nu)_{\nu<\lambda}$ is of algebraic type, and for some $\mu < \lambda$, the values $(vf(c_\nu))_{\mu<\nu<\lambda}$ are strictly increasing. If in addition, the extension of v from K to $K(a)$ is unique, then the sequence of values is cofinal in $v\,\mathrm{im}_K(f)$, and in particular, $v\,\mathrm{im}_K(f)$ has no maximal element.*

Proof. Write $f(X) = \prod_{i=1}^n (X - a_i)$ with $a = a_1$ and $a_i \in \tilde{K}$. Since a is a limit of $(c_\nu)_{\nu<\lambda}$ we have that $v(a - c_\nu) = v(c_{\nu+1} - c_\nu)$ is strictly increasing with ν. The same holds for a_i in the place of a if a_i is also a limit of $(c_\nu)_{\nu<\lambda}$. If it is not, then by Lemma 3 of [12] there is some $\mu_i < \lambda$ such that $v(a - a_i) \leq v(a - c_{\mu_i})$. For $\mu_i < \nu < \lambda$, we have that $v(a - c_{\mu_i}) < v(a - c_\nu)$, which by the ultrametric triangle law yields $v(a_i - c_\nu) = \min\{v(a - a_i), v(a - c_\nu)\} =$

$v(a-a_i)$, which does not depend on ν. So if we take μ to be the maximum of all these μ_i, then for $\mu<\nu<\lambda$, the value

$$vf(c_\nu)=v\prod_{i=1}^{n}(c_\nu-a_i)=\sum_{i=1}^{n}v(c_\nu-a_i)$$

is strictly increasing with ν. Consequently, $(c_\nu)_{\nu<\lambda}$ is of algebraic type.

Now assume in addition that the extension of v from K to $K(a)$ is unique. Thus, for all $\sigma\in\operatorname{Aut}(\tilde{K}|K)$, the valuations v and $v\circ\sigma$ agree on $K(a)$. Choosing σ such that $\sigma a=a_i$, we find that

$$v(a_i-c)=v(\sigma a-c)=v\sigma(a-c)=v(a-c)$$

for all $c\in K$. So we obtain that

$$vf(c)=\sum_{i=1}^{n}v(c-a_i)=nv(a-c).$$

Suppose that there is some $c\in K$ such that

$$nv(a-c)=vf(c)>vf(c_\nu)=nv(a-c_\nu)$$

for all ν. This implies that $v(a-c)>v(a-c_\nu)$ which by Lemma 3 of [12] means that $c\in K$ is a limit of $(c_\nu)_{\nu<\lambda}$, contradicting our hypothesis. This proves that the sequence $(vf(c_\nu))_{\mu<\nu<\lambda}$ is cofinal in $v\operatorname{im}_K(f)$. Since it has no last element, it follows that $v\operatorname{im}_K(f)$ has no maximal element. □

Corollary 2.12. *If K admits a proper immediate algebraic extension, then there is a pseudo Cauchy sequence of algebraic type in K without a limit in K.*

Proof. Suppose that K admits a proper immediate algebraic extension $L|K$, and pick $a\in L\setminus K$. Then by Theorem 1 of [12], there is a pseudo Cauchy sequence in K without a limit in K, but having a as a limit. By the foregoing lemma, this pseudo Cauchy sequence is of algebraic type. □

2.3. Cuts and distances. Take any totally ordered set $(S,<)$. A *cut* Λ in S is a pair of sets $(\Lambda^{\mathrm{L}},\Lambda^{\mathrm{R}})$, where:

(a) Λ^{L} is an *initial segment* of S, i.e., if $\alpha\in\Lambda^{\mathrm{L}}$ and $\beta<\alpha$, then $\beta\in\Lambda^{\mathrm{L}}$,
(b) $\Lambda^{\mathrm{L}}\cup\Lambda^{\mathrm{R}}=S$ and $\Lambda^{\mathrm{L}}\cap\Lambda^{\mathrm{R}}=\emptyset$ (or equivalently, $\Lambda^{\mathrm{R}}=S\setminus\Lambda^{\mathrm{L}}$).

Note that then, Λ^{R} is a *final segment* of S, i.e., if $\alpha\in\Lambda^{\mathrm{R}}$ and $\beta>\alpha$, then $\beta\in\Lambda^{\mathrm{R}}$.

If Λ_1 and Λ_2 are cuts in S, then we will write $\Lambda_1<\Lambda_2$ if $\Lambda_1^{\mathrm{L}}\subsetneq\Lambda_2^{\mathrm{L}}$, and $\Lambda_1=\Lambda_2$ if $\Lambda_1^{\mathrm{L}}=\Lambda_2^{\mathrm{L}}$. But we also want to compare two cuts Λ_1 and Λ_2 if Λ_2 is a cut in a totally ordered set $(T,<)$ and Λ_1 is a cut in some subset S of T, endowed with the restriction of $<$. Here, we have at least two canonical ways of comparison. What suits our purposes best is what could be called *initial segment comparison* or just *left comparison*. (We leave it to the reader to figure out the analogous definition for the final segment comparison and to show that this leads to different results.) For every cut Λ in S, we define $\Lambda^{\mathrm{L}}\uparrow T$ to be the least initial segment of T containing Λ^{L}, that is, $\Lambda^{\mathrm{L}}\uparrow T$ is the unique initial segment of T in which Λ^{L} forms a cofinal subset. Then we set

$$\Lambda\uparrow T := \big(\Lambda^{\mathrm{L}}\uparrow T, T\setminus(\Lambda^{\mathrm{L}}\uparrow T)\big).$$

Observe that $\Lambda\mapsto\Lambda\uparrow T$ is an order preserving embedding of the set of cuts of S in the set of cuts in T. Now we can write $\Lambda_1<\Lambda_2$ if $\Lambda_1\uparrow T<\Lambda_2$, $\Lambda_1=\Lambda_2$ if $\Lambda_1\uparrow T=\Lambda_2$, and $\Lambda_1>\Lambda_2$ if $\Lambda_1\uparrow T>\Lambda_2$. That is, $\Lambda_1<\Lambda_2$ if Λ_1^{L} is contained but not cofinal in Λ_2^{L}, and $\Lambda_1=\Lambda_2$ if Λ_1^{L} is a cofinal subset of Λ_2^{L}.

We can embed S in the set of all cuts of S by sending $s\in S$ to the cut

$$s^+ := (\{t\in S| t\le s\}, \{t\in S| t>s\}).$$

We identify s with s^+. Then for any cut Λ, we have $s\le\Lambda$ if and only if $s\in\Lambda^{\mathrm{L}}$, and equality holds if and only if s is the maximal element of Λ^{L}. We also define

$$s^- := (\{t\in S| t<s\}, \{t\in S| t\ge s\}).$$

For any subset $M\subseteq S$, we let M^+ denote the cut

$$M^+ = (\{s\in S|\exists m\in M: s\le m\}, \{s\in S| s>M\}).$$

That is, if $M^+=(\Lambda^{\mathrm{L}},\Lambda^{\mathrm{R}})$ then Λ^{L} is the least initial segment of S which contains M, and Λ^{R} is the largest final segment which does not meet M. If $M=\emptyset$ then $\Lambda^{\mathrm{L}}=\emptyset$ and $\Lambda^{\mathrm{R}}=M$, and if $M=S$, then $\Lambda^{\mathrm{L}}=M$ and $\Lambda^{\mathrm{R}}=\emptyset$. Symmetrically, we set

$$M^- = (\{s\in S| s<M\}, \{s\in S|\exists m\in M: s\ge m\}).$$

Take two cuts $\Lambda_1=(\Lambda_1^{\mathrm{L}},\Lambda_1^{\mathrm{R}})$ and $\Lambda_2=(\Lambda_2^{\mathrm{L}},\Lambda_2^{\mathrm{R}})$ in some ordered abelian group. We let $\Lambda_1+\Lambda_2$ be the cut $(\Lambda^{\mathrm{L}},\Lambda^{\mathrm{R}})$ defined by $\Lambda^{\mathrm{L}}:=\Lambda_1^{\mathrm{L}}+\Lambda_2^{\mathrm{L}}$ (note that the sum of two initial segments is always an initial segment). This is called the *left sum* of two cuts; the *right sum* is defined by setting $\Lambda^{\mathrm{R}}:=\Lambda_1^{\mathrm{R}}+\Lambda_2^{\mathrm{R}}$. In general, left and right sum are not equal. For instance, the left sum of 0^- and 0^+ is 0^-, and the right sum is 0^+. In this paper, we will only use the left sum, without mentioning this any further.

We call a cut Λ *idempotent* if $\Lambda+\Lambda=\Lambda$. Lemma 2.14 below will show that in a divisible ordered abelian group, idempotency of a cut does not depend on whether we take left or right sums.

We leave the easy proof of the following observation to the reader.

LEMMA 2.13. *If $G'\supset G$ is an extension of ordered abelian groups, then the operation $\uparrow$ is an addition preserving embedding of the ordered set of cuts in G in the ordered set of cuts in G'.*

If S is a subset of an ordered abelian group G and $n \in \mathbb{N}$, we set $n \cdot S := \{n\alpha \mid \alpha \in S\}$ and $nS := \{\alpha_1 + \cdots + \alpha_n \mid \alpha_1, \ldots, \alpha_n \in S\}$. If Λ^{L} is an initial segment, then $n\Lambda^{\mathrm{L}}$ is again an initial segment, and $n \cdot \Lambda^{\mathrm{L}}$ is cofinal in $n\Lambda^{\mathrm{L}}$. If in addition G is n-divisible, then $n \cdot \Lambda^{\mathrm{L}} = n\Lambda^{\mathrm{L}}$. Corresponding assertions hold for Λ^{R} in the place of Λ^{L}.

In every ordered abelian group, $n \cdot \Lambda^{\mathrm{L}}$ and $n\Lambda^{\mathrm{L}}$ define the same cut $n\Lambda := (n \cdot \Lambda^{\mathrm{L}})^+ = (n\Lambda^{\mathrm{L}})^+$. Note that $n\Lambda$ coincides with the n-fold (left) sum of Λ.

Lemma 2.14. *Let $\Lambda = (\Lambda^{\mathrm{L}}, \Lambda^{\mathrm{R}})$ be a cut in some ordered abelian group Γ, and $n > 1$ a fixed natural number. The following assertions are equivalent*:

(a) Λ *is idempotent*,
(b) $\Lambda^{\mathrm{L}} + \Lambda^{\mathrm{L}} = \Lambda^{\mathrm{L}}$,
(c) $i\Lambda = \Lambda$ *for every natural number* $i > 1$,
(d) $n\Lambda = \Lambda$.

If Γ is divisible, then these assertions are also equivalent to each of the following:

(e) $\Lambda^{\mathrm{R}} + \Lambda^{\mathrm{R}} = \Lambda^{\mathrm{R}}$,
(f) $n \cdot \Lambda^{\mathrm{L}} = \Lambda^{\mathrm{L}}$,
(g) $n \cdot \Lambda^{\mathrm{R}} = \Lambda^{\mathrm{R}}$,
(h) $\forall \alpha \in \Gamma : \alpha \in \Lambda^{\mathrm{L}} \Leftrightarrow n\alpha \in \Lambda^{\mathrm{L}}$,
(i) $\forall \alpha \in \Gamma : \alpha \in \Lambda^{\mathrm{R}} \Leftrightarrow n\alpha \in \Lambda^{\mathrm{R}}$,
(k) $\Lambda = H^+$ *or* $\Lambda = H^-$ *for some convex subgroup H of* Γ.

Proof. The equivalence of (a) and (b) holds by definition. (a) $\Rightarrow$ (c) is proved by induction on i. Further, (c) $\Rightarrow$ (d) is trivial. We have that $\Lambda \leq 2\Lambda \leq \cdots \leq n\Lambda$ or $\Lambda \geq 2\Lambda \geq \cdots \geq n\Lambda$, depending on whether $0 \in \Lambda^{\mathrm{L}}$ or $0 \in \Lambda^{\mathrm{R}}$. Thus, $n\Lambda = \Lambda$ implies $\Lambda = 2\Lambda = \Lambda + \Lambda$; that is, (d) implies (a).

Now assume that Γ is divisible. Then $\alpha \mapsto n\alpha$ and $\alpha \mapsto \frac{1}{n}\alpha$ are order preserving isomorphisms. Therefore, (f) and (g) are equivalent. The equivalence of (f) with (d) holds since the divisibility implies that $n \cdot \Lambda^{\mathrm{L}} = n\Lambda^{\mathrm{L}}$. Further, (f) is equivalent to $n\Lambda^{\mathrm{L}} \subseteq \Lambda^{\mathrm{L}} \wedge \Lambda^{\mathrm{L}} \subseteq n\Lambda^{\mathrm{L}}$, and this in turn is equivalent to $n\Lambda^{\mathrm{L}} \subseteq \Lambda^{\mathrm{L}} \wedge \frac{1}{n}\Lambda^{\mathrm{L}} \subseteq \Lambda^{\mathrm{L}}$. This is equivalent to $\forall \alpha \in \Gamma : \alpha \in \Lambda^{\mathrm{L}} \Rightarrow n\alpha \in \Lambda^{\mathrm{L}} \wedge \alpha \in \Lambda^{\mathrm{L}} \Rightarrow \frac{1}{n}\alpha \in \Lambda^{\mathrm{L}}$. But the latter implication can be reformulated as $n\alpha \in \Lambda^{\mathrm{L}} \Rightarrow \alpha \in \Lambda^{\mathrm{L}}$. This proves that (f) and (h) are equivalent. In the same way, the equivalence of (g) with (i) is proved.

As for Λ^{L}, divisibility also implies that $n \cdot \Lambda^{\mathrm{R}} = n\Lambda^{\mathrm{R}}$. Taking $n = 2$ in what we have already proved, we see that (e) is equivalent to the $n = 2$ case of (g), and hence to (a).

Finally, it remains to show the equivalence of (k) with the other conditions. Set

$$H := \{\pm\alpha \mid 0 \leq \alpha \in \Lambda^{\mathrm{L}}\} \cup \{\pm\alpha \mid 0 \geq \alpha \in \Lambda^{\mathrm{R}}\}.$$

Note that exactly one of the two sets is empty, depending on whether $0 \in \Lambda^{\mathrm{L}}$ or $0 \in \Lambda^{\mathrm{R}}$. It is easy to see that $\Lambda = H^+$ if $0 \in \Lambda^{\mathrm{L}}$ and $\Lambda = H^-$ if $0 \in \Lambda^{\mathrm{R}}$. Hence, it suffices to prove that H is a convex subgroup if and only if $\Lambda^{\mathrm{L}}+\Lambda^{\mathrm{L}}=\Lambda^{\mathrm{L}}$. Observe that H is always convex and closed under $\alpha\mapsto -\alpha$. Hence, H is a convex subgroup if and only if it is closed under addition. In the case of $0\in\Lambda^{\mathrm{L}}$, this holds if and only if $\Lambda^{\mathrm{L}}+\Lambda^{\mathrm{L}}=\Lambda^{\mathrm{L}}$, and in the case of $0\in\Lambda^{\mathrm{R}}$, this holds if and only if $\Lambda^{\mathrm{R}}+\Lambda^{\mathrm{R}}=\Lambda^{\mathrm{R}}$. But as we have already shown that (b) and (e) are equivalent, we see that (k) is equivalent with (b). ☐

In a nondivisible group, a condition like $\forall i\in\mathbb{N}: i\Lambda^{\mathrm{L}}=\Lambda^{\mathrm{L}}$ can only hold if Λ^{L} is empty, and condition (a) is in general not equivalent to (e), (h), (i), (k).

