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A PARABOLIC VERSION OF CORONA DECOMPOSITIONS

JORGE RIVERA-NORIEGA

ABSTRACT. Let E be a subset in $(n+1)$-dimensional Euclidian space with parabolic homogeneity, codimension 1, and with an appropriate surface measure σ associated to it. We define a parabolic version of Corona decomposition of E and establish two results on sufficient conditions for the existence of parabolic Corona decomposition for E. Both results are parabolic versions of well-known results due to G. David and S. Semmes.

1. Introduction

The monograph [2] deals with the question of finding conditions on a set $E \subset \mathbb{R}^n$, $n \geq 3$, with Hausdorff dimension $d < n$, so that Calderón–Zygmund operators with nice kernels are bounded in the spaces $L^2(E, dH^d)$, where H^d denotes the d-dimensional Hausdorff measure. The main theorem in that work establishes the equivalence of several conditions with this L^2 boundedness, some conditions of geometric nature and some more described through analytical properties.

Our motivation for this work is to explore a parabolic version of portions of that theorem for sets in $\mathbb{R}^{n+1}$ with parabolic homogeneity and codimension 1. Besides [2], [3], our source of motivation for focusing on parabolic problems is the theory developed in [5], [6], [8], [9], [10] for parabolic singular integrals, and parabolic uniformly rectifiable sets.

The fundamental works [2], [3] include many ideas, techniques and constructions that could be easily adapted to the parabolic setting. However, when trying to obtain appropriate parabolic versions of those results, there are some adaptations to be carefully considered:

(a) For the so called *Corona decompositions* of [2], [3], one uses Lipschitz graphs for a local approximation to the set E. In parabolic problems

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some more specific regularity for the approximating graphs is required, as it was essentially addressed in [5], [7].

(b) In the constructions of [2], [3], some planes are used as a reference for the aforementioned approximation process. The good planes in parabolic problems must include lines parallel to the time axis, as observed in [8], [9]. This suggests that considerations about planes in some arguments of [2] should be adapted accordingly.

(c) The construction of the approximating Lipschitz graphs in [2], [3] is performed using a condition on the angle between planes. However, it is not clear that this condition is explicitly needed in parabolic constructions, as noted for instance in [8], [11], where parabolic Lipschitz graph are constructed through Whitney type extensions, and n-dimensional planes in $\mathbb{R}^{n+1}$ parallel to $\mathbb{R}^n$.

These and some other remarks are considered in constructions of subsequent sections of this paper.

In view of this, we focused on arguments that required adjustments beyond considerations on the parabolic homogeneity, and still providing the important steps to prove a parabolic version of the main theorem in [2], including only some of the several conditions addressed therein. To be more precise in our statements, we introduce some notations.

For $(X,t)\in\mathbb{R}^n\times\mathbb{R}\equiv\mathbb{R}^{n+1}$, denoted by $\mathcal{C}_r(X,t)$ the *cylinder of radius* $r>0$ *and centered at* (X,t) given by $\{(Y,s)\in\mathbb{R}^{n+1}:|X-Y|<r,|t-s|<r^2\}$. The *parabolic distance* between $(X,t),(Y,s)\in\mathbb{R}^{n+1}$ is defined by $d(X,t;Y,s)=|X-Y|+|t-s|^{1/2}\equiv\|X-Y,t-s\|$. This last expression defines what is called the *parabolic norm* of points in $\mathbb{R}^{n+1}$, i.e., $\|X,t\|=d(X,t;\vec{0})$. By extension, we define the parabolic distance between sets E and F as $d(E;F)=\inf\{d(X,t;Y,s):(X,t)\in E,(Y,s)\in F\}$, where E and F are either both in $\mathbb{R}^{n+1}$ or both in $\mathbb{R}^n$, and containing the variable t.

In some cases, we denote points in $\mathbb{R}^{n+1}$ by $(x_0,x,t)\in\mathbb{R}\times\mathbb{R}^{n-1}\times\mathbb{R}$, to stress that in graph coordinates x_0 is the variable depending on (x,t). From time to time, especially in certain parts of this work where the t variable is irrelevant, we use the notation $\mathbf{X}$ for points in $\mathbb{R}^{n+1}$.