Example 2.15. Take $\Gamma:=\mathbb{Z}\times\mathbb{Q}$ with the lexicographic ordering, and set

$$\Lambda:=\big(\{(m,q)\mid -1\geq m\in\mathbb{Z},q\in\mathbb{Q}\},\{(m,q)\mid 0\leq m\in\mathbb{Z},q\in\mathbb{Q}\}\big).$$

Then Λ satisfies (e), (h), (i), and $\Lambda=H^-$ for the convex subgroup $H=\{0\}\times\mathbb{Q}$ of Γ. But Λ is not idempotent since

$$\Lambda+\Lambda=\big(\{(m,q)\mid -2\geq m\in\mathbb{Z},q\in\mathbb{Q}\},\{(m,q)\mid -1\leq m\in\mathbb{Z},q\in\mathbb{Q}\}\big)<\Lambda.$$

On the other hand, the cut induced by Λ in the divisible hull $\tilde{\Gamma}$ of Γ is

$$\Lambda\uparrow\tilde{\Gamma}=\big(\{(m,q)\mid -1\geq m\in\mathbb{Q},q\in\mathbb{Q}\},\{(m,q)\mid -1< m\in\mathbb{Q},q\in\mathbb{Q}\}\big).$$

Since $(-\frac{1}{2},0)$ is in the right cut set while $2(-\frac{1}{2},0)=(1,0)$ is in the left cut set, this cannot be equal to H^+ or H^- for any convex subgroup H of $\tilde{\Gamma}$.

Take any extension $(L|K,v)$ of valued fields, and $z\in L$. We define

$$\Lambda^{\mathrm{L}}(z,K):=\{v(z-c)\mid c\in K \text{ and } v(z-c)\in vK\}.$$

Further, we set $\Lambda^{\mathrm{R}}(z,K):=vK\setminus\Lambda^{\mathrm{L}}(z,K)$.

Lemma 2.16. *$\Lambda^{\mathrm{L}}(z,K)$ is an initial segment of vK. Thus, $\Lambda^{\mathrm{R}}(z,K)=\{\alpha\in vK|\forall c\in K: v(z-c)<\alpha\}$, and $(\Lambda^{\mathrm{L}}(z,K),\Lambda^{\mathrm{R}}(z,K))=\Lambda^{\mathrm{L}}(z,K)^+$ is a cut in vK.*

Proof. Take $\alpha\in\Lambda^{\mathrm{L}}(z,K)$ and $\beta\in vK$ such that $\beta<\alpha$. Pick $c,d\in K$ such that $v(z-c)=\alpha$ and $vd=\beta$. Then $\beta=vd=\min\{vd,v(z-c)\}=v(z-c-d)\in\Lambda^{\mathrm{L}}(z,K)$. This proves that $\Lambda^{\mathrm{L}}(z,K)$ is an initial segment of vK. ☐

We have seen in Lemma 2.14 that in a divisible ordered abelian group we have many nice characterizations of idempotent cuts; in particular, we are very interested in characterization k) which shows that idempotent cuts correspond to upper or lower edges of convex subgroups. This is the reason for the following definition. Take an element z in any valued field extension (L,v) of (K,v). Then the *distance of z from K* is the cut

$$\operatorname{dist}(z,K):=(\Lambda^{\mathrm{L}}(z,K),\Lambda^{\mathrm{R}}(z,K))\uparrow\widetilde{vK}$$

in the divisible hull $\widetilde{vK}$ of vK.

Take two elements y, z in some valued field extension of (K, v). We define

$$z \sim_K y$$

to mean that $v(z-y) > \operatorname{dist}(z, K)$. Note that by our identification of the value $v(z-y)$ with the cut $v(z-y)^+$,

$$v(z-y) > \operatorname{dist}(z, K) \quad \text{if and only if} \quad v(z-y) > \Lambda^{\mathrm{L}}(z, K).$$

LEMMA 2.17.

(1) *If $z \sim_K y$ then $v(z-c) = v(y-c)$ for all $c \in K$ such that $v(z-c) \in vK$, whence $\Lambda^{\mathrm{L}}(z, K) = \Lambda^{\mathrm{L}}(y, K)$ and $\operatorname{dist}(z, K) = \operatorname{dist}(y, K)$.*

(2) *If $\Lambda^{\mathrm{L}}(z, K)$ has no maximal element, then the following are equivalent:*
   (a) $z \sim_K y$,
   (b) $v(z-c) = v(y-c)$ *for all* $c \in K$ *such that* $v(z-c) \in vK$,
   (c) $v(z-y) \geq \operatorname{dist}(z, K)$.

Proof. (1) Assume that $z \sim_K y$. If $v(z-c) \in vK$, then $v(z-c) \in \Lambda^{\mathrm{L}}(z, K)$ and therefore, $v(z-y) > v(z-c)$. Hence, $v(y-c) = \min\{v(z-c), v(z-y)\} = v(z-c)$.

(2) The implication (a)⇒(b) follows from part (1). Now assume that (b) holds, and take any $c \in K$ such that $v(z-c) \in vK$. Then because of $v(z-c) = v(y-c)$, we obtain $v(z-y) \geq \min\{v(z-c), v(y-c)\} = v(z-c)$. This shows that $v(z-y) \geq \operatorname{dist}(z, K)$. We have proved that (b) implies (c).

Since $\Lambda^{\mathrm{L}}(z, K)$ has no maximal element, $v(z-c)$ cannot be the maximal element of $\Lambda^{\mathrm{L}}(z, K)$. Thus, $v(z-y) \geq \operatorname{dist}(z, K)$ implies $v(z-y) > \operatorname{dist}(z, K)$, which proves the implication (c) ⇒ (a). □

LEMMA 2.18. *If $(K, v) \subseteq (L, v) \subseteq (L(z), v)$, then*

$$\operatorname{dist}(z, L) \geq \operatorname{dist}(z, K). \tag{2.6}$$

If "$>$" holds, then there exists an element $y \in L$ such that $z \sim_K y$.

Proof. Since $K \subseteq L$, we have that $\Lambda^{\mathrm{L}}(z, K) \subseteq \Lambda^{\mathrm{L}}(z, L)$, whence (2.6). If "$>$" holds, then there exists an element $y \in L$ such that $v(z-y) > \Lambda^{\mathrm{L}}(z, K)$, i.e., $z \sim_K y$. □

We define

$$v(z-K) := \{v(z-c) \mid c \in K\} = v\operatorname{im}_K(X-z).$$

Note that

$$\Lambda^{\mathrm{L}}(z, K) = v(z-K) \cap vK.$$

Hence, if $vK(z) = vK$, then $\Lambda^{\mathrm{L}}(z, K) = v(z-K)$.

THEOREM 2.19 (cf. Theorem 1 of [12]). *Let L be an immediate extension of K. Then for every element $z \in L \setminus K$ it follows that $v(z-K)$ has no maximal element and that $v(z-K) = \Lambda^{\mathrm{L}}(z, K)$. In particular, vz is not maximal in $\Lambda^{\mathrm{L}}(z, K)$ and therefore, $vz < \operatorname{dist}(z, K)$.*

Proof. Take $z \in L \setminus K$. Then $\infty \notin v(z-K)$. If $(L|K,v)$ is immediate, then $vL = vK$ and therefore, $v(z-K) = \Lambda^{\mathrm{L}}(z,K)$. Take any $c \in K$. Then $v(z-c) \in vL = vK$ and thus there exists $d \in K$ such that $vd(z-c) = 0$. So $d(z-c)v \in Lv = Kv$. Hence, there exists $d' \in K$ such that $(d(z-c)-d')v = 0$, which means that $v(z-c-d'd^{-1}) > -vd = v(z-c)$. Since $c + d'd^{-1} \in K$ and $v(z-c-d'd^{-1}) \in vL = vK$, this shows that $v(z-c)$ was not the maximal element of $v(z-K)$. This proves that $v(z-K)$ has no maximal element. □

The following is a corollary to Lemma 2.11.

COROLLARY 2.20. *If $(K(z)|K,v)$ is an algebraic extension and (K,v) is algebraically maximal, then $v(z-K)$ has a maximum.*

Proof. If $v(z-K)$ has no maximum, then there is a sequence $(c_\nu)_{\nu<\lambda}$ without last element (so λ is a limit ordinal) and such that $(v(z-c_\nu))_{\nu<\lambda}$ is strictly increasing and cofinal in $v(z-K)$. The former implies that $(c_\nu)_{\nu<\lambda}$ is a pseudo Cauchy sequence with z as a limit. The latter implies that $(c_\nu)_{\nu<\lambda}$ has no limit in K since by Lemma 3 of [12], any limit b satisfies $v(z-b) > v(z-c_\nu)$ for all $\nu < \lambda$. By Lemma 2.11, $(c_\nu)_{\nu<\lambda}$ is of algebraic type. Hence, by Theorem 3 of [12], there is a nontrivial immediate algebraic extension of K, which shows that K cannot be algebraically maximal. □

Does the converse of Theorem 2.19 also hold, that is, if $v(z-K)$ has no maximal element, is then the extension $(K(z)|K,v)$ immediate? This is far from being true. Under certain additional conditions however, the converse holds.

LEMMA 2.21. *Take an extension $(K(z)|K,v)$ of valued fields of degree $p = \mathrm{char}(Kv)$ and such that the extension of v from K to $K(z)$ is unique.*

(1) *If $v(z-K)$ has no maximal element, then $(K(z)|K,v)$ is immediate.*
(2) *If $(K(y)|K,v)$ is an immediate extension and if $y \sim_K z$ in some common valued extension field of $K(z)$ and $K(y)$, then $(K(z)|K,v)$ is also an immediate extension.*

Proof. (1) Since the extension of v from K to $K(z)$ is unique, we have

$$p = [K(z):K] = \big(vK(z):vK\big)[K(z)v:Kv]p^\nu$$

by (1.1). Assume that $(K(z)|K,v)$ is not immediate. Then $(vK(z):vK) = p$ or $[K(z)v:Kv] = p$. If $(vK(z):vK) = p$, then we can choose $b_1,\ldots,b_p \in K(z)$ such that the values $vb_1,\ldots,vb_p$ belong to distinct cosets modulo vK. If $[K(z)v:Kv] = p$, then we can choose $b_1,\ldots,b_p \in K(z)$ such that $vb_1 = \cdots = vb_p = 0$ and the residues $b_1v,\ldots,b_pv$ form a basis of $K(z)v|Kv$. In both cases, we obtain that $b_1,\ldots,b_p$ is a valuation basis of $(K(z)|K,v)$. In the first case, given any $c_1,\ldots,c_p \in K$, this follows from the ultrametric triangle law since all values vc_ib_i, $1 \le i \le p$, must be distinct. In the second case, we may assume w.l.o.g. (after suitable renumbering) that $vc_1 = \min_i vc_i$; then $vc_1^{-1}c_ib_i \geq 0$ and we obtain

$$\left(\sum_{i=1}^{p} c_1^{-1}c_ib_i\right)v = b_1v + \sum_{i=2}^{p}(c_1^{-1}c_i)vb_iv \neq 0.$$

This yields $v\sum_{i=1}^{p} c_1^{-1}c_ib_i = 0$ and thus $v\sum_{i=1}^{p} c_ib_i = vc_1 = \min_i vc_i = \min_i vc_ib_i$.

Without loss of generality, we can choose $b_1 = 1$. We write $z = \sum_{i=1}^{p} c_ib_i$. For every $c \in K$, we obtain

$$v(z-c) = \min\{v(c_1-c), vc_2, \ldots, vc_p\} \leq \min\{vc_2, \ldots, vc_p\}.$$

The maximum value $\min\{vc_2, \ldots, vc_p\}$ is assumed for $c_1 - c = 0$. That is, $v(z - c_1)$ is the maximum of $v(z - K)$.

(2) Assume that $(K(y)|K,v)$ is an immediate extension. Then by Theorem 2.19, $v(y-K) = \Lambda^{\mathrm{L}}(y,K)$ has no maximal element. By Lemma 2.17, $z \sim_K y$ implies that $v(y-c) = v(z-c)$ for all $c \in K$ such that $v(y-c) \in vK$, that is, for all $c \in K$. It follows that $v(z-K)$ has no maximal element. Now part (1) shows that $(K(z)|K,v)$ is an immediate extension. □

LEMMA 2.22. *Assume that (K,v) is henselian, that $(K(z)|K,v)$ is an immediate extension, and that $z \sim_K y$ in some common valued extension field of $K(z)$ and $K(y)$. Take a polynomial $f \in K[X]$ of degree smaller than $p = \mathrm{char}Kv$. Then $f(z) \sim_K f(y)$.*

Proof. Let $f \in K[X]$ be a polynomial of degree $< p$. Since $(K(z)|K,v)$ is immediate and $f(z) \in K(z)$, we know from Theorem 2.19 that $\Lambda^{\mathrm{L}}(f(z),K)$ has no maximal element. Hence, by part (2) of Lemma 2.17 it suffices to show that $v(f(z)-c) = v(f(y)-c)$ for all $c \in K$. Since $f - c$ is again of degree $< p$, we see that it suffices to show that $vf(z) = vf(y)$ for all polynomials f of degree $< p$.

Again by Theorem 2.19, we know that $\Lambda^{\mathrm{L}}(z,K)$ has no maximal element. Since $z \sim_K y$, part (1) of Lemma 2.17 shows that $\Lambda^{\mathrm{L}}(z,K) = \Lambda^{\mathrm{L}}(y,K)$. As in the proof of Corollary 2.20, we find a pseudo Cauchy sequence $(c_\nu)_{\nu<\lambda}$ in (K,v) that has both z and y as a limit, but no limit in K. For every polynomial of degree $< p$, the value of the sequence $(f(c_\nu))_{\nu<\lambda}$ must eventually be fixed since otherwise, Theorem 3 of [12] would show the existence of an immediate extension of (K,v) of degree less than p. But since (K,v) is henselian, the Lemma of Ostrowski shows that this is impossible, since the defect must be a power of p. Now one shows like in the proof of Theorem 2 of [12] that both $vf(z)$ and $vf(y)$ are equal to the eventually fixed value of the sequence $(f(c_\nu))_{\nu<\lambda}$. □

We will show that we can drop the condition that (K,v) be henselian if the element y is purely inseparable over K. To this end, we need the following result which is proved in [22].

LEMMA 2.23. *Let (K,v) be a valued field, K^h its henselization w.r.t. a fixed extension of v to the algebraic closure $\tilde{K}$, and $y \in \tilde{K}$. If*

$$\mathrm{dist}(y,K^h) > \mathrm{dist}(y,K),$$

then y is not purely inseparable over K.

LEMMA 2.24. *Assume that $(K(z)|K,v)$ is an immediate extension and that $z \sim_K y$ in some common valued extension field of $K(z)$ and $K(y)$. Suppose that y is purely inseparable over K. Take a polynomial $f \in K[X]$ of degree smaller than $p = \mathrm{char} Kv$. Then $f(z) \sim_K f(y)$.*

Proof. Since henselizations are immediate extensions and since $K^h(z)$ lies in the henselization of $K(z)$, we know that $(K^h(z)|K^h,v)$ is an immediate extension. From the previous lemma, we infer that $\mathrm{dist}(y,K^h) = \mathrm{dist}(y,K)$. Hence, $z \sim_K y$ implies that $z \sim_{K^h} y$. From Lemma 2.22, we obtain that $f(z) \sim_{K^h} f(y)$, whence $f(z) \sim_K f(y)$. □

If $\alpha \in vK$ and Λ is a cut in vK, then $\alpha + \Lambda := (\alpha + \Lambda^{\mathrm{L}}, \alpha + \Lambda^{\mathrm{R}})$. Since addition of α is an order preserving isomorphism of vK, this is again a cut. The proof of the following lemma is straightforward.

LEMMA 2.25. *For every $c \in K$,*

$$\Lambda^{\mathrm{L}}(z+c,K) = \Lambda^{\mathrm{L}}(z,K) \quad \textit{and} \quad \mathrm{dist}(z+c,K) = \mathrm{dist}(z,K),$$
$$\Lambda^{\mathrm{L}}(cz,K) = vc + \Lambda^{\mathrm{L}}(z,K) \quad \textit{and} \quad \mathrm{dist}(cz,K) = vc + \mathrm{dist}(cz,K),$$
$$z \sim_K y \quad \Rightarrow \quad z + c \sim_K y + c,$$
$$c \neq 0 \wedge z \sim_K y \quad \Rightarrow \quad cz \sim_K cy.$$

2.4. Properties of Artin–Schreier extensions. In this section, we collect a few facts about Artin–Schreier extensions of valued fields. Throughout this section, we assume that $K(\vartheta)|K$ is an Artin–Schreier extension of degree p with $\vartheta^p - \vartheta = a \in K$.

LEMMA 2.26. *If $\mathrm{char} K = p$, then ϑ' is another Artin–Schreier generator of $L|K$ if and only if $\vartheta' = i\vartheta + c$ for some $i \in \{1,\ldots,p-1\}$ and $c \in K$.*

Proof. If ϑ, ϑ' are roots of the same polynomial $X^p - X - a$, then $\vartheta - \vartheta'$ is a root of $X^p - X$, whose roots are $0,1,\ldots,p-1 \in \mathbb{F}_p$. Hence, $\vartheta, \vartheta+1,\ldots,\vartheta+p-1$ are all roots of $X^p - X - a$. Pick a nontrivial $\sigma \in \mathrm{Gal}\, L|K$. We then have that $\sigma\vartheta - \vartheta = j$ for some $j \in \mathbb{F}_p^\times$. If ϑ, ϑ' are any two Artin–Schreier generators of $L|K$ such that $\sigma\vartheta - \vartheta = \sigma\vartheta' - \vartheta'$, then we have $\sigma(\vartheta - \vartheta') = \vartheta - \vartheta'$. Since σ is a generator of $\mathrm{Gal}\, L|K \simeq \mathbb{Z}/p\mathbb{Z}$, it follows that $\tau(\vartheta - \vartheta') = \vartheta - \vartheta'$ for all $\tau \in \mathrm{Gal}\, L|K$, that is, $\vartheta - \vartheta' \in K$. If ϑ, ϑ' are any two Artin–Schreier generators of $L|K$ such that $\sigma\vartheta - \vartheta = j \in \mathbb{F}_p^\times$ and $\sigma\vartheta' - \vartheta' = j' \in \mathbb{F}_p^\times$, then there is some $i \in \{1,\ldots,p-1\}$ such that $ij = j'$ and therefore, $\sigma i\vartheta - i\vartheta = ij = j'$. Then by what we have shown before, $\vartheta' = i\vartheta + c$ for some $c \in K$.

Conversely, if ϑ is an Artin–Schreier generator of $L|K$ and if $i \in \{1,\dots,p-1\}$ and $c \in K$, then $(i\vartheta+c)^p-(i\vartheta+c)=i(\vartheta^p-\vartheta)+c^p-c \in K$. But $i\vartheta+c$ cannot lie in K, so $K(i\vartheta+c)=L$ since $[L:K]$ is a prime. This shows that also $i\vartheta+c$ is an Artin–Schreier generator of $L|K$. $\square$

We will frequently use the following easy observation.

LEMMA 2.27. *If $va \le 0$, then $v\vartheta = \frac{1}{p}va$, and if $v\vartheta > 0$, then $v\vartheta = va$. If $va > 0$, then exactly one of the conjugates $\vartheta, \vartheta+1, \dots, \vartheta+p-1$ has value va and the others have value* 0.

Proof. We have that $\vartheta^p - \vartheta = a$. If $v\vartheta \neq 0$, then $v\vartheta^p = pv\vartheta \neq v\vartheta$ and therefore, $va = v(\vartheta^p-\vartheta) = \min\{pv\vartheta, v\vartheta\} \neq 0$ by the ultrametric triangle law. If $va = 0$, we thus have $v\vartheta = 0 = \frac{1}{p}va$. For $va < 0$, we must have $v\vartheta < 0$, whence $pv\vartheta < v\vartheta$ and $va = pv\vartheta$. For $v\vartheta > 0$, we have $v\vartheta < pv\vartheta$ and $va = v\vartheta$.

Now assume that $va > 0$. As a is the product of the conjugates of ϑ, one of them must have positive value; by what we have shown before, this value is va. If we denote this conjugate by ϑ', then the other conjugates are of the form $\vartheta'+i$ with $i \in \{1,\dots,p-1\}$ and hence of value 0. $\square$

The following lemma gives a first classification of Artin–Schreier extensions of valued fields.

LEMMA 2.28. *Assume that* $\mathrm{char} Kv = p$. *If $va > 0$ or if $va = 0$ and $X^p - X - av$ has a root in Kv, then ϑ lies in the henselization of K (with respect to every extension of the valuation to the algebraic closure of K) and there are precisely p many distinct extensions of v from K to $K(\vartheta)$; hence, equality holds in the fundamental inequality* (1.2).

If $va = 0$ and $X^p - X - av$ has no root in Kv, then $K(\vartheta)v|Kv$ is a separable extension of degree p and $(K(\vartheta)|K, v)$ is defectless.

If $(K(\vartheta)|K, v)$ has nontrivial defect, then $va < 0$.

Proof. If $va > 0$, then the reduction of $X^p - X - a$ modulo v is $X^p - X$ which splits completely in Kv and has p many distinct roots since $\mathrm{char} Kv = p > 0$. Then by Hensel's Lemma, $X^p - X - a$ splits completely in every henselization of K.

If $va = 0$ and $X^p - X - av$ has a root in Kv, then $X^p - X - av$ splits completely in Kv and has p many distinct roots. Hence, again, $X^p - X - a$ splits completely in every henselization of K.

In both cases, pick one extension of v to $K(\vartheta)$ and call it again v. The roots of $X^p - X - a$ are in one-to-one correspondence with the roots of $X^p - X - av$. Hence, the roots $\eta_1, \dots, \eta_p$ of $X^p - X - a$ have distinct residues in Kv, say, $c_1 v, \dots, c_p v$ with $c_i \in K$. If σ_i is the automorphism of $K(\vartheta)|K$ which sends η_1 to η_i, then $v \circ \sigma_i(\eta_1 - c_i) = v(\sigma_i \eta_1 - c_i) = v(\eta_i - c_i) > 0$ and $v \circ \sigma_j(\eta_1 - c_i) = v(\eta_j - c_i) = 0$ for $j \neq i$. This shows that the extensions $v \circ \sigma_i$ are distinct for

$1 \leq i \leq p$. Since all extensions are conjugate and therefore must be of the form $v \circ \sigma_i$, we find that there are precisely p many distinct extensions.

If $va = 0$ and $X^p - X - av$ has no root in Kv, then $[Kv(\vartheta v) : Kv] = p$. We obtain that $p = [K(\vartheta) : K] \geq [K(\vartheta)v : Kv] \geq [Kv(\vartheta v) : Kv] = p$ and see that equality must hold everywhere. So $K(\vartheta)v = Kv(\vartheta v)$ is a separable extension of Kv, and $K(\vartheta)|K$ is defectless.

By what we have proved, $K(\vartheta)|K$ is defectless whenever $va \geq 0$. This yields the last assertion of our lemma. □

If $\mathrm{char} K = p > 0$, then the Artin–Schreier polynomial is additive. If ϑ is a root of $X^p - X - a$ and if $c \in K$, then

$$(\vartheta - c)^p - (\vartheta - c) = \vartheta^p - \vartheta - c^p + c = a - c^p + c,$$

that is, $\vartheta - c$ is a root of the polynomial $X^p - X - (a - c^p + c)$.

REMARK 2.29. Since $K(\vartheta) = K(\vartheta - c)$, this shows that summands appearing in some presentation of a that are pth powers can be replaced by their pth roots without changing the extension. This allows to deduce normal forms for a that serve various purposes. They are key tools in [19] and [23] and in related work of S. Abhyankar [1] and H. Epp [7].

COROLLARY 2.30. *Assume that* $\mathrm{char} K = p$ *and that* $(K(\vartheta)|K, v)$ *has non-trivial defect. Then* $v(\vartheta - c) < 0$ *for every* $c \in K$, *and consequently,* $\mathrm{dist}(\vartheta, K) \leq 0^-$.

Proof. If there exists $c \in K$ such that $v(\vartheta - c) \geq 0$, then $\vartheta - c$ is a root of the polynomial $X^p - X - (a - c^p + c)$ and by Lemma 2.27 we have $v(a - c^p + c) \geq 0$. But then by Lemma 2.28, the field $K(\vartheta) = K(\vartheta - c)$ cannot be a defect extension of K. □

The converse is also true, in the following sense.

LEMMA 2.31. *Assume that* $\mathrm{char} K = p$. *If* $\mathrm{dist}(\vartheta, K) \leq 0^-$ *and* $v(\vartheta - K)$ *has no maximal element, then the extension of* v *from* K *to* $K(\vartheta)$ *is unique,* $(K(\vartheta)|K, v)$ *is immediate and consequently,* $K(\vartheta)|K$ *is an Artin–Schreier defect extension.*

Proof. In [22], we show that the assumption that $\mathrm{dist}(\vartheta, K) \leq 0^-$ implies that the extension of v from K to $K(\vartheta)$ is unique. Since $v(\vartheta - K)$ has no maximal element, Lemma 2.21 yields that $(K(\vartheta)|K, v)$ is immediate. □

We will also need the following fact.

LEMMA 2.32. *Let* K *be an Artin–Schreier closed field of characteristic* $p > 0$. *Then also every purely inseparable extension of* K *is Artin–Schreier closed.*

Proof. If $\mathrm{char} K = 0$, then every purely inseparable extension is trivial and there is nothing to show. So let $\mathrm{char} K = p > 0$. Assume L to be a purely inseparable extension of the Artin–Schreier closed field K. Take $a \in L$ and let $\vartheta \in \tilde{L}$ be a root of $X^p - X - a$. Let $m \geq 0$ be the minimal integer such that $a^{p^m} \in K$. Then $(\vartheta^{p^m})^p - \vartheta^{p^m} = (\vartheta^p - \vartheta)^{p^m} = a^{p^m}$. Since K is Artin–Schreier closed by assumption, it follows that $\vartheta^{p^m} \in K$. The field $K(\vartheta)$ contains $a = \vartheta^p - \vartheta$ and thus, $[K(\vartheta) : K] \geq [K(a) : K] = p^m$. On the other hand, $p^m \geq [K(\vartheta) : K]$ since $\vartheta^{p^m} \in K$. Consequently, $[K(\vartheta) : K] = [K(a) : K]$, showing that $\vartheta \in K(a) \subseteq L$. □

3. Inseparably defectless fields

In this section, we shall give a characterization of inseparably defectless fields. Throughout, we assume that $\mathrm{char} K = p$. Recall that every purely inseparable algebraic extension admits a unique extension of the valuation. Every defectless field and in particular every trivially valued field is inseparably defectless. Note that a valued field can be *inseparably maximal*, that is, it does not admit proper immediate purely inseparable extensions, without being inseparably defectless. The field (F, v) of Example 3.25 in [20] is of this kind.

By definition, (K, v) is an inseparably defectless field if and only if the extension $(K^{1/p^\infty}|K, v)$ is defectless. For this to hold, it is already sufficient that $(K^{1/p}|K, v)$ is defectless:

LEMMA 3.1. *The field (K, v) is inseparably defectless if and only if $(K^{1/p}|K, v)$ is defectless, and this holds if and only if $(K|K^p, v)$ is defectless.*

Proof. The first implication "⇒" is trivial. Assume that $(K^{1/p}|K, v)$ is defectless. The Frobenius endomorphism sends the extension

$$(K^{1/p^2}|K^{1/p}, v)$$

onto the extension $(K^{1/p}|K, v)$ and is valuation preserving. Consequently, also the former extension is defectless. Replacing the Frobenius by its mth power, we find that $(K^{1/p^{m+1}}|K^{1/p^m}, v)$ is defectless for every $m \geq 1$. By a repeated application of Lemma 2.3, also $(K^{1/p^m}|K, v)$ is defectless. Since every finite subextension of $K^{1/p^\infty}|K$ is already contained in K^{1/p^m} for some m, it follows that $(K^{1/p^\infty}|K, v)$ is defectless.

The second equivalence is proved again by use of the Frobenius endomorphism. □

LEMMA 3.2. *The field (K, v) is inseparably defectless if and only if for every finite (possibly trivial) subextension $L|K^p$ of $K|K^p$ and every subextension $L(b)|L$ of $K|L$ of degree p, the set $v(b - L)$ has a maximal element.*

Proof. By the previous lemma, (K,v) is inseparably defectless if and only if every finite subextension $(F|K^p,v)$ of $(K|K^p,v)$ is defectless. But $F|K$ is a tower of purely inseparable extensions of degree $p=\mathrm{char}K$, so the latter holds if and only if each extension in the tower is defectless. So by a repeated application of Lemma 2.1 we see that (K,v) is inseparably defectless if and only if for every finite subextension $L|K^p$ of $K|K^p$ and every subextension $L(b)|L$ of $K|L$ of degree p, the extension $(L(b)|L,v)$ is defectless. The latter is equivalent to $v(b-L)$ having a maximal element. Indeed, if $(L(b)|L,v)$ is a defect extension, then it is immediate and by Theorem 2.19, $v(b-L)$ has no maximal element. The converse holds by part (1) of Lemma 2.21 since the extension of v from L to $L(b)$ is unique. □

Now we are able to give the proof of Theorem 1.3.

Proof of Theorem 1.3. The first assertion is an easy consequence of the last lemma. Given L and b as in that lemma, we take $b_1,\dots,b_{p^\nu}$, $\nu\ge 0$, to be a K^p-basis of L. Then $c\in L$ if and only if $c=\sum_{i=1}^{p^\nu} b_i c_i^p$ for some $c_1,\dots,c_{p^\nu}\in K$. Hence, $v(b-L)$ has a maximum if and only if (K,v) is K-extremal with respect to the polynomial (1.5).

To prove the second assertion of Theorem 1.3, we assume that vK is divisible or a $\mathbb{Z}$-group. The same is then true for vK^p and for vL for every L as in the previous lemma. Further, we note that in the previous lemma, we can restrict the scope to all $b\in\mathcal{O}$. As well, we can restrict the scope to all defectless extensions $(L|K^p,v)$. So we can choose $b_1,\dots,b_{p^\nu}$ to be a valuation basis of $(L|K^p,v)$. If vK^p is divisible, then $vL=vK^p$ and we can assume in addition that $vb_1=\cdots=vb_n=0$. If vK^p is a $\mathbb{Z}$-group with least positive element α, then we can assume in addition that for $1\le i\le p^\nu$, $vb_i=\frac{\ell_i}{p^m}\alpha$ for some $\ell_i\in\{0,\dots,p^m-1\}$, with $m\ge 0$ fixed; so $0\le vb_i<\alpha$. Now it remains to show that $v(b-L)$ has a maximal element if and only if (K,v) is $\mathcal{O}$-extremal with respect to the polynomial (1.5). We observe that $0\le v(b-0)\in v(b-L)$. Take $c\in L$ such that $v(b-c)\ge 0$. We write $c=\sum_{i=1}^{p^\nu} b_i c_i^p$ with $c_1,\dots,c_{p^\nu}\in K$. It follows that

$$0\le vc=v\sum_{i=1}^{p^\nu} b_i c_i^p=\min_i vb_i c_i^p.$$

Hence, for $1\le i\le n$, $vb_i+vc_i^p\ge 0$, and by our assumptions on the values vb_i, this implies that $vc_i^p\ge 0$ and hence $c_i\in\mathcal{O}$. This shows that the image of $\mathcal{O}^{p^\nu}$ under the polynomial (1.5) is a final segment of $v(b-L)$, hence one of the sets has a maximal element if and only if the other has. □

Corollary 3.3. *Every extremal field with value group a divisible or a $\mathbb{Z}$-group is inseparably defectless.*

Corollary 3.4. *The property "inseparably defectless" is elementary in the language of valued fields.*

Proof. The property can be axiomatized by an infinite scheme of axioms where n runs through all powers p^ν of p. Each of the axioms quantifies over all $b \in K$ and all bases of finite extensions of K^p. The latter is done by quantifying over all choices of $a_1, \ldots, a_\nu \in K$ such that the elements $a_1^{e_1} \cdot \cdots \cdot a_\nu^{e_\nu}$, $0 \le e_i < p$, are linearly independent over K^p (which can be expressed by an elementary sentence). □

Note that also the additional conditions in Theorem 1.3 concerning the elements $b_1, \ldots, b_n$ are elementary in the language of valued fields.

For a valued field of finite p-degree, one knows several properties which are equivalent to "inseparably defectless". The following theorem is due to F. Delon [4].

THEOREM 3.5. *Let K be a field of characteristic $p > 0$ and finite p-degree. Then for the valued field (K, v), the property of being inseparably defectless is equivalent to each of the following properties*:

(a) $[K : K^p] = (vK : pvK)[Kv : Kv^p]$, *i.e.*, $(K|K^p, v)$ *is a defectless extension*,
(b) $(K^{1/p}|K, v)$ *is a defectless extension*,
(c) *every immediate extension of* (K, v) *is separable*,
(d) *there is a separable maximal immediate extension of* (K, v).