To start the description of the type of hypersurfaces considered, we define once and for all the appropriate adaptation of surface measure. Given a Borel set $F\subset\mathbb{R}^{n+1}$, let $\sigma(F)=\int_F d\sigma_t\,dt$, where σ_t is the $(n-1)$-dimensional Hausdorff measure of $F_t\equiv F\cap\mathbb{R}^n\times\{t\}$.

In the remaining, we denote by H^d the Hausdorff measure of dimension $0<d<n+2$. Also, to shorten notations, if P is any n-dimensional plane containing a line parallel to the t axis then it will be called a *t-plane*. Notice now that if P is any t-plane through $(X,t)\in\mathbb{R}^{n+1}$ then $\sigma(\mathcal{C}_r(X,t)\cap P)=\alpha_n r^{n+1}$, for certain dimensional constant α_n. This is indeed the behavior we want for a surface measure with parabolic homogeneity.

If we now fix a subset E in $\mathbb{R}^{n+1}$, let $d(X,t)=d(X,t;E)$ and define

$$\gamma(Z,\tau;r)=\inf_P\left[\frac{1}{r^{n+3}}\int_{E\cap C_r(Z,\tau)} d((Y,s),P)^2\,d\sigma(Y,s)\right], \tag{1}$$

where the infimum is taken over all t-planes P. Finally, define the measure

$$d\nu(Z,\tau;r)=\gamma(Z,\tau;r)\,d\sigma(Z,\tau)\frac{dr}{r}. \tag{2}$$

DEFINITION 1.1. We say that $E\subset\mathbb{R}^{n+1}$ is *uniformly rectifiable in the parabolic sense* (URPS), or that E is *parabolic uniformly rectifiable*, if the following conditions hold:

- For some $M\geq 1$ and $R>0$, E satisfies an (M,R) *Ahlfors condition* for $0<\rho\leq R$ and $(X,t)\in E$

$$\rho^{n+1}M^{-1}\leq\sigma\big(\mathcal{C}_\rho(X,t)\cap E\big)\leq M\rho^{n+1}. \tag{3}$$

- For every $(X,t)\in E$ and $\mathcal{C}_\rho(X,t)\subset\mathcal{C}_R(X,t)$, the following Carleson measure condition holds:

$$\nu\big([\mathcal{C}_\rho(X,t)\cap E]\times(0,\rho)\big)\leq C\rho^{n+1} \tag{4}$$

  for some constant C.

The smallest constant for which (4) holds will be denoted by $\|\nu\|_+$. In general, when a measure μ defined on Borel sets of $E\times(0,\infty)$ satisfies an estimate as (4) then one says that μ is a *Carleson measure* and $\|\mu\|_+$ is referred to as the *Carleson norm* of μ.

From now on, given $E\subset\mathbb{R}^{n+1}$ satisfying an (M,R) Ahlfors condition, we let $\Delta_r(X,t)$ denote the *surface cube* $\mathcal{C}_r(X,t)\cap E$. Also, to shorten notation, we call $E\subset\mathbb{R}^{n+1}$ a *parabolic hypersurface* if it satisfies an (M,R) Ahlfors condition as (3) and $\mathbb{R}^{n+1}\setminus E$ has exactly two connected components. These components will be denoted by $\Omega_1\equiv\Omega_1(E)$ and $\Omega_2\equiv\Omega_2(E)$.

For any real valued function ψ defined on an n-dimensional plane P, taking values in the orthogonal complement of P in $\mathbb{R}^{n+1}$, the *graph of the function* is $\{Z+\psi(Z): Z\in P\}$. Notice that it may occur that the plane P coincides with $\mathbb{R}^n$.

A function $\psi:\mathbb{R}^n\longrightarrow\mathbb{R}$ is a Lip$(1,1/2)$ function with constant $A_1>0$ if for $(x,t),(x,s)\in\mathbb{R}^n$, $|\psi(x,t)-\psi(y,s)|\leq A_1\|x-y,t-s\|$. The function ψ is called a *parabolic Lipschitz function* if it satisfies the following two conditions:

- ψ satisfies a Lipschitz condition in the space variable

$$|\psi(x,t)-\psi(y,t)|\leq A_2|x-y| \tag{5}$$

  uniformly on $t\in\mathbb{R}$.
- For every interval $I\subseteq\mathbb{R}$, every $x\in\mathbb{R}^n$, and with a uniform constant A_3

$$\frac{1}{|I|}\int_I\int_I\frac{|\psi(x,t)-\psi(x,s)|^2}{|s-t|^2}\,dt\,ds\leq A_3<\infty. \tag{6}$$

The *character* of a parabolic Lipschitz function ψ or its graph is $\max\{\tilde{A}_2, \tilde{A}_3\}$, where $\tilde{A}_2$ and $\tilde{A}_3$ are respectively, the minimal constants for which (5) and (6) hold.