Proof. The equivalence of "(K, v) inseparably defectless" with properties (a) and (b) follows readily from Lemma 3.1. By Lemma 2.5, every immediate extension of an inseparably defectless field is linearly disjoint from $K^{1/p^\infty}|K$, i.e., it is separable. This proves that "(K, v) inseparably defectless" implies property (c). Since every valued field admits a maximal immediate extension (cf. [15] and [11]), it follows that property (c) implies property (d).

It now suffices to show that property (d) implies property (a). Let (L, v) be a separable maximal immediate extension of (K, v). The separability implies that $[L : L^p] = [L^{1/p} : L] \ge [L.K^{1/p} : L] = [K^{1/p} : K] = [K : K^p]$. On the other hand, we have that $vL = vK$ and $Lv = Kv$. Since (L, v) is a maximal immediate extension, it is a maximal field. Since every maximal field is a defectless field (cf. [30], Theorem 31.21), the extension $(L^{1/p}|L, v)$ is defectless, and by Lemma 3.1 we conclude that also $(L|L^p, v)$ is defectless. Since $(vL : pvL) = (vK : pvK)$ and $[Lv : Lv^p] = [Kv : Kv^p]$ are finite (as $[K : K^p]$ is finite), it follows that $[L : L^p]$ is finite and equal to $(vL : pvL)[Lv : Lv^p]$. Consequently,

$$\begin{aligned}[L : L^p] &= (vL : pvL)[Lv : Lv^p] = (vK : pvK)[Kv : Kv^p]\\ &\le [K : K^p] \le [L : L^p].\end{aligned}$$

Thus, equality holds everywhere, showing that (a) holds. □

From the proof, we also obtain the following corollary.

COROLLARY 3.6. *Let K be a field of characteristic $p > 0$ and finite p-degree. A given maximal immediate extension of the valued field (K, v) has the same p-degree as K if and only if (K, v) is an inseparably defectless field.*

The very useful upward direction of the following lemma was also stated by F. Delon ([4], Proposition 1.44).

LEMMA 3.7. *Let $(L|K, v)$ be a finite extension of valued fields. Then (K, v) is inseparably defectless and of finite p-degree if and only if (L, v) is.*

Proof. The p-degree of a field does not change under finite extensions. Assume that one and hence both fields have finite p-degree. Since $[L : K]$ is finite, also $(vL : vK)$ and $[Lv : Kv]$ are finite. Hence, also the p-degree of Lv is equal to that of Kv. The same can be shown for ordered abelian groups: $(vL : pvL) = (vK : pvK)$ (the details are left to the reader). It follows that $[K : K^p] = (vK : pvK)[Kv : Kv^p]$ if and only if $[L : L^p] = (vL : pvL)[Lv : Lv^p]$, which by Theorem 3.5 means that (K, v) is inseparably defectless if and only if (L, v) is. □

In Lemma 4.15 in Section 4.5, we will generalize the upward direction to the case of arbitrary p-degree.

4. Artin–Schreier defect extensions

4.1. Classification of Artin–Schreier defect extensions. We will consider the following situation:

- $(L|K, v)$ an Artin–Schreier defect extension of valued fields of characteristic $p > 0$,
- $\vartheta \in L \setminus K$ an Artin–Schreier generator of $L|K$,
- $a = \wp(\vartheta) = \vartheta^p - \vartheta \in K$,
- $\delta = \mathrm{dist}(\vartheta, K)$.

Since $(L|K, v)$ is immediate and nontrivial, we know that $v(\vartheta - K) = \Lambda^{\mathrm{L}}(\vartheta, K)$ has no maximal element and that $\delta > v\vartheta$ (cf. Theorem 2.19). An element $\vartheta' \in L$ is another Artin–Schreier generator of $L|K$ if and only if

$$\vartheta' = i\vartheta + c \quad \text{with } c \in K \text{ and } 1 \leq i \leq p - 1. \tag{4.1}$$

(cf. Lemma 2.26). Consequently, using Lemma 2.25, we see that δ is an invariant of the extension $(L|K, v)$.

LEMMA 4.1. *The distance δ does not depend on the choice of the Artin–Schreier generator ϑ.*

So we can call δ the *distance of the Artin–Schreier defect extension* $(L|K, v)$. From Corollary 2.30, we know that

$$\delta \leq 0^-.$$

We will now distinguish two types of Artin–Schreier defect extensions. We will call $(L|K,v)$ a *dependent Artin–Schreier defect extension* if there exists an immediate purely inseparable extension $K(\eta)|K$ of degree p such that

$$\eta \sim_K \vartheta. \tag{4.2}$$

Otherwise, we will speak of an *independent Artin–Schreier defect extension.* For the definition and properties of the equivalence relation "$\sim_K$", see Section 2.3. We will now show that independent Artin–Schreier defect extensions are characterized by idempotent distances δ. See Lemma 2.14 for a bunch of different criteria which are all equivalent to "δ is idempotent".

PROPOSITION 4.2. *In the situation as described above, the Artin–Schreier defect extension $(L|K,v)$ is independent if and only if its distance δ is idempotent*:

$$\delta = p\delta.$$

Proof. Assume that $K(\eta)|K$ is purely inseparable of degree p, that is, $\eta^p \in K \setminus K^p$. By definition, (4.2) is equivalent to $v(\vartheta-\eta) > \delta$. Since $v(\vartheta^p - \eta^p) = v(\vartheta-\eta)^p = pv(\vartheta-\eta)$, this in turn is equivalent to

$$v(\vartheta^p - \eta^p) > p\delta.$$

Here, the left-hand side is equal to $v(\vartheta + a - \eta^p) = v(\vartheta - (\eta^p - a))$ which is a value in $\Lambda^{\mathrm{L}}(\vartheta, K)$, and hence is $\leq \delta$. Consequently, if (4.2) holds with $K(\eta)|K$ a purely inseparable extension of degree p, then $p\delta < \delta$, that is, δ is not idempotent.

For the converse, assume that δ is not idempotent. Since $\delta \leq 0^-$, this implies that $p\delta < \delta$. Then there is $c \in K$ such that $p\delta < v(\vartheta - c) \leq \delta$. Choose $\eta \in \tilde{K}$ such that $\eta^p = a + c$. Then $v(\vartheta^p - \eta^p) = v(\vartheta + a - \eta^p) = v(\vartheta - c) > p\delta$. Hence, $v(\vartheta - \eta) > \delta$, and it follows that $\eta \sim_K \vartheta$. Consequently, $\eta \notin K$, and we obtain that $K(\eta)|K$ is a purely inseparable extension of degree p. Finally, we deduce from Lemma 2.21 that this extension is immediate. □

If K admits no proper immediate purely inseparable extension, then by definition, K admits no dependent Artin–Schreier defect extension. The converse is not true: every separable-algebraically closed nontrivially valued field K of characteristic $p > 0$ which is not algebraically closed is a counterexample. Indeed, its value group is divisible and its residue field is algebraically closed (see, e.g., [17], Lemma 2.16) and hence, the proper purely inseparable extension $\tilde{K}|K$ is immediate. But a closer look shows that the irreversibility comes only from immediate purely inseparable extensions which lie in the completion K^c of K.

PROPOSITION 4.3. *Assume that K admits an immediate purely inseparable extension $K(\eta)|K$ of degree p such that $\eta \notin K^c$, and set*

$$\varepsilon := \operatorname{dist}(\eta, K).$$

Then K admits a dependent Artin–Schreier defect extension $K(\vartheta)|K$. More precisely, given any $b\in K^\times$, then

$$(p-1)vb+v\eta>p\varepsilon \tag{4.3}$$

if and only if there is an Artin–Schreier generator ϑ such that $\vartheta^p-\vartheta=(\eta/b)^p$ and

$$\begin{aligned}\vartheta &\sim_K \frac{\eta}{b},\\ v\vartheta &= v\eta-vb,\\ \mathrm{dist}(\vartheta,K) &= \mathrm{dist}(\eta,K)-vb.\end{aligned}$$

All Artin–Schreier defect extensions obtained in this way are dependent.

Proof. Let ϑ be a root of the polynomial

$$X^p-X-\left(\frac{\eta}{b}\right)^p\in K[X]. \tag{4.4}$$

Assume that (4.3) holds. Then we have

$$(p-1)vb+v\eta>p\varepsilon>pv\eta, \tag{4.5}$$

where the last inequality holds since $\varepsilon>v\eta$ by Theorem 2.19. This gives $vb>v\eta$, showing that

$$v\left(\frac{\eta}{b}\right)^p<0.$$

Hence, by Lemma 2.27,

$$v\vartheta=v\frac{\eta}{b}=v\eta-vb. \tag{4.6}$$

By definition of ϑ,

$$\eta^p+b^p\vartheta=\eta^p+b^{p-1}b\vartheta=(b\vartheta)^p. \tag{4.7}$$

Let c be an arbitrary element of K. By (4.6), (4.3) and the definition of ε,

$$vb^p\vartheta=pvb+v\eta-vb=(p-1)vb+v\eta\geq p\varepsilon>pv(\eta-c)=v(\eta^p-c^p)$$

which yields, using the ultrametric triangle inequality and (4.7),

$$\begin{aligned}v(\eta-c)&=\frac{1}{p}v(\eta^p-c^p)=\frac{1}{p}\min\{v(\eta^p-c^p),vb^p\vartheta\}\\ &=\frac{1}{p}v(\eta^p+b^p\vartheta-c^p)=\frac{1}{p}v\big((b\vartheta)^p-c^p\big)=v(b\vartheta-c).\end{aligned}$$

By Lemma 2.17, this implies that $b\vartheta\sim_K\eta$, which by Lemma 2.25 implies that

$$\vartheta\sim_K\frac{\eta}{b}.$$

From this, the assertion on the distance of ϑ follows by virtue of Lemma 2.25, while the value $v\vartheta$ has already been determined in (4.6). By Lemma 2.31, the extension of v from K to $K(\vartheta)$ is unique and $(K(\vartheta)|K,v)$ is an Artin–Schreier defect extension. By definition, it is dependent.

For the converse, assume that (4.3) does not hold, i.e., $(p-1)vb+v\eta\leq p\varepsilon$. If $v(\frac{\eta}{b})^p>0$, then by Lemma 2.27, $v\vartheta=0<v\frac{\eta}{b}$ or $v\vartheta=v(\frac{\eta}{b})^p=pv\frac{\eta}{b}>v\frac{\eta}{b}$ and we cannot have $\vartheta\sim_K\frac{\eta}{b}$. If $v(\frac{\eta}{b})^p\leq 0$, then again by Lemma 2.27, (4.6) holds, and so we have

$$vb^p\vartheta=pvb+v\eta-vb=(p-1)vb+v\eta\leq p\varepsilon.$$

Therefore, and since $\Lambda^{\mathrm{L}}(\eta,K)$ has no last element, there is some $c\in K$ such that $vb^p\vartheta<pv(\eta-c)=v(\eta^p-c^p)$. But then, by the ultrametric triangle inequality,

$$v(\eta-c)>\frac{1}{p}vb^p\vartheta=\frac{1}{p}v(\eta^p+b^p\vartheta-c^p)=v(b\vartheta-c),$$

which again shows that $\vartheta\sim_K\frac{\eta}{b}$ cannot be true.

Since $\eta\notin K^c$ by assumption, there is some b satisfying (4.3). Hence, by what we have shown, K admits a dependent Artin–Schreier defect extension $K(\vartheta)|K$. $\square$

The following proposition shows an even stronger independence property than what is expressed in the definition.

Proposition 4.4. *Let $(L|K,v)$ be an independent Artin–Schreier defect extension, and take any element $\zeta\in L\setminus K$. Then there exists no purely inseparable extension $K(\eta)|K$ such that $\zeta\sim_K\eta$. In particular, it follows that*

$$\operatorname{dist}(\zeta,K)=\operatorname{dist}(\zeta,K^{1/p^\infty}). \tag{4.8}$$

Proof. Since $\zeta\in L\setminus K$, $[K(\zeta):K]=p=[K(\vartheta):K]$ and therefore, there is a polynomial $f\in K[X]$ of degree smaller than p such that $\vartheta=f(\zeta)$. Suppose that there exists a purely inseparable extension $K(\eta)|K$ such that $\zeta\sim_K\eta$. But then by Lemma 2.24, $\vartheta=f(\zeta)\sim_K f(\eta)$. Since also $K(f(\eta))|K$ is a purely inseparable extension, this is impossible since $(L|K,v)$ is assumed to be independent.

Equation (4.8) is deduced as follows. If it does not hold, then $\operatorname{dist}(\zeta,K)<\operatorname{dist}(\zeta,K^{1/p^\infty})$ in view of $K\subset K^{1/p^\infty}$. But then by virtue of Lemma 2.18, there would exist some $\eta\in K^{1/p^\infty}$ such that $\zeta\sim_K\eta$, which we have just shown not to be the case. $\square$

4.2. Deformation of Artin–Schreier defect extensions. For the proof of Proposition 4.3, we have transformed an immediate purely inseparable extension into an immediate separable extension. This was done by changing the minimal polynomial $Y^p-\eta^p$ to the minimal polynomial

$$Y^p-b^{p-1}Y-\eta^p\in K[Y] \tag{4.9}$$

through addition of the summand $b^{p-1}Y$. This polynomial has $b\vartheta$ as its root. The condition (4.3) on the value of b means that it is large enough to guarantee that $b\vartheta \sim_K \eta$. For this condition to be satisfied, it is necessary that η is not contained in the completion of K. On the other hand, an immediate purely inseparable extension with a generator η in the completion of K cannot be transformed into any immediate separable extension with a generator ϑ such that $\vartheta \sim_K \eta$. Indeed, if $\eta \in K^c$ and $\eta \sim_K \eta'$, then $v(\eta - \eta') > \widetilde{vK}$, that is, $\eta = \eta'$. Moreover, every henselian field K is separable-algebraically closed in its completion (cf. [30], Theorem 32.19).

The general idea of the transformation of the minimal polynomial can be expressed as follows: if $y \notin K^c$ is a root of the polynomial $f \in K[X]$, then for a given polynomial $g \in K[X]$, a root z of g will satisfy $y \sim_K z$ as soon as the coefficients of the polynomial $f - g$ have large enough values. This follows in general from the principle of Continuity of Roots. But we wanted to give a self-contained proof for our special case, because it is particularly simple and explicit and leads to the following deformation theory.

For any fixed $a \in K$, we consider the following family of polynomials defined over K:

$$f_{a,b}(Y) := Y^p - b^{p-1}Y - a, \quad b \in K^\times. \tag{4.10}$$

This family can be viewed as a deformation of the polynomial $Y^p - a$, with this polynomial as its limit for $vb \to \infty$:

$$\begin{aligned} Y^p - b^{p-1}Y - a &\longrightarrow Y^p - a, \\ vb &\longrightarrow \infty. \end{aligned}$$

But it is not necessarily true that the ramification theoretical properties are preserved in the limit, as Example 4.16 in Section 4.6 will show.

Associated with this family through the transformation $Y = bX$ is the family

$$g_{a,b}(X) := X^p - X - \frac{a}{b^p}, \quad b \in K^\times, \tag{4.11}$$

where $\vartheta_{a,b}$ is a root of $g_{a,b}$ if and only if $b\vartheta_{a,b}$ is a root of $f_{a,b}$.

We summarize the properties of these families in the following theorem.

THEOREM 4.5. (a) *If $pvb \le va$, then the polynomial $g_{a,b}(X)$ induces a trivial extension or an Artin–Schreier extension for which equality holds in the fundamental inequality* (1.2); *if $pvb < va$, then this extension lies in the henselization of K.*

(b) *Suppose that the polynomial $Y^p - a$ induces an immediate extension which does not lie in the completion of K. Then for each $b \in K^\times$ of large enough value, the polynomial $g_{a,b}(X)$ induces a dependent Artin–Schreier defect extension*; *every root $b\vartheta_{a,b}$ of $f_{a,b}(X)$ will then satisfy*

$$b\vartheta_{a,b} \sim_K a^{1/p}.$$

"Large enough value" means that

$$(p-1)vb + \frac{va}{p} > p\,\mathrm{dist}(a^{1/p}, K). \tag{4.12}$$

If this condition is violated, then $b\vartheta_{a,b} \sim_K a^{1/p}$ *does not hold.*

(c) *Suppose that a root* $\vartheta_{a,1}$ *of the polynomial* $f_{a,1}(X) = X^p - X - a$ *satisfies*

$$v\vartheta_{a,1} > p\,\mathrm{dist}(\vartheta_{a,1}, K). \tag{4.13}$$

Then the polynomial $X^p - a$ *induces an immediate extension which does not lie in the completion, and for every* b *in the valuation ring* $\mathcal{O}$ *of* K *and every root* $\vartheta_{a,b}$ *of* $g_{a,b}$, $K(\vartheta_{a,b})|K$ *is a dependent Artin–Schreier defect extension with* $b\vartheta_{a,b} \sim_K a^{1/p}$. *If condition* (4.13) *is violated, then* $\vartheta_{a,1} \sim_K a^{1/p}$ *does not hold.*

Proof. (a) Both assertions follow from Lemma 2.28.

(b) All assertions follow from Proposition 4.3 where $\eta = a^{1/p}$.

(c) Assume that condition (4.13) holds. Then it follows from the second part of the proof of Proposition 4.2, where we set $c = 0$ and $\vartheta = \vartheta_{a,1}$, that the polynomial $X^p - a$ induces an immediate extension which does not lie in the completion, and that $\vartheta_{a,1} \sim_K a^{1/p}$. The latter implies that $v\vartheta_{a,1} = va^{1/p} = \frac{va}{p}$ and that $\mathrm{dist}(\vartheta_{a,1}, K) = \mathrm{dist}(a^{1/p}, K)$; hence, it implies that (4.13) is equivalent to

$$\frac{va}{p} > p\,\mathrm{dist}(a^{1/p}, K). \tag{4.14}$$

Consequently, (4.12) will hold for every $b \in \mathcal{O}$, so it follows from part (b) that for every root $\vartheta_{a,b}$ of $g_{a,b}$, $K(\vartheta_{a,b})|K$ is a dependent Artin–Schreier defect extension with $b\vartheta_{a,b} \sim_K a^{1/p}$.

The last assertion of part (c) is seen as follows. We have shown that if $\vartheta_{a,1} \sim_K a^{1/p}$ holds, then (4.13) and (4.14) are equivalent. But if (4.14) is violated, then by part (b), $\vartheta_{a,1} \sim_K a^{1/p}$ cannot hold. □

Note that

$$p\,\mathrm{dist}(a^{1/p}, K) = \mathrm{dist}(a, K^p). \tag{4.15}$$

A deformation which at first sight seems to be different from the above has been used by B. Teissier in [29]. Starting from the Artin–Schreier polynomial $X^p - X - a$, we set $X = aY$ and then divide the polynomial by a^p, which leads to the polynomial

$$Y^p - a^{1-p}Y - a^{1-p} = Y^p - a^{1-p}(1+Y).$$

Hence, $\vartheta^p - \vartheta = a$ if and only if for $\tilde{\vartheta} = \vartheta/a$,

$$\tilde{\vartheta}^p - a^{1-p}(1+\tilde{\vartheta}) = 0. \tag{4.16}$$

We assume that $va < 0$. Then by Lemma 2.27, $\vartheta^p - \vartheta = a$ implies that $va = pv\vartheta < v\vartheta$ and therefore,

$$v\tilde{\vartheta} = v\vartheta - va > 0.$$

That is, $1 + \tilde{\vartheta}$ is a 1-unit in $\mathcal{O}$. Reducing this 1-unit to 1 deforms equation (4.16) to

$$\overline{\vartheta}^p - a^{1-p} = 0,$$

viewed as an equation in an associated graded ring. In fact, we have reduced equation (4.16) modulo the $\mathcal{O}$-ideal

$$a^{1-p}\tilde{\vartheta}\mathcal{O} = a^{-p}\vartheta\mathcal{O} = \vartheta^{1-p^2}\mathcal{O}.$$

Analyzing the above transformation, one sees that its advantage is that it leads to equations with integral coefficients. However, if we multiply the polynomial $Y^p - a^{1-p}$ by a^p and then set $X = aY$, we obtain the polynomial $X^p - a$. So we have just replaced the polynomial $X^p - X - a$ by $X^p - a$. From Theorem 4.5 together with (4.15), we see that this procedure preserves the valuation theoretical behaviour of the associated roots if and only if

$$va > p\,\mathrm{dist}(a, K^p).$$

4.3. Fields without dependent Artin–Schreier defect extensions. If K admits any immediate purely inseparable extension that does not lie in the completion K^c of K, then K satisfies the hypothesis of Proposition 4.3. To show this, suppose that $\tilde{\eta} \in K^{1/p^\infty} \setminus K^c$ such that $K(\tilde{\eta})|K$ is an immediate extension. We may assume that $\tilde{\eta}^p \in K^c$ (otherwise, we replace $\tilde{\eta}$ by $\tilde{\eta}^{p^\nu}$ for a suitable $\nu \geq 1$). Since $\tilde{\eta} \notin K^c$, we have that $\Lambda^{\mathrm{L}}(\tilde{\eta}, K)$ is bounded from above in vK and $\Lambda^{\mathrm{L}}(\tilde{\eta}^p, K^p) = p\Lambda^{\mathrm{L}}(\tilde{\eta}, K)$ is bounded from above in $vK^p = pvK$. On the other hand, since $\tilde{\eta}^p \in K^c$, there is some $b \in K$ such that $v(\tilde{\eta}^p - b) > \Lambda^{\mathrm{L}}(\tilde{\eta}^p, K^p)$. We choose $\eta \in K^{1/p}$ such that $\eta^p = b$. Then $v(\tilde{\eta} - \eta) = \frac{1}{p}v(\tilde{\eta}^p - b) > \Lambda^{\mathrm{L}}(\tilde{\eta}, K)$, that is,

$$\eta \sim_K \tilde{\eta}.$$

By Lemma 2.21, this shows that $K(\eta)|K$ is an immediate extension; since $\Lambda^{\mathrm{L}}(\eta, K) = \Lambda^{\mathrm{L}}(\tilde{\eta}, K) \neq vK$, it is not contained in K^c. We may now apply Proposition 4.3 to obtain the following corollary.

Corollary 4.6. *Assume that K does not admit any dependent Artin–Schreier defect extension. Then every immediate purely inseparable extension lies in the completion of K.*

Lemma 4.7. *If K is Artin–Schreier closed, then so is K^c. If K admits no dependent (or no independent) Artin–Schreier defect extension, then the same holds for K^c.*

Proof. Assume that $K^c(\vartheta)|K^c$ is an Artin–Schreier extension generated by a root ϑ of the polynomial $X^p - X - a$ over K^c. Since $\vartheta \notin K^c$, we have that $\mathrm{dist}(\vartheta, K^c) < \infty$. Since $a \in K^c$, we may choose an element $\tilde{a} \in K$ such that $v(a-\tilde{a}) > \mathrm{dist}(\vartheta, K^c)$ with $v(a-\tilde{a}) \geq 0$. Let $\tilde{\vartheta}$ be a root of the polynomial $X^p - X - \tilde{a} \in K[X]$. Using Lemma 2.27 and observing that adding $i \in \{1, \ldots, p-1\}$ produces the conjugates of both $\tilde{\vartheta}$ and $\vartheta + \tilde{\vartheta}$, we can choose $\tilde{\vartheta}$ such that the root $\vartheta - \tilde{\vartheta}$ of the polynomial $X^p - X - (a - \tilde{a})$ has value $v(\vartheta - \tilde{\vartheta}) = v(a - \tilde{a}) > \mathrm{dist}(\vartheta, K^c) \geq \mathrm{dist}(\vartheta, K)$. Thus, $\mathrm{dist}(\tilde{\vartheta}, K) = \mathrm{dist}(\vartheta, K) \leq \mathrm{dist}(\vartheta, K^c) < \infty$, which shows that $K(\tilde{\vartheta})|K$ is nontrivial and hence an Artin–Schreier extension. This proves the first assertion of our lemma.

Now assume that $(K^c(\vartheta)|K, v)$ is an Artin–Schreier defect extension. By Corollary 2.30, we have that $\mathrm{dist}(\vartheta, K^c) \leq 0^-$. With $\tilde{\vartheta}$ as before, we obtain that $\mathrm{dist}(\tilde{\vartheta}, K) = \mathrm{dist}(\vartheta, K) \leq 0^-$. By Lemma 2.31, this shows that also $(K(\tilde{\vartheta})|K, v)$ is an Artin–Schreier defect extension. The equality of the distances shows that $K^c(\vartheta)|K^c$ is independent if and only if $K(\tilde{\vartheta})|K$ is. □

An immediate consequence of this lemma and the preceding corollary is the following corollary.

Corollary 4.8. *If K does not admit any dependent Artin–Schreier defect extension, then K^c does not admit any proper immediate purely inseparable extension. In particular, this holds if K is separable-algebraically maximal.*

We can now give the following proof.

Proof of Theorem 1.11. Every Artin–Schreier closed nontrivially valued field K of characteristic $p > 0$ has p-divisible value group and perfect residue field (cf. Corollary 2.17 of [17]). Therefore, every purely inseparable extension of K is immediate. Hence, by the last corollary, the perfect hull of K lies in the completion of K, i.e., K lies dense in its perfect hull.

An alternative proof of this fact can be given in the following way. We represent the extension $K^{1/p^\infty}|K$ as an infinite tower of purely inseparable extensions $K_{\mu+1}|K_\mu$ ($\mu < \nu$ where ν is some ordinal). Then we only have to show that $(K_{\mu+1}, v)$ lies in $(K_\mu, v)^c$ for every $\mu < \nu$. In view of Proposition 4.3, it suffices to show that K_μ is Artin–Schreier closed. But this holds by Lemma 2.32.

Since K^c has the same value group and the same residue field as K, also every purely inseparable extension of K^c is immediate. By the preceding corollary, this yields that K^c must be perfect. □

4.4. Persistence results. Another property of independent Artin–Schreier defect extensions is their persistence in maximal immediate extensions, in the following sense.

LEMMA 4.9. *If K admits an independent Artin–Schreier defect extension $(K(\vartheta)|K,v)$ with Artin–Schreier generator ϑ of distance $\delta = 0^-$, then every algebraically maximal immediate extension (and in particular, every maximal immediate extension) M of K contains also an independent Artin–Schreier defect extension of K with an Artin–Schreier generator $\tilde{\vartheta}$ of distance 0^- such that $\tilde{\vartheta} \sim_K \vartheta$.*

Proof. If $\vartheta \in M$, there is nothing to show. Assume that $\vartheta \notin M$. Then $M(\vartheta)|M$ is also an Artin–Schreier extension with Artin–Schreier generator ϑ. Since M is algebraically maximal, Corollary 2.20 shows that there exists an element $u \in M$ satisfying

$$v(\vartheta - u) \geq \Lambda^{\mathrm{L}}(\vartheta, M).$$

On the other hand, $K \subseteq M$ implies

$$\Lambda^{\mathrm{L}}(\vartheta, K) \subseteq \Lambda^{\mathrm{L}}(\vartheta, M).$$

Since $vM = vK$, this shows that $v(\vartheta - u) \geq 0$. We put

$$a_u := \wp(\vartheta - u) = \wp(\vartheta) - \wp(u) \in M$$

and note that $va_u \geq 0$. Since $M|K$ is immediate, there exists $b \in K$ such that

$$v(a_u - b) > v(a_u) \geq 0$$

and $vb = va_u \geq 0$. Consequently, the polynomial $X^p - X - (a_u - b) \in M[X]$ admits a root ϑ' in the henselian field M. But then,

$$\tilde{\vartheta} := \vartheta' + u \in M$$

is a root of the polynomial $X^p - X - (\wp(\vartheta) - b) \in K[X]$. We compute:

$$\wp(\vartheta - \tilde{\vartheta}) = \wp(\vartheta) - \wp(\vartheta' + u) = \wp(\vartheta) - \big(\wp(\vartheta) - b\big) = b.$$

This shows $v(\vartheta - \tilde{\vartheta}) \geq 0$, whence $\tilde{\vartheta} \sim_K \vartheta$. In particular, this shows that $\tilde{\vartheta} \notin K$ so that $K(\tilde{\vartheta})|K$ is nontrivial and hence an Artin–Schreier extension. By Lemma 2.31, the extension of v from K to $K(\tilde{\vartheta})$ is unique and $K(\tilde{\vartheta})|K$ is an Artin–Schreier defect extension. Finally, $\tilde{\vartheta} \sim_K \vartheta$ implies that $\mathrm{dist}(\tilde{\vartheta}, K) = \mathrm{dist}(\vartheta, K) = 0^-$ (Lemma 2.17) and therefore, $K(\tilde{\vartheta})|K$ is an independent Artin–Schreier defect extension. □

From this lemma, we deduce the following corollary.

COROLLARY 4.10. *If there exists a maximal immediate extension in which K is separable-algebraically closed, then K admits no independent Artin–Schreier defect extension of distance 0^-.*

We will now consider independent Artin–Schreier defect extensions $(K(\vartheta)|K,v)$ with Artin–Schreier generator ϑ of distance $\delta < 0^-$. In this case, Proposition 4.2 and Lemma 2.14 show that $\delta = H^-$ for some nontrivial convex subgroup H of $\widetilde{vK}$. This means that $v(\vartheta - K) = \Lambda^{\mathrm{L}}(\vartheta, K)$ is cofinal in $(\widetilde{vK})^{<0} \setminus H$.

We denote by v_δ the coarsening of v on $\tilde{K}$ with respect to H. Then $v_\delta(\vartheta - K)$ is cofinal in $(\widetilde{vK})^{<0}/H = (\widetilde{v_\delta K})^{<0}$. Thus, $v_\delta(\vartheta - K)$ has no maximal element. Since the extension of v from K to $K(\vartheta)$ is unique, the same must hold for v_δ; cf. the proof of Lemma 2.4. Now Lemma 2.21 shows that also $(K(\vartheta)|K, v_\delta)$ is an immediate Artin–Schreier extension. As its distance is 0^-, it is covered by the case treated in Lemma 4.9. From this, we obtain the following lemma.

LEMMA 4.11. *Assume that for every coarsening w of v (including v itself), there exists a maximal immediate extension (M_w, w) of (K, w) such that K is separable-algebraically closed in M_w. Then K admits no independent Artin–Schreier defect extensions.*

The condition of Lemma 4.11 is preserved under finite defectless extensions.

LEMMA 4.12. *Assume that for every coarsening w of v (including v itself), K_0 admits a maximal immediate extension $(N_w|K_0, w)$ such that K_0 is relatively algebraically closed (or separable-algebraically closed) in N_w. If the extension $(K|K_0, v)$ is finite and defectless, then for every coarsening w of v (including v itself), $(M_w, w) = (N_w.K, w)$ is a maximal immediate extension of (K, w) such that K is relatively algebraically closed (or separable-algebraically closed, respectively) in M_w.*

Proof. Since $(K|K_0, v)$ is defectless by hypothesis, the same is true for the extension $(K|K_0, w)$ by Lemma 2.4. We note that (K_0, w) is henselian since it is assumed to be separable-algebraically closed in the henselian field (N_w, w). So we may apply Lemma 2.5: since $(N_w|K_0, w)$ is immediate and $(K|K_0, w)$ is defectless, $(N_w.K|K, w)$ is immediate and N_w is linearly disjoint from K over K_0. The latter shows that K is relatively algebraically closed (or separable-algebraically closed, respectively) in $N_w.K$. On the other hand, $(M_w, w) = (N_w.K, w)$ is a maximal field, being a finite extension of a maximal field. □

PROPOSITION 4.13. *If K_0 is a separable-algebraically maximal field and $K|K_0$ is a finite defectless extension, then K admits no independent Artin–Schreier defect extensions.*

Proof. Let w be any coarsening of v. Since (K_0, v) is separable-algebraically maximal, the same is true for (K_0, w) since every finite separable immediate extension of (K_0, w) would also be immediate for the finer valuation v. Now let (N_w, w) be a maximal immediate extension of (K_0, w). Since (K_0, w) is separable-algebraically maximal, it is separable-algebraically closed in N_w. Hence, K_0 satisfies the condition of Lemma 4.12. So our proposition is a consequence of Lemma 4.12 together with Lemma 4.11. □

4.5. Generalization of Lemma 3.7 and proof of Theorem 1.2. For the generalization of Lemma 3.7, we will need the following result.