We recall that it is a well-established fact (see e.g., [5]) that every parabolic Lipschitz function is a $\mathrm{Lip}(1,1/2)$ function.

Now we describe the *dyadic grid* associated to an (M,R) Ahlfors regular set, regardless of the homogeneity or the surface measure associated to it (see e.g., [1]). Given $E \subset \mathbb{R}^{n+1}$ satisfying an (M,R) Ahlfors condition as (3), there exists a family of partitions Δ_j of E (called the *generations* of the dyadic grid), $j \in \mathbb{Z}$, into sets Q called *parabolic dyadic surface cubes* (or simply *dyadic cubes*) with the following properties:

(DG1) If $j \le k$, $Q \in \Delta_j$ and $Q' \in \Delta_k$ then either $Q \cap Q' = \emptyset$ or $Q \subseteq Q'$;

(DG2) there exists a constant $G > 0$ such that if $Q \in \Delta_j$ then

$$2^j/G \le \operatorname{diam} Q \le G 2^j \quad \text{and} \quad 2^{j(n+1)}/G \le \sigma(Q) \le G 2^{j(n+1)};$$

(DG3) if $Q \in \Delta_j$ and $\tau > 0$, then with the same constant $G > 0$ as above

$$\sigma\big(\big\{(X,t) \in Q : d\big((X,t); E \setminus Q\big) \le \tau 2^j\big\}\big) \le G \tau^{1/G} 2^{j(n+1)}.$$

This last property is recalled as a *relative smallness property of the boundary of the cubes*. In (DG2), $\operatorname{diam} Q = \sup\{\|x-y, t-s\| : (x,t),(y,s) \in Q\}$ denotes the *parabolic diameter* of Q.

From now on, we adopt the following standard notations: $A \lesssim B$ means that $A \le kB$ with a constant $k > 0$, that may depend at most on the constants involved in previous definitions, or whose dependance is clear from the context. In any case, the dependance may be explicitly stated using the notation $k = k(\cdots)$. Similarly, $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$ hold simultaneously. Finally, the constant in a chain of inequalities may change from line to line, as long as the dependance of such a constant does not interfere in the essence of the argument.

Let $\Delta = \bigcup \Delta_j$ and for $Q \in \Delta$, $t \in \mathbb{R}$ and $x \in \mathbb{R}^{n-1}$ define:

- the *t-slice* of Q as $Q^t = (\mathbb{R}^n \times \{t\}) \cap E$ and
- the *x-slice* of Q as $Q^x = \{t \in \mathbb{R} : (x,t) \in Q \text{ for some } x \in \mathbb{R}^n\}$.

Observe that if $Q^t \neq \emptyset$ then (DG1)–(DG3) imply $H^{n-1}(Q^t) \approx (\operatorname{diam} Q)^{n-1}$, and similarly if $Q^x \neq \emptyset$ then by (DG1)–(DG3) one has $H^1(Q^x) \approx [\operatorname{diam} Q]^2$.

When E coincides with a t-plane then the dyadic grid may actually be chosen to coincide (after a rotation, if needed) with the grid of cubes of the form

$$Q_{2^j}(X,t) = \{(Z,\tau) : |Z - X| < 2^j, |\tau - t| < 2^{2j}\} \quad \text{for certain } (X,t) \in \mathbb{R}^{n+1}.$$

We will keep the notation $Q_r(p)$ for the cube contained in a t-plane centered at p with radius $r > 0$.

Let $Q \in \Delta_j$ and $R \in \Delta$. We say that R is a *descendant of* Q if $R \in \Delta_{j+1}$ and $R \subset Q$. We say that R is a *sibling of* Q if $R \in \Delta_j$ and both R and Q are descendants of the same $\tilde{Q} \in \Delta_{j-1}$. The family of siblings of Q is denoted by $\varsigma(Q)$.