Lemma 4.14. *Let $K \subset K_1 \subset K_2$ be extensions of valued fields of characteristic $p > 0$ such that $K_1|K$ is finite and purely inseparable and $K_2|K_1$ is an independent Artin–Schreier defect extension. Then there exists an Artin–Schreier extension $L|K$ such that $K_2 = K_1.L$, and every such extension $L|K$ is an independent Artin–Schreier defect extension.*

Proof. Let $\tilde{\vartheta}$ be an Artin–Schreier generator of $K_2|K_1$ and choose $\nu \geq 1$ such that

$$K_1^{p^\nu} \subseteq K.$$

Then

$$\wp(\tilde{\vartheta}^{p^\nu}) = (\wp(\tilde{\vartheta}))^{p^\nu} \in K,$$

hence,

$$K(\tilde{\vartheta}^{p^\nu})|K$$

is an Artin–Schreier extension: it is nontrivial since $K(\tilde{\vartheta})|K$ is not purely inseparable. Comparing degrees, we see that $K_2 = K_1(\tilde{\vartheta}^{p^\nu}) = K_1.K(\tilde{\vartheta}^{p^\nu})$.

Now let $L|K$ be any such Artin–Schreier extension. Let ϑ be an Artin–Schreier generator of $L|K$ and hence of $K_2|K_1$ too. Using $\vartheta^p = \vartheta + a$ with $a \in K$, we compute

$$\vartheta^{p^\nu} = \vartheta + a' \quad \text{where } a' = a + \cdots + a^{p^{\nu-1}} \in K. \tag{4.17}$$

Hence,

$$\operatorname{dist}(\vartheta^{p^\nu}, K_1) = \operatorname{dist}(\vartheta, K_1).$$

Further,

$$\delta := \operatorname{dist}(\vartheta, K_1) = p^\nu \delta = \operatorname{dist}(\vartheta^{p^\nu}, K_1^{p^\nu})$$

since δ is idempotent by hypothesis;

$$\operatorname{dist}(\vartheta^{p^\nu}, K_1^{p^\nu}) \leq \operatorname{dist}(\vartheta^{p^\nu}, K) \leq \operatorname{dist}(\vartheta^{p^\nu}, K_1)$$

because $K_1^{p^\nu} \subseteq K \subset K_1$. Putting these three equations together, we find that equality holds everywhere. In particular,

$$\operatorname{dist}(\vartheta, K_1) = \operatorname{dist}(\vartheta^{p^\nu}, K) = \operatorname{dist}(\vartheta, K),$$

where the second equality again holds because of (4.17). This shows that $\Lambda^{\mathrm{L}}(\vartheta, K)$ is cofinal in $\Lambda^{\mathrm{L}}(\vartheta, K_1)$. Since $K_1(\vartheta)|K_1$ is immediate, we know from Theorem 2.19 that $\Lambda^{\mathrm{L}}(\vartheta, K_1) = v(\vartheta - K_1)$ has no maximal element. Now we have that $\Lambda^{\mathrm{L}}(\vartheta, K) \subseteq v(\vartheta - K) \subseteq v(\vartheta - K_1)$ and that $\Lambda^{\mathrm{L}}(\vartheta, K)$ is cofinal in $v(\vartheta - K_1)$; this yields that $v(\vartheta - K)$ is cofinal in $v(\vartheta - K_1)$ and thus has no maximal element. Now Lemma 2.21 shows that $K(\vartheta)|K$ is immediate. Since $\operatorname{dist}(\vartheta, K) = \operatorname{dist}(\vartheta, K_1)$ is idempotent, $K(\vartheta)|K$ is independent. □

Lemma 4.15. *Every finite extension of an inseparably defectless field of characteristic $p > 0$ is again an inseparably defectless field.*

Proof. By Corollary 2.2, every finite purely inseparable extension of an inseparably defectless field is again an inseparably defectless field. Thus, it remains to show the lemma in the case of a finite separable extension L of an inseparably defectless field K. We fix an extension of v to K^{sep} and consider the ramification fields K^r and L^r of K and L with respect to that extension. By Proposition 2.8, we know that K is inseparably defectless if and only if K^r is inseparably defectless, and the same holds for L and L^r. By Lemma 2.7, we have $L^r = L.K^r$, and therefore $L^r|K^r$ is a finite separable extension. The same proposition shows that $K^{\mathrm{sep}}|K^r$ is a p-extension, so $L^r|K^r$ is a tower of Artin–Schreier extensions (cf. Lemma 2.9). Hence, replacing K and L by their ramification fields, we may assume from the start that they are henselian and that $L|K$ is a tower of Artin–Schreier extensions. Now it suffices to prove that L is inseparably defectless under the additional assumption that $L|K$ itself is an Artin–Schreier extension since then, our assertion will follow by induction. Since $L^{1/p^\infty} = L.K^{1/p^\infty}$, it suffices to show for every finite purely inseparable extension $K_1|K$ (which itself is defectless by hypothesis), that $K_2 = K_1.L$ is a defectless extension of L. This follows immediately if $K_2|K_1$ and thus $K_2|K$ are defectless. Now assume that $K_2|K_1$ is immediate. Note that K_1 is an inseparably defectless field, being a finite purely inseparable extension of the inseparably defectless field K. In particular, this yields that K_1 admits no immediate purely inseparable extension and hence by virtue of Proposition 4.2, no dependent Artin–Schreier defect extension. The immediate Artin–Schreier extension $K_2|K_1$ is thus independent. An application of Lemma 4.14 now shows that $L|K$ is immediate. But then, $K_2|L$ is defectless by Corollary 2.5. Hence, we have proved that L is an inseparably defectless field. □

In both of the preceding lemmas, the finiteness conditions cannot be dropped, as Examples 4.16 and 4.19 in the next section will show.

We are now able to give the following proof.

Proof of Theorem 1.2. Assume that the valued field K of characteristic $p > 0$ is separable-algebraically maximal and inseparably defectless. We note that K is henselian since it is separable-algebraically maximal. Let $(L|K, v)$ be a finite extension. We want to show that it is defectless. Since any subextension of a defectless extension is defectless too, we may assume w.l.o.g. that $L|K$ is normal. Hence, there exists an intermediate field K_1 such that $L|K_1$ is separable and $K_1|K$ is purely inseparable. By hypothesis, we know that $K_1|K$ is defectless. It remains to prove that $L|K_1$ is defectless.

Using Lemma 2.9, choose a finite tame extension N of K_1 such that $L.N|N$ is a tower of Artin–Schreier extensions. By Proposition 2.8, $L|K_1$ is defectless if and only if $L.N|N$ is defectless. Since $K_1|K$ is defectless and $N|K_1$ is tame and hence defectless, both extensions being finite, $N|K$ is finite and defectless. Using Lemma 4.15, we conclude that N is inseparably defectless too and therefore does not admit immediate purely inseparable extensions. This shows that every immediate Artin–Schreier extension of the henselian field N must be independent. Moreover, from Proposition 4.13 we infer that N does not admit independent Artin–Schreier defect extensions. Consequently, given an Artin–Schreier extension $L'|N$ contained in $L.N|N$, this extension must be defectless. In view of Lemma 4.15 and Proposition 4.13, L' will again be inseparably defectless and will not admit any independent Artin–Schreier defect extension. By induction, we conclude that all Artin–Schreier extensions in the tower $L.N|N$ are defectless, hence $L.N|N$ and thus $L|K_1$ and $L|K$ are defectless, as asserted.

Conversely, every defectless field is immediately seen to be separable-algebraically maximal and inseparably defectless. □

4.6. Examples.

EXAMPLE 4.16 (For an independent Artin–Schreier defect extension with distance 0^-). Let k be an algebraically closed field of characteristic $p>0$, and $K=k(t)^{1/p^\infty}$ the perfect hull of the rational function field $k(t)$. Further, let $v=v_t$ be the unique extension of the t-adic valuation from $k(t)$ to K; we write $vt=1$. Note that vK is p-divisible and $Kv=k$ is algebraically closed.

We consider the Artin–Schreier extension $L_0=k(t,\vartheta)$ of $k(t)$ generated by a root ϑ of the polynomial

$$X^p-X-\frac{1}{t}.$$

As $v\vartheta=-1/p\notin\mathbb{Z}=vk(t)$, we see that $[L_0:k(t)]=p=(vL_0:vk(t))$. Thus, the extension of v from $k(t)$ to L_0 is unique. Further, the extension of v from L_0 to its perfect hull is unique. But the latter is equal to $L_0.K$, so we find that the extension of v from K to $L:=L_0.K$ is unique. On the other hand, the extension $L|K$ is immediate since vK is p-divisible and $Kv=k$ is algebraically closed. Therefore, $L|K$ is an Artin–Schreier defect extension. Since K is perfect, it is independent by definition.

For

$$a_n:=\sum_{i=1}^{n}\frac{1}{t^{p^{-i}}}$$

we have

$$a_n^p-a_n=\frac{1}{t}-\frac{1}{t^{p^{-n}}},$$

whence

$$(\vartheta-a_n)^p-(\vartheta-a_n)=\vartheta^p-\vartheta-(a_n^p-a_n)=\frac{1}{t}-\left(\frac{1}{t}-\frac{1}{t^{p^{-n}}}\right)=\frac{1}{t^{p^{-n}}}.$$

By Lemma 2.27, this yields

$$v(\vartheta-a_n)=\frac{1}{p}v\frac{1}{t^{p^{-n}}}=-\frac{1}{p^{n+1}}.$$

Since this increases with n, we see that $(a_n)_{n\in\mathbb{N}}$ is a pseudo Cauchy sequence with limit ϑ. By Corollary 2.30, $\mathrm{dist}(\vartheta,K)\le 0^-$. On the other hand, the values $v(\vartheta-a_n)$ are cofinal in $\widetilde{vK}^{<0}$. Therefore,

$$\mathrm{dist}(\vartheta,K)=0^-.$$

This example shows that the condition in Lemma 4.14 that $K_1|K$ be finite cannot be dropped. Indeed, it is known that $(k(t),v_t)$ is a defectless field (for instance, this is a consequence of the Generalized Stability Theorem, cf. [19]). So it does not admit any Artin–Schreier defect extension. But the infinite extension K of $k(t)$ admits an independent Artin–Schreier defect extension.

The example also shows that ramification theoretical properties of a polynomial are not necessarily preserved in the limit. As above, one shows that for every $n\in\mathbb{N}$, a root of the polynomial

$$X^p-X-\frac{1}{t^{np+1}}$$

generates a nontrivial immediate extension of K. The same is true for a root of the polynomial

$$Y^p-t^{n(p-1)}Y-\frac{1}{t}.$$

Under $n\to\infty$ (which implies $vt^{n(p-1)}\to\infty$), the limit of this polynomial is

$$Y^p-\frac{1}{t}.$$

But this polynomial does not induce a nontrivial extension of K since K is perfect.

This example works even for nonalgebraically closed fields k. In [17] we presented it with $k=\mathbb{F}_p$. See also [20].

Example 4.17 (For an independent Artin–Schreier defect extension with distance smaller than 0^-). In the previous example, we may choose k such that it admits a nontrivial valuation $\overline{v}$. Now we consider the valuation $v':=v\circ\overline{v}$ on L. As $(L|K,v)$ is immediate and $Lv=k=Kv$, it follows that also $(L|K,v')$ is immediate. The value group $\overline{v}k$ is canonically isomorphic to a nontrivial convex subgroup H of $v'L$ (such that $v'L/H\simeq vL$). If there would exist some $c\in K$ and an element $\beta\in H$ such that $v'(\vartheta-c)\ge\beta$, then $v(\vartheta-c)\ge 0$ which is impossible. On the other hand, the values $v'(\vartheta-a_n)$ are cofinal in $\{\alpha\in\widetilde{v'K}\mid\alpha<H\}$ since the values $v(\vartheta-a_n)$ are cofinal in $vK^{<0}$. This shows that the distance $\mathrm{dist}(\vartheta,K)$ with respect to v' is the cut

$$H^-=(\{\alpha\in\widetilde{v'K}\mid\alpha<H\},\{\alpha\in\widetilde{v'K}\mid\exists\beta\in H:\beta\le\alpha\})$$

which is smaller than 0^- since H is nontrivial.

Example 4.18 (For a dependent Artin–Schreier defect extension). With $k(t)$ as before, we take K_0 to be the separable-algebraic closure of $k(t)$, with any extension v_t of the t-adic valuation of $k(t)$. Being separable-algebraically closed, K_0 does in particular not admit any Artin–Schreier extension. But we can build a field admitting a dependent Artin–Schreier defect extension by taking $K = K_0(x)$ and endowing it with the (unique) extension v of v_t such that $vx > vK_0$. (This means that K has the x-adic valuation v_x with residue field K_0, and $v = v_x \circ v_t$ is the composition of v_x with v_t.) We take any $\eta \in K_0^{1/p} \setminus K_0$. Then v has a unique extension from K to $K(\eta)$. Since η lies in the completion of (K_0, v) by Theorem 1.11, we have $\Lambda^{\mathrm{L}}(\eta, K_0) = v_t K_0 = vK_0$. It follows that $\Lambda^{\mathrm{L}}(\eta, K)$ is the least initial segment of vK containing vK_0. That is, the cut $\operatorname{dist}(\eta, K)$ is the cut $(vK_0)^+$ induced in $\widetilde{vK}$ by the upper edge of the convex subgroup vK_0 of vK. In particular, η does not lie in the completion of (K, v). Now Proposition 4.3 shows that K admits a dependent Artin–Schreier defect extension. According to this proposition, it can for instance be generated by a root ϑ of the polynomial $X^p - X - (\eta/x)^p$, as $vx > \operatorname{dist}(\eta, K) = p \operatorname{dist}(\eta, K)$. Then $\operatorname{dist}(\vartheta, K) = \operatorname{dist}(\eta, K) - vx = (vK_0)^+ - vx = (-vx + vK_0)^+$ is the cut induced by the upper edge of the coset $-vx + vK_0$ in $\widetilde{vK}$. Note that in vK, which is the lexicographic product $\mathbb{Z}vx \times vK_0$, the cut $(-vx + vK_0)^+$ is equal to the cut vK_0^- induced by the lower edge of the convex subgroup vK_0 of vK. Nevertheless, the cut $\operatorname{dist}(\vartheta, K)$ in $\widetilde{vK}$ is *not* equal to H^- or H^+ for any convex subgroup H of vK or of $\widetilde{vK}$ (cf. Example 2.15 in Section 2.3).

Enlarging the rank of the valuation in order to obtain a dependent Artin–Schreier defect extension may appear to be a dirty trick. Therefore, we add a further example which shows that such extensions can also appear for valuations of rank one.

Example 4.19 (For a dependent Artin–Schreier defect extension in rank 1). With $(k(t), v)$ as before, we take a_1 to be a root of the Artin–Schreier polynomial $X^p - X - 1/t$. Then $va_1 = -1/p < 0$. By induction on i, we take a_{i+1} to be a root of the Artin–Schreier polynomial $X^p - X + a_i$, for all $i \in \mathbb{N}$. Then $va_i = -1/p^i < 0$. Note that $t, a_1, \ldots, a_i \in k(a_{i+1})$ for every i, because $a_i = a_{i+1} - a_{i+1}^p$. We have $1/p \in vk(a_1) \setminus vk(t)$. Since $p \leq (vk(a_1) : vk(t)) \leq [k(a_1) : k(t)] \leq p$, equality holds everywhere and we find that $vk(a_1) = \frac{1}{p} vk(t)$. Repeating this argument by induction on $i > 1$, we obtain $1/p^i \in vk(a_i) \setminus vk(a_{i-1})$ and thus, $vk(a_i) = \frac{1}{p} vk(a_{i-1}) = \frac{1}{p^i} vk(t)$. Therefore, the value group of $K := k(a_i \mid i \in \mathbb{N})$ is the p-divisible hull $\frac{1}{p^\infty}\mathbb{Z}$ of $\mathbb{Z}$ (an ordered abelian group of rank 1).

Finally, we choose η such that $\eta^p = 1/t$. Since vK is p-divisible and $Kv = k$ is algebraically closed, the extension $K(\eta)|K$ with the unique extension of the valuation v is immediate. We wish to determine $\operatorname{dist}(\eta, K)$. We set $c_i :=$

$a_1+\cdots+a_{i-1}\in k(a_{i-1})$ for $i>1$. Using that $a_1^p=\frac{1}{t}+a_1$ and $a_{i+1}^p=a_{i+1}-a_i$ for $i\in\mathbb{N}$, we compute:

$$\begin{aligned}0=\eta^p-\frac{1}{t}&=(\eta-c_i+a_1+\cdots+a_{i-1})^p-\frac{1}{t}\\&=(\eta-c_i)^p+a_1^p+\cdots+a_{i-1}^p-\frac{1}{t}=(\eta-c_i)^p+a_{i-1}.\end{aligned}$$

It follows that $v(\eta-c_i)^p=va_{i-1}$, that is, $v(\eta-c_i)=\frac{1}{p}va_{i-1}=va_i=-1/p^i$. Hence, $-1/p^i\in\Lambda^{\mathrm{L}}(\eta,K)$ for all i. Assume that there is some $c\in K$ such that $v(\eta-c)>-1/p^i$ for all i. Then $v(c-c_i)=\min\{v(\eta-c_i),v(\eta-c)\}=-1/p^i$ for all i. On the other hand, there is some i such that $c\in k(a_{i-1})$ and thus, $c-c_i\in k(a_{i-1})$. But this contradicts the fact that $v(c-c_i)=-1/p^i\notin vk(a_{i-1})$. This proves that the values $-1/p^i$ are cofinal in $\Lambda^{\mathrm{L}}(\eta,K)$. Hence, $\Lambda^{\mathrm{L}}(\eta,K)=vK^{<0}$ and $\mathrm{dist}(\eta,K)=0^-$.

Now Proposition 4.3 shows that K admits a dependent Artin–Schreier defect extension. According to this proposition, it can for instance be generated by a root ϑ of the polynomial $X^p-X-(\eta/t)^p$, as $vt=1>\mathrm{dist}(\eta,K)=p\,\mathrm{dist}(\eta,K)$. Then $\mathrm{dist}(\vartheta,K)=\mathrm{dist}(\eta,K)-1=0^--1=(-1)^-$.

This example shows that the condition in Lemma 4.15 that the extension be finite cannot be dropped. Indeed, as we have noted in Example 4.16, $(k(t),v_t)$ is a defectless and hence inseparably defectless field. But the infinite extension K of $k(t)$ is not an inseparably defectless field.

Example 4.20 (For a field having a dependent but no independent Artin–Schreier defect extension). We do not know whether the field K of the last example admits any independent Artin–Schreier defect extension; this an open problem. But in any case, we can construct from it a field which has a dependent but no independent Artin–Schreier defect extension. Indeed, by Zorn's Lemma there is an extension field of K within its algebraic closure not admitting any independent Artin–Schreier defect extension; such an extension field can be found by a (possibly transfinitely) repeated extension by independent Artin–Schreier defect extensions. We choose such an extension field and call it L. Since it is a separable algebraic extension of K, the extension $L(\eta)|L$ is still nontrivial and purely inseparable, and by our hypothesis on the value group and residue field of K, it is also immediate.

We wish to show that $\mathrm{dist}(\eta,L)=\mathrm{dist}(\eta,K)$. Assume that this is not true. Then there is an element $\zeta\in L$ such that $v(\eta-\zeta)>\mathrm{dist}(\eta,K)$. We write $L=\bigcup_{\mu<\nu}K_\mu$ where ν is some ordinal, $K_{\mu+1}|K_\mu$ is an independent Artin–Schreier defect extension whenever $0\le\mu<\nu$, and $K_\lambda=\bigcup_{\mu<\lambda}K_\mu$ for every limit ordinal $\lambda<\nu$. Let μ_0 be the minimal ordinal for which K_{μ_0} contains such an element ζ. Then μ_0 must be a successor ordinal, and we have that $\mathrm{dist}(\eta,K)=\mathrm{dist}(\eta,K_{\mu_0-1})$. Hence, $v(\eta-\zeta)>\mathrm{dist}(\eta,K_{\mu_0-1})$, that is, $\zeta\sim_{K_{\mu_0-1}}\eta$. But this is a contradiction since by construction, $K_{\mu_0}|K_{\mu_0-1}$

is an independent Artin–Schreier defect extension. This proves that

$$\mathrm{dist}(\eta, L) = \mathrm{dist}(\eta, K) = 0^- .$$

Now Corollary 4.6 shows that L admits a dependent Artin–Schreier defect extension $L'|L$. On the other hand, by construction it does not admit any independent Artin–Schreier defect extension.

This example shows once more that Lemma 4.14 becomes false if the finiteness condition is dropped. To see this, note that $L'.L^{1/p^\infty}|L^{1/p^\infty}$ is still an Artin–Schreier defect extension, since $L'|L$ is linearly disjoint from $L^{1/p^\infty}|L$, vL^{1/p^∞} is p-divisible and $L^{1/p^\infty}v$ is algebraically closed, and the extension of v from L to $L'.L^{1/p^\infty}$ and thus also the extension of v from L^{1/p^∞} to $L'.L^{1/p^\infty}$ is unique. On the other hand, L^{1/p^∞} admits no purely inseparable extensions at all, so by definition, such an Artin–Schreier defect extension can only be independent. We have thus shown that L^{1/p^∞} admits an independent Artin–Schreier defect extension whereas L does not. In view of Lemma 4.14, this is only possible since $L^{1/p^\infty}|L$ is an infinite extension. In contrast to Example 4.16, here we have the case where the lower field is *not* defectless.

Example 4.21 (For a field which is not relatively algebraically closed in any maximal immediate extension, but has no independent Artin–Schreier defect extension). If we replace $k(t)$ by its absolute ramification field $k(t)^r$ (with respect to an arbitray extension of v to the separable-algebraic closure of $k(t)$), then the constructions of Examples 4.19 and 4.20 can be taken over literally. Since $vk(t)^r$ is divisible by every prime different from p, the value group of K, L and L' will then be divisible. Since their residue fields are algebraically closed and all fields are henselian, it follows that K, L and L' are equal to their ramification fields.

Observe that now L' will be contained in every maximal immediate extension of L. This is true because vL is divisible and Lv is algebraically closed, which implies that every maximal immediate extension of L is algebraically closed. We have thus shown that L is not separable-algebraically closed in any of its maximal immediate extensions, whereas it doesn't admit independent Artin–Schreier defect extensions.

Since $L'|L$ is linearly disjoint from $L^c|L$, we may replace L by its completion L^c. By Lemma 4.7, L^c still cannot admit independent Artin–Schreier defect extensions. As the completion of a henselian field is again henselian (cf. [30], Theorem 32.19) and is an immediate extension, it follows that the completion of a field which is equal to its absolute ramification field has the same property. The same argument as before shows that again, $L'.L^c$ will be contained in every maximal immediate extension of L^c. Hence, L^c is an example of a complete field, equal to its absolute ramification field, which is not relatively algebraically closed in any maximal immediate extension, but has no independent Artin–Schreier defect extension.

5. Another characterization of defectless fields

Theorem 5.1. *Let (K,v) be a separably defectless field of characteristic $p>0$. If in addition $K^c|K$ is separable, then (K,v) is a defectless field.*

Proof. Assume that $K^c|K$ is separable, but that (K,v) is not a defectless field. We have to show that (K,v) is not separably defectless. Let $(F|K,v)$ be a finite defect extension of minimal degree of inseparability. If this extension is separable, then we are done. Suppose it is not. We wish to deduce a contradiction by constructing a defect extension of smaller degree of inseparability. Let $E|K$ be the maximal separable subextension. By assumption, it is defectless, so the purely inseparable extension $(F|E,v)$ must be a defect extension. Using the arguments of the proof of Theorem 1.3 (with K replaced by E), one shows that there exists a subextension $L|E$ of $F|E$ and an element $\eta\in L^{1/p}\setminus L$ such that the extension $(L(\eta)|L,v)$ is immediate.

Since a finite extension of a complete field is again complete and since L^c must contain both K^c and L, we find that $L^c=L.K^c$. Together with the fact that $K^c|K$ is separable, this yields that also $L^c|L$ is separable (see [26], Chapter X, §6, Corollary 4). It follows that $\eta\notin L^c$. By an application of Proposition 4.3, we now obtain an immediate separable extension $(L(\vartheta)|L,v)$. Altogether, we have constructed a defect extension $(L(\vartheta)|K,v)$ which has smaller degree of inseparability than $(F|K,v)$. This is the desired contradiction. □

We use this theorem to show the following theorem.

Theorem 5.2. *Let K be a henselian field of characteristic $p>0$. Then K is a separably defectless field if and only if K^c is a defectless field.*

Proof. Since K is henselian, the same holds for K^c (cf. [30], Theorem 32.19). By virtue of the preceding theorem, K^c is a defectless field if and only if it is a separably defectless field. Thus, it suffices to prove that K^c is a separably defectless field if and only if K is.

Let $L|K$ be an arbitrary finite separable extension. The henselian field K is separable-algebraically closed in K^c (cf. [30], Theorem 32.19). Consequently, every finite separable extension of K is linearly disjoint from K^c over K, whence

$$[L.K^c:K^c]=[L:K]. \tag{5.1}$$

On the other hand, $L.K^c=L^c$ is the completion of L and thus an immediate extension of L. Consequently,

$$\begin{aligned}(vL.K^c:vK^c)\cdot[L.K^cv:K^cv]&=(vL^c:vK^c)\cdot[L^cv:K^cv]\\&=(vL:vK)\cdot[Lv:Kv].\end{aligned} \tag{5.2}$$

Assume that K^c is a separably defectless field. Then $L.K^c|K^c$ is defectless, i.e., $[L.K^c:K^c]=(vL.K^c:vK^c)\cdot[L.K^cv:K^cv]$. Hence, $[L:K]=(vL:vK)\cdot$

$[Lv : Kv]$, showing that $L|K$ is defectless. Since $L|K$ was an arbitrary finite separable extension, we have shown that K is a separably defectless field.

Now assume that K^c is not a separably defectless field. Then there exists a finite Galois extension $L'|K^c$ with nontrivial defect. Take an irreducible polynomial $f = X^n + c_{n-1}X^{n-1} + \cdots + c_0 \in K^c[X]$ of which L' is the splitting field. For every $\alpha \in vK$, there are $d_{n-1}, \ldots, d_0 \in K$ such that $v(c_i - d_i) \geq \alpha$. If α is large enough, then by Theorem 32.20 of [30], the splitting fields of f and $g = X^n + d_{n-1}X^{n-1} + \cdots + d_0$ over the henselian field K^c are the same. Consequently, if L denotes the splitting field of g over K, then $L' = L.K^c = L^c$. We obtain

$$\begin{aligned}[L : K] &\geq [L.K^c : K^c] = [L' : K^c] \\ &> (vL' : vK^c)[L'v : K^cv] = (vL^c : vK^c)[L^cv : K^cv] \\ &= (vL : vK)[Lv : Kv].\end{aligned}$$

That is, the separable extension $L|K$ is not defectless. Hence, K is not a separably defectless field. □

6. Algebraically and separable-algebraically maximal fields

6.1. Algebraically maximal fields. We will now give a characterization of algebraically maximal fields which has been presented by F. Delon [4]. We need the following fact, which was proved by Yu. Ershov in [8] by a different method. Note that the proof in [4] has gaps since it is not immediately clear that if $\sum_{i=1}^n \alpha_{i,\nu}$ is increasing with ν, then there is an increasing cofinal subsequence of $(\alpha_{i,\nu})_\nu$ for some i. Ershov solves this problem by invoking Ramsey theory. We will avoid this by further analyzing the valuation theoretical situation.

LEMMA 6.1. *Let (K, v) be any valued field with valuation ring $\mathcal{O}$, and $f \in K[X]$ a polynomial in one variable.*

(1) *If $v\,\mathrm{im}_K(f)$ has no maximum, then there is a pseudo Cauchy sequence $(c_\nu)_{\nu<\lambda}$ of algebraic type in (K, v) without limit in K but admitting a root of f as a limit, and such that $(vf(c_\nu))_{\nu<\lambda}$ is a strictly increasing cofinal sequence in $v\,\mathrm{im}_K(f)$.*

(2) *If $v\,\mathrm{im}_{\mathcal{O}}(f)$ has no maximum, then there is a pseudo Cauchy sequence $(c_\nu)_{\nu<\lambda}$ of algebraic type in $\mathcal{O}$ without limit in K but admitting a root of f as a limit, and such that $(vf(c_\nu))_{\nu<\lambda}$ is a strictly increasing cofinal sequence in $v\,\mathrm{im}_{\mathcal{O}}(f)$.*

Proof. (1) We choose a sequence $(c_\nu)_{\nu<\lambda}$ of elements in K such that the values $vf(c_\nu)$ are strictly increasing and cofinal in $v\,\mathrm{im}_K(f)$. We write $f(X) = \prod_{i=1}^n (X - a_i)$ with $a_1, \ldots, a_n \in \tilde{K}$ and choose some extension of v to $\tilde{K}$.

We introduce a symbol $-\infty$ and define $-\infty < \alpha$ for all $\alpha \in v\tilde{K}$. Now we consider all balls $B^\circ_\alpha(a_i) = \{a \in \tilde{K} \mid v(a_i - a) > \alpha\}$ with center a root a_i of

f, $1 \le i \le n$, and radius α in the finite set $\mathcal{D} := \{v(a_i - a_j) \mid 1 \le i < j \le n\} \cup \{-\infty\}$; note that $B^\circ_{-\infty}(a_i) = \tilde{K}$. These are finitely many balls, with $\tilde{K}$ one of them, so there is at least one among them in which there lies some cofinal subsequence of $(c_\nu)_{\nu<\lambda}$. Take such a ball with α maximal. After renaming our elements if necessary, we may assume that this ball is $B^\circ_\alpha(a_1)$, that the subsequence is again called $(c_\nu)_{\nu<\lambda}$, and that exactly $a_1, \dots, a_m$ $(m \le n)$ are the roots of f which lie in $B^\circ_\alpha(a_1)$. Then for every $\nu < \lambda$ and $m < i \le n$, we have that

$$v(c_\nu - a_i) = \min\{v(c_\nu - a_1), v(a_1 - a_i)\} = v(a_1 - a_i).$$

On the other hand, by the maximality of α we have the following: if $\mathcal{D}$ contains elements $> \alpha$ (which is the case if $B^\circ_\alpha(a_1)$ contains at least two roots of f) and if β is the least of these elements, then there is no cofinal subsequence of $(c_\nu)_{\nu<\lambda}$ which lies in any of the balls $B^\circ_\beta(a_i)$. This even remains true if we replace $B^\circ_\beta(a_i)$ by $B_\beta(a_i) = \{a \in \tilde{K} | v(a_i - a) \ge \beta\}$. Indeed, by our choice of β we have for $1 \le i \le m$ that $B_\beta(a_i)$ contains $a_1, \dots, a_m$ and thus, $c \in B_\beta(a_i)$ implies $v(c - a_j) \ge \beta$ for $1 \le j \le m$. If in addition c does not lie in any $B^\circ_\beta(a_j)$, then $v(c - a_j) = \beta$ for $1 \le j \le m$. Hence, if a cofinal subsequence of $(c_\nu)_{\nu<\lambda}$ would lie in $B_\beta(a_i)$, then the value

$$vf(c_\nu) = v \prod_{i=1}^{n} (c_\nu - a_i) = \sum_{i=1}^{n} v(c_\nu - a_i) = m\beta + \sum_{i=m+1}^{n} v(a_1 - a_i)$$

would be fixed for all c_ν in this subsequence, a contradiction.

After deleting elements from $(c_\nu)_{\nu<\lambda}$, we may thus assume that $v(c_\nu - a_i) < \beta \le v(a_1 - a_i)$ for all ν and $1 \le i \le m$. It follows that $v(c_\nu - a_i) = \min\{v(c_\nu - a_1), v(a_1 - a_i)\} = v(c_\nu - a_1)$ for all ν and $1 \le i \le m$. Now we compute:

$$vf(c_\nu) = \sum_{i=1}^{n} v(c_\nu - a_i) = mv(c_\nu - a_1) + \sum_{i=m+1}^{n} v(a_1 - a_i).$$

If $\mu < \nu < \lambda$, then $vf(c_\mu) < vf(c_\nu)$ and hence we must have $v(c_\mu - a_1) < v(c_\nu - a_1)$. This shows that $(c_\nu)_{\nu<\lambda}$ is a pseudo Cauchy sequence with limit a_1.