Definition 1.2. A Borel set $E \subset \mathbb{R}^{n+1}$ admits a *parabolic Corona decomposition* if for each $\eta > 0$ there is a constant $C = C(\eta) > 0$ such that we can partition $\Delta = \bigcup \Delta_j$ into a *good set* $\mathcal{G} \subset \Delta$ and a *bad set* $\mathcal{B} \subset \Delta$ with the following properties:

(CD1) The bad set satisfies a *Carleson packing condition*:

$$\sum_{\substack{Q \in \mathcal{B} \\ Q \subset R}} \sigma(Q) \leq C\sigma(R).$$

The good set in turn can be partitioned into a family $\mathcal{F}$ of subsets S of $\mathcal{G}$ such that:

(CD2) Each S has a maximal element denoted by $Q(S)$;
(CD3) If $Q \in S$, $Q' \in \Delta$ with $Q \subseteq Q' \subseteq Q(S)$, then $Q' \in S$;
(CD4) If $Q \in S$ then either all of the descendants of Q lie in S or none of them do;
(CD5) The maximal cubes satisfy the *Carleson packing condition*

$$\sum_{\substack{S \in \mathcal{F} \\ Q(S) \subset R}} \sigma(Q(S)) \leq C\sigma(R).$$

An additional property of the elements in the family $\mathcal{F}$ is the following:

(CD6) For each $S \in \mathcal{F}$ there exists a parabolic Lipschitz graph Γ with character η such that for every $Q \in S$ if $\mathbf{X} \in E$, $d(\mathbf{X}, Q) \leq \operatorname{diam}(Q)$ then $d(\mathbf{X}; \Gamma) \leq \eta \operatorname{diam}(Q)$.

Following [2], we refer to the $S \in \mathcal{F}$ as the *stopping-time regions*, since they are usually constructed through algorithms using stopping time arguments.

The triple $(\mathcal{B}, \mathcal{G}, \mathcal{F})$ satisfying (CD1)–(CD5) will be called a *coronization of E* (see [3, p. 55]). Observe that our notion of parabolic Corona decomposition differs from the original one introduced by [2, p. 18], only in property (CD6), as the type of approximating graph we require is of a different nature as that of Lipschitz functions.

At the same time, the so called *generalized Corona decompositions* in [3, p. 63ff] allow more general sets (not necessarily Lipschitz graphs) to be the approximating sets alluded to in (CD6). However, this generalization is essentially in a different sense, and in particular, we still have to obtain the precise regularity condition addressed by (6) in the definition of parabolic Lipschitz functions.

A *basic parabolic Lipschitz domain* is a domain of the form

$$\Omega(\psi) = \{(x_0, x, t) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} : x_0 > \psi(x, t)\}$$

for some parabolic Lipschitz function ψ. We say that an (M,R) Ahlfors–David regular set E *contains big pieces of parabolic Lipschitz graphs of character at most* $B_1 \geq 1$ (or in short notation $E \in \mathrm{BPPLG}(B_1)$), if there exist a constant $\vartheta > 0$ such that for every $(Q,s) \in E$ and $r > 0$ there exists a basic parabolic Lipschitz domain $D = \Omega(\psi)$, with character at most B_1, and such that after a rotation in space variable, if we set

$$D_r(X,t) = \{(z,\tau) \in \mathbb{R}^{n-1} \times \mathbb{R} : |z - x| < r, \tau \in (t - r^2, t + r^2)\}$$

then the following estimate holds

$$\sigma\big(E \cap \{(\psi(z,\tau), z, \tau) : (z,\tau) \in D_r(X,t)\}\big) \geq \vartheta r^{n+1}. \tag{7}$$

Using the fundamental results of [3, I Chapter 3] on coronizations of Ahlfors–David regular sets, we can obtain the following adaptation of [3, Theorem I.3.42].

Proposition 1.3. *Suppose that E admits a generalized parabolic Corona decomposition, in which* (CD6) *above is substituted by*:

(GCD6) *For each $S \in \mathcal{F}$ there exists a set $E_S \in \mathrm{BPPLG}(C(\eta))$ such that for every $Q \in S$ the estimate $d(\mathbf{X}; E_S) \leq \eta \operatorname{diam}(Q)$ holds whenever $\mathbf{X} \in E$ and $d(\mathbf{X}, Q) \leq \operatorname{diam}(Q)$.*

Then E admits a parabolic Corona decomposition in the sense of Definition 1.2. The constant $C(\eta) > 0$ may depend on η but not on S.

Certainly the opposite of this theorem is immediate. We will have a use for Proposition 1.3 in the proof of Theorem 1.5 below.

In our first theorem, we relate the concepts introduced in Definitions 1.1 and 1.2 as follows.

Theorem 1.4. *Let E be a parabolic hypersurface in $\mathbb{R}^{n+1}$. If E is uniformly rectifiable in the parabolic sense, then E admits a parabolic Corona decomposition.*

The technique of the proof follows closely that of [2]. In fact, Theorem 1.4 may be viewed, in the spirit of [2], as an intermediate step in order to develop a theory of singular integrals of parabolic type on sets E uniformly rectifiable in the parabolic sense. With techniques from [2, Chapter 15] and the constructions in Section 2, it may not be difficult to prove a reciprocal of Theorem 1.4, but we will not pursue the implementation of these adaptations in this work.

Moving on to the description of our next result, consider convolution type singular integral operators

$$Tf(\mathbf{X}) = \int_E K(\mathbf{X} - \mathbf{Y}) f(\mathbf{Y}) \, d\sigma(\mathbf{Y}), \tag{8}$$

$\mathbf{X} = (X,t)$, $\mathbf{Y} = (Y,s)$, where the kernel $K(X,t)$ is odd as a function of X, it satisfies the following properties:

- $K(X,t) \leq C_1/\|X,t\|^{n+1}$;
- $|\nabla_X K(X,t)| \leq C_2/\|X,t\|^{n+2}$;
- $|\nabla_X^2 K(X,t)|, |\partial_t K(X,t)| \leq C_2/\|X,t\|^{n+3}$,

where $|\nabla_X^2 K|$ denotes the norm of the vector of all the second order derivatives with respect to X of K. When all of the above conditions hold, we say that K is a *good kernel*.

The choice of T guarantees its L^2 boundedness over parabolic Lipschitz graphs by standard techniques and ideas in [6] (see [12, Theorem 2.1]). When dealing with singular integral operators over (M,R)-Ahlfors regular sets E, we say that T is *bounded on* $L^2(E,d\sigma)$ if the operator

$$T^* f(\mathbf{X}) = \sup_{\epsilon>0} \int_{E\cap\{\|X,t\|>\epsilon\}} K(\mathbf{X}-\mathbf{Y}) f(\mathbf{Y})\, d\sigma(\mathbf{Y}),$$

originally defined for instance on $C_0(E)$, extends as a bounded operator on $L^2(E,d\sigma)$. This convention is adopted in order to avoid issues of defining principal value operators over E.

THEOREM 1.5. *Let E be a set in $\mathbb{R}^{n+1}$ satisfying a (M,R) David–Ahlfors condition. If any operator T as described in* (8) *defines a bounded operator on $L^2(E,d\sigma)$ then E admits a parabolic Corona decomposition.*

For the proof of this theorem, while following the technique from [2, Chapters 3–5] and [3], we find it necessary to prove an adaptation of a theorem originally proved in the nonparabolic setting in [4]. The precise statement is at the end of Section 3, and its proof is provided in the last section.

Finally, it may be conjectured that Theorem 1.5 has a reciprocal, and that it may be obtained using techniques from [13], but we do not address this issue here.

2. Proof of Theorem 1.4

In order to obtain a parabolic Corona decomposition from the parabolic uniform rectifiability of E, we follow several steps and the lines of [2]. The steps are indicated as subsections.

2.1. Construction of $\mathcal{F}$, $\mathcal{B}$ and the stopping-time regions. Let $0 < \varepsilon < \delta$ be two small positive numbers so that ε/δ is also small. Let k be a large constant to be determined in the bulk of the proof of the theorem (more precisely, right after Lemma 2.6). Denote by $\mathcal{G} \equiv \mathcal{G}(\varepsilon)$ the set of cubes $Q \in \Delta$ such that there is a t-plane P_Q such that $d(\mathbf{X}, P_Q) \leq \varepsilon \operatorname{diam} Q$ for all $\mathbf{X} \in kQ$. Define now $\mathcal{B} = \Delta \setminus \mathcal{G}$ so that we already have a decomposition $\Delta = \mathcal{B} \cup \mathcal{G}$.

We now record an almost-uniqueness property for the t-planes in the previous definitions. The proof is straightforward from the definition, and details can be found in [2, p. 32].