Any limit $a \in \tilde{K}$ of this sequence satisfies $v(a - a_1) > v(c_\nu - a_1)$ and hence also $v(a - a_i) \ge \min\{v(a - a_1), v(a_1 - a_i)\} > v(c_\nu - a_1)$ for $1 \le i \le m$ and all ν. Thus,

$$vf(a) = \sum_{i=1}^{n} v(a - a_i) > mv(c_\nu - a_1) + \sum_{i=m+1}^{n} v(a_1 - a_i) = vf(c_\nu)$$

for all ν. This shows that a cannot lie in K. Hence, $(c_\nu)_{\nu<\lambda}$ is a pseudo Cauchy sequence without limit in K, and by construction, it is of algebraic type.

(2) We proceed as in (1), but choose the sequence $(c_\nu)_{\nu<\lambda}$ in $\mathcal{O}$ such that the values $vf(c_\nu)$ are strictly increasing and cofinal in $v\,\mathrm{im}_{\mathcal{O}}(f)$. We only have to note in addition that if $a\in K$ would be a limit of the sequence, than it would also lie in $\mathcal{O}$. $\square$

COROLLARY 6.2. *Assume that (K,v) is not K-extremal with respect to the polynomial $f(X)\in K[X]$. Then for all $c\in K$ of large enough value, (K,v) is not $\mathcal{O}$-extremal with respect to the polynomial $f(c^{-1}X)$. Hence, if (K,v) is $\mathcal{O}$-extremal with respect to every polynomial in one variable, then (K,v) is K-extremal with respect to every polynomial in one variable. The same holds for "separable polynomial" in the place of "polynomial".*

Proof. Take the pseudo Cauchy sequence $(c_\nu)_{\nu<\lambda}$ as in Lemma 6.1. For large enough $\nu_0<\lambda$, the values of the c_ν with $\nu_0<\nu<\lambda$ are constant, say, α. For every c of value $\geq -\alpha$, we have that $cc_\nu\in\mathcal{O}$ for $\nu_0<\nu<\lambda$. Hence, (K,v) is not $\mathcal{O}$-extremal with respect to the polynomial $f(c^{-1}X)$. $\square$

The first part of the following result was proved by Yu. Ershov in [8].

PROPOSITION 6.3. *A valued field is algebraically maximal if and only if it is henselian and K-extremal with respect to every polynomial in one variable. The same holds with "$\mathcal{O}$-extremal" in the place of "K-extremal".*

Proof. Suppose that (K,v) is henselian, but not algebraically maximal. Then there is a proper immediate algebraic extension $L|K$. Take $a\in L\setminus K$. By Theorem 1 of [12], there is a pseudo Cauchy sequence in K without limit in K, having a as a limit. Let $f\in K[X]$ be the minimal polynomial of a over K. Since K is henselian, the extension of v from K to $K(a)$ is unique. Now it follows from Lemma 2.11 that $v\,\mathrm{im}_K(f)$ has no maximal element. That is, K is not K-extremal with respect to f. Hence, by Corollary 6.2, K is also not $\mathcal{O}$-extremal with respect to every polynomial in one variable.

For the converse, suppose that there is a polynomial $f\in K[X]$ such that $v\,\mathrm{im}_K(f)$ or $v\,\mathrm{im}_{\mathcal{O}}(f)$ has no maximal element. Then by Lemma 6.1, (K,v) admits a pseudo Cauchy sequence of algebraic type in (K,v) without limit in K. Now Theorem 3 of [12] shows that there is a proper immediate algebraic extension of (K,v), i.e., (K,v) is not algebraically maximal. $\square$

Theorem 1.4 and its "K-extremal" version will follow from Proposition 6.3 once we have proved the following proposition.

PROPOSITION 6.4. *If a valued field is K- or $\mathcal{O}$-extremal with respect to every separable polynomial in one variable, then it is henselian.*

Proof. In view of Corollary 6.2, we only have to prove the assertion for "K-extremal". Suppose that the valued field (K,v) with valuation ring $\mathcal{O}$ is not henselian. Then there is a polynomial $f\in\mathcal{O}[X]$ and an element $b\in\mathcal{O}$ such that $vf(b)>2vf'(b)$, but f has no root in K. We take K_0 to be a finitely generated subfield of K containing b and all coefficients of f, and K_1 to be the relative algebraic closure of K_0 in K. Then f has no root in K_1, which shows that K_1 is not henselian. Since K_1 has finite transcendence degree over its prime field, it has finite rank, which means that $v|_{K_1}$ is a composition $v|_{K_1} = v_1 \circ \cdots \circ v_k$ of valuations v_i with archimedean value groups. By a repeated application of Theorem 32.15 of [30], it follows that (K_1, v_1) is not henselian or for some $i \leq k$ and $v^i := v_1 \circ \cdots \circ v_{i-1}$, $(K_1 v^i, v_i)$ is not henselian. In the first case, there is a monic separable and irreducible polynomial $g \in K_1[X]$ with v_1-integral coefficients and a v_1-integral element $c \in K_1$ such that $v_1 g(c) > 2v_1 g'(c)$, but g does not have a zero in K_1. It follows that $vg(c) > 2vg'(c)$.

In the second case, there is a monic separable and irreducible polynomial $\overline{g} \in K_1 v^i[X]$ with v_i-integral coefficients and a v_i-integral element $\overline{c} \in K_1 v^i$ such that $v_i \overline{g}(\overline{c}) > 2v_i \overline{g}'(\overline{c})$, but $\overline{g}$ does not have a zero in $K_1 v^i$. We take some monic polynomial $g \in K_1[X]$ with v^i-integral coefficients such that its v^i-reduction is equal to $\overline{g}$. Also, we pick a v^i-integral element $c \in K_1$ whose v^i-reduction is $\overline{c}$. Then it follows that $v^{i+1} g(c) > 2v^{i+1} g'(c)$, whence $vg(c) > 2vg'(c)$.

It is well known that if w is any valuation for which the polynomial g has w-integral coefficients and $wg(c) > 2wg'(c)$ holds, then a repeated application of the Newton algorithm

$$c_{n+1} := c_n - \frac{g(c_n)}{g'(c_n)},$$

starting with $c_0 = c$, leads to a strictly increasing sequence of values $wg(c_n)$; this sequence is cofinal in the value group of w in case this value group is archimedean. Hence, in the first case, we obtain a sequence of elements $c_n \in K_1$ such that the sequence $v_1 g(c_n)$ is cofinal in $v_1 K_1$. This implies that if $d \in K$ is such that $vg(d)$ is the maximum of $v\,\mathrm{im}_K(g)$ and H denotes the convex subgroup of vK generated by $v_1 K_1$, then $vg(d) > H$. Let v_H be the coarsening of v with respect to H. Then $v_H g(d) > 0$, i.e., $g(d)v_H = 0$. On the other hand, the reduction modulo v_H induces an isomorphism on K_1, and since g was chosen to be separable and irreducible, we thus have that $g'(d)v_H \neq 0$, i.e., $v_H g'(d) = 0$. But then by the Newton algorithm, if $g(d) \neq 0$, then there is some $d' \in K$ such that $v_H g(d') > v_H g(d)$ and hence, $vg(d') > vg(d)$. This contradiction shows that $g(d) = 0$. But this contradicts our choice of g. Hence, $v\,\mathrm{im}_K(g)$ does not have a maximum.

In the second case, the Newton algorithm provides elements $\overline{c}_n \in K_1 v^i$ such that the sequence $v_i \overline{g}(\overline{c}_n)$ is cofinal in $v_i(K_1 v^i)$. We choose v^i-integral elements $c_n \in K_1$ whose v^i-reductions are $\overline{c}_n$. Then it follows that the values $vg(c_n)$ are cofinal in a convex subgroup H of vK which is the convex hull of the convex subgroup of vK_1 which corresponds to the coarsening v^i of $v|_{K_1}$. This implies that if $d \in K$ is such that $vg(d)$ is the maximum of $v\,\mathrm{im}_K(g)$, then $vg(d) > H$. Let v_H be the coarsening of v with respect to H. Then again, $v_H g(d) > 0$ and $g(d)v_H = 0$. On the other hand, the reduction of g modulo v_H is $\overline{g}$, so $0 = g(d)v_H = \overline{g}(dv_H)$. Since $\overline{g}$ was chosen to be separable and irreducible, we thus have that $g'(d)v_H = \overline{g}'(dv_H) \neq 0$, i.e., $v_H g'(d) = 0$. Arguing as in the first case, we show that $v\,\mathrm{im}_K(g)$ does not have a maximum. Hence, we find that K is not K-extremal with respect to every separable polynomial in one variable. □

The following are corollaries to Theorem 1.4.

COROLLARY 6.5. *The property "algebraically maximal" is elementary in the language of valued fields.*

COROLLARY 6.6. *Every extremal field is algebraically maximal.*

We will now give the following proof.

Proof of Theorem 1.5. and its "K-extremal" version: In view of Theorem 1.4, it suffices to prove that if K is a henselian but not algebraically maximal field, then there is a p-polynomial f in one variable with coefficients in K with respect to which K is not K-extremal. By Corollary 6.2, for suitable $c \in K$, K is then also not $\mathcal{O}$-extremal with respect to the p-polynomial $f(c^{-1}X)$.

Take a proper immediate algebraic extension of K. Since K is assumed henselian, it follows that this extension is purely wild and hence linearly disjoint over K from the absolute ramification field K^r of K. We may assume that this extension is minimal, that is, it does not admit any proper subextension. Then by Theorem 13 of [18], it is generated by a root of a p-polynomial f. As in the first part of the proof of Proposition 6.3, it follows that $v\,\mathrm{im}_K(f)$ has no maximal element, that is, K is not K-extremal with respect to the p-polynomial f. □

6.2. Separable-algebraically maximal fields. The *breadth* of a pseudo Cauchy sequence $(c_\nu)_{\nu<\lambda}$ in a valued field (K,v) with valuation ring $\mathcal{O}$ is the (integral or fractional) $\mathcal{O}$-ideal of all $c \in K$ such that $vc \geq v(c_{\nu+1} - c_\nu)$ for all $\nu < \lambda$. We call it nontrivial if it contains nonzero elements.

The following is a further consequence of Proposition 4.3.

COROLLARY 6.7. *Take a separable-algebraically maximal field (K,v). Every immediate algebraic extension of (K,v) is purely inseparable and lies in its completion. Every pseudo Cauchy sequence of algebraic type in (K,v) without limit in K has breadth $\{0\}$, and its unique limit in $\tilde{K}$ is purely inseparable over K.*

Proof. Every immediate algebraic extension of K must be purely inseparable since otherwise, it would contain a proper immediate separable-algebraic subextension. Since K in particular does not admit any dependent Artin–Schreier defect extensions, we thus obtain from Corollary 4.6 that every immediate algebraic extension of K must lie in K^c.

Take a pseudo Cauchy sequence $(c_\nu)_{\nu<\lambda}$ of algebraic type in (K,v) without limit in K. By Theorem 3 of [12], this pseudo Cauchy sequence gives rise to a proper immediate algebraic extension of K, in which it has a limit. By what we have just shown, this extension is purely inseparable and lies in the completion of K. The latter shows that $(c_\nu)_{\nu<\lambda}$ has breadth $\{0\}$ and therefore has a unique limit in the algebraic closure of K. The former shows that this limit must be purely inseparable over K. □

The following result has been presented by F. Delon in [4].

Corollary 6.8. *The completion of a separable-algebraically maximal field is algebraically maximal.*

Proof. Take any valued field (K,v) and suppose that K^c admits a proper immediate algebraic extension. Then by Corollary 2.12 there is a pseudo Cauchy sequence $(c_\nu)_{\nu<\lambda}$ of algebraic type in K^c without limit in K^c. This must have nontrivial breadth, that is, there is some $\gamma\in vK$ such that $v(c_{\nu+1}-c_\nu)<\gamma$ for all ν (because otherwise, Theorem 3 of [12] would render a proper immediate extension of K^c within K^c, which is absurd). Since $c_\nu\in K^c$, there is $c_\nu^*\in K$ such that $v(c_\nu-c_\nu^*)\geq\gamma$ and hence $v(c_{\nu+1}^*-c_\nu^*)=v(c_{\nu+1}-c_\nu)$ for all ν. It follows that $(c_\nu^*)_{\nu<\lambda}$ is a pseudo Cauchy sequence in K without limit in K and with the same nontrivial breadth as $(c_\nu)_{\nu<\lambda}$.

Let $f\in K^c[X]$ be a polynomial such that for some $\mu<\lambda$, the sequence $(vf(c_\nu))_{\mu<\nu<\lambda}$ is strictly increasing. Such a polynomial must exist since by assumption, $(c_\nu)_{\nu<\lambda}$ is of algebraic type. Since $(c_\nu)_{\nu<\lambda}$ has nontrivial breadth, it follows from Lemma 8 of [12] that the sequence $(vf(c_\nu))_{\mu<\nu<\lambda}$ is bounded from above in vK. Hence, we can choose a polynomial $f^*\in K[X]$ with coefficients so close to the corresponding coefficients of f that $vf^*(c_\nu)=vf(c_\nu)$ whenever $\mu<\nu<\lambda$. This shows that also $(c_\nu^*)_{\nu<\lambda}$ is of algebraic type. Hence, by the foregoing corollary, K cannot be separable-algebraically maximal. □

Now we give the following proof.

Proof of Theorem 1.6 and its "K-extremal" version. Assume that (K,v) is K-extremal or $\mathcal{O}$-extremal with respect to every separable polynomial in one variable. Then by Proposition 6.4, K is henselian. Suppose that (K,v) is not separable-algebraically maximal. Then there is a proper immediate separable-algebraic extension $L|K$. Take $a\in L\setminus K$, and let $f\in K[X]$ be the minimal polynomial of a over K. By Theorem 1 of [12], there is a pseudo Cauchy sequence in K without limit in K, having a as a limit. Since K is henselian, the extension of v from K to $K(a)$ is unique. Now it follows from Lemma 2.11 that $v\,\mathrm{im}_K(f)$ has no maximal element, that is, K is not

K-extremal with respect to f. By Corollary 6.2, it follows that K is not $\mathcal{O}$-extremal with respect to the separable polynomial $f(c^{-1}X)$ for some $c \in K$. This contradicts our assumption that K is K-extremal or $\mathcal{O}$-extremal with respect to every separable polynomial in one variable. Hence, K is separable-algebraically maximal.

For the converse, assume that (K,v) is separable-algebraically maximal. Suppose that there is a separable polynomial $f \in K[X]$ such that $v\,\mathrm{im}_K(f)$ or $v\,\mathrm{im}_{\mathcal{O}}(f)$ has no maximal element. Then by Lemma 6.1, (K,v) admits a pseudo Cauchy sequence $(c_\nu)_{\nu<\lambda}$ of algebraic type in (K,v) without limit in K, but with a root $a \notin K$ of f as a limit. By Corollary 6.7, a is purely inseparable over K. But this contradicts the fact that a is a root of a separable polynomial over K. Hence, K is K-extremal and $\mathcal{O}$-extremal with respect to every separable polynomial in one variable. □

Corollary 6.9. *The property "separable-algebraically maximal" is elementary in the language of valued fields.*

We turn to the following proof.

Proof of Theorem 1.7. The proof is the same as for Theorem 1.5, except that the immediate algebraic extension of K can be taken to be separable, and hence the p-polynomial f is separable. □

Finally, we note that Theorem 1.8 has also been proved, as our above proofs of Theorems 1.4, 1.5, 1.6 and 1.7 have all dealt simultaneously with both K- and $\mathcal{O}$-extremality.

Acknowledgments. I thank Peter Roquette for his invaluable help and support, and Olivier Piltant and Bernard Teissier for inspiring discussions. Parts of this paper were written while I was a guest of the Equipe Géométrie et Dynamique, Institut de Mathématiques de Jussieu, Paris, and of the Laboratoire de Mathématiques at the Université de Versailles Saint-Quentin-en-Yvelines. I gratefully acknowledge their hospitality and support. I thank the referee for the exceptionally careful reading and an impressive number of useful remarks and corrections which helped to improve the paper significantly.

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