LEMMA 2.1. *Let $Q \in \Delta$ and suppose that P_1 and P_2 are two t-planes such that $d(\mathbf{X}, P_i) \leq \varepsilon \operatorname{diam} Q$ for all $\mathbf{X} \in kQ$, $i = 1,2$. Then $d(P_1, P_2) \lesssim \varepsilon \operatorname{diam} Q$.*

Now define

$$\gamma_\infty(Z,\tau;r) = \inf_P \sup\left\{ \frac{d(Y,s;P)}{r} : (Y,s) \in \Delta_r(Z,\tau) \right\}, \tag{9}$$

where the infimum is taken over all t-planes P, and recall that in [8, p. 359] it is proved that $\gamma_\infty(Z,\tau;r)^{n+3} \leq C(n)\gamma(Z,\tau;2r)$. Then it can be easily verified that (4) in the definition of parabolic uniform rectifiability implies

$$\sum_{\substack{\gamma_\infty(Q) > \epsilon \\ Q \subseteq R}} \sigma(Q) \leq C(\epsilon)\sigma(R), \tag{10}$$

where R is any surface cube of E. Here in analogy with (9), for a dyadic cube Q we define

$$\gamma_\infty(Q) = \inf_P \sup\left\{ \frac{d(Y,s;P)}{\operatorname{diam} Q} : (Y,s) \in 2Q \right\}.$$

Estimate (10) can be proved with a well known technique of associating dyadic cubes on E with certain rectangles in $E \times (0,\infty)$. In turn, (10) implies the Carleson packing condition for $\mathcal{B}$ in (CD1) above.

Now we organize the sets in $\mathcal{G}$ and define the family $\mathcal{F}$ of regions S. Then we will obtain the remaining properties of a parabolic Corona decomposition.

For the next construction, we localize our work in a fixed surface cube $R_0 \subset E$. Let $\Delta(R_0) = \bigcup_{\Delta_j \subset R_0} \Delta_j$ and denote by $\mathcal{G}(R_0)$ the family of subcubes of R_0 in $\mathcal{G}$. Let Q_0 be a maximal cube in $\mathcal{G}(R_0)$ and let π_0 denote the projection of points in $\mathbb{R}^{n+1}$ on P_0, the t-plane that is associated to Q_0 according to the definition of $\mathcal{G}$. Define for $Q \subset Q_0$ the set $P_0(Q) = \pi_0^{-1}(\pi_0(Q))$, which is a cylindrical region perpendicular to P_0 and with level surfaces given by $\pi_0(Q)$.

Now we construct a family $K(Q_0)$ of cubes as follows:

- $Q_0 \in K(Q_0)$
- $R \in \Delta(R_0)$ is added to $K(Q_0)$ if all of the following holds:
  - R is a descendant of Q, for certain $Q \in K(Q_0)$;
  - every sibling of R, including R itself, is an element of $\mathcal{G}(R_0)$;
  - $d(P_0(R) \cap P_R; P_0(kR) \cap P_0) \leq \delta \operatorname{diam} R$.

This last inequality refers indirectly to the angle between P_0 and P_Q, as originally considered in [2]. However, we will not use specifically the angle between these planes, but rather we measure how close the portions of planes $P_0(R) \cap P_R$ and $P_0(kR) \cap P_0$ are from each other.

Now define

$$S(Q_0) = \left(\bigcup_{Q \in K(Q_0)} Q \right) \cup \left(\bigcup_{\substack{R \in \varsigma(Q) \\ Q \in K(Q_0)}} R \right)$$

and repeat the procedure inductively, choosing at each stage a maximal cube in $\mathcal{G}(R_0)$ not contained in any of the previously constructed $S(Q)$. Let $\mathcal{F}$ denote the collection of all the regions so obtained, and let us introduce the notation $Q_0 = Q(S)$ if and only if $S = S(Q_0)$.

Observe that if $S \in \mathcal{F}$ then the only options for $\mathbf{X} \in S$ is that either $\mathbf{X}$ belongs to a minimal cube, or there exists an infinite sequence of elements in S such that $\mathbf{X}$ belongs to all of the terms of the sequence.

2.2. Construction of the approximating graphs. According to (CD6), we must associate to each $S \in \mathcal{F}$ the graph of a parabolic Lipschitz function, and so in the next construction we fix $S \in \mathcal{F}$. Given $\mathbf{X} \in \mathbb{R}^{n+1}$, define

$$d(\mathbf{X}) = \inf_{Q \in S} [d(\mathbf{X}, Q) + \operatorname{diam} Q] \tag{11}$$

and let $P = P_{Q(S)}$. Let $P^{\perp}$ denote the normal vector to P pointing towards the region Ω_1 and let π and $\pi^{\perp}$ denote the canonical projections from $\mathbb{R}^{n+1}$ onto P and $P^{\perp}$, respectively. Note that $\pi^{\perp}$ might be negative by the chosen orientation of $P^{\perp}$. Define for $p \in P$

$$D(p) = \inf_{\mathbf{X} \in \pi^{-1}(p)} d(\mathbf{X}) = \inf_{Q \in S} [d(p, \pi(Q)) + \operatorname{diam} Q]. \tag{12}$$

In this subsection, we construct the graph approximating S and prove some of its basic properties, leaving the conclusion of the proof of the next lemma to Section 2.3.

LEMMA 2.2. *There exists a parabolic Lipschitz function $\psi : P \longrightarrow P^{\perp}$ with character of the order of δ, and such that for every $\mathbf{X} \in Q(S)$ one has*

$$d(\mathbf{X}; (\pi(\mathbf{X}), \psi(\pi(\mathbf{X})))) \lesssim \varepsilon d(\mathbf{X}). \tag{13}$$

To begin the construction of the graph, let $\mathcal{Z} = \{\mathbf{Z} \in E : d(\mathbf{Z}) = 0\}$ and for $\mathbf{Z} \in \mathcal{Z}$ define

$$\psi(\pi(\mathbf{Z})) = \pi^{\perp}(\mathbf{Z}).$$

LEMMA 2.3. *The function ψ is of* $\operatorname{Lip}(1, 1/2)$ *type on $\pi(\mathcal{Z})$, with constant of the order of δ.*

The proof of this lemma follows from the following result for which we have some applications later on.

LEMMA 2.4. *If $\mathbf{Z}_1, \mathbf{Z}_2 \in 2Q(S)$ satisfy $\|\mathbf{Z}_1 - \mathbf{Z}_2\| \geq 10^{-3} \min\{d(\mathbf{Z}_1), d(\mathbf{Z}_2)\}$, then*

$$|\pi^{\perp}(\mathbf{Z}_1) - \pi^{\perp}(\mathbf{Z}_2)| \lesssim 2\delta \|\pi(\mathbf{Z}_1) - \pi(\mathbf{Z}_2)\|.$$

Proof. We can assume that $\|\mathbf{Z}_1 - \mathbf{Z}_2\| \geq 10^{-3} d(\mathbf{Z}_1)$ and choose $Q \in S$ such that $d(\mathbf{Z}_1, Q) + \operatorname{diam} Q \leq d(\mathbf{Z}_1)$. In fact, we can replace it by one of its ancestors (if necessary) so that $\operatorname{diam} Q \approx \|\mathbf{Z}_1 - \mathbf{Z}_2\|$. Now, by construction, there is a t-plane P_Q such that $d(\mathbf{X}, P_Q) \le \varepsilon \operatorname{diam} Q$ for all $\mathbf{X} \in 2Q$. This implies that

$$d(\mathbf{Z}_1, P_Q) + d(\mathbf{Z}_2, P_Q) \lesssim \varepsilon \|\mathbf{Z}_1 - \mathbf{Z}_2\| < \delta \|\mathbf{Z}_1 - \mathbf{Z}_2\|.$$

On the other hand, by definition of S one has $\delta \|\mathbf{Z}_1 - \mathbf{Z}_2\| \lesssim \varepsilon\delta \operatorname{diam} Q + \delta^2 \|\pi(\mathbf{Z}_1) - \pi(\mathbf{Z}_2)\|$, which proves the lemma. □

The issue is now to define ψ outside of $\pi(\mathcal{Z})$ and for that purpose we use a Whitney-type extension. Note that $\{\mathbf{P} + \psi(\mathbf{P}) : \mathbf{P} \in \pi(\mathcal{Z})\} \subset E$, and since E is URPS then we can take advantage of the Carleson measure estimate (4) (as in [8], [9]) to conclude that this extension yields a parabolic Lipschitz function.

As observed before, over the n-dimensional t-plane P one can choose a dyadic grid of *parabolic cubes*, which after a rotation in space variables have the form

$$Q_r(z,\tau) = \{(y,s) \in P : |y_i - z_i| < r, i = 1, \dots, n-1, |s - \tau| < r^2\}.$$

Let $\mathbf{X} \in P$ be such that $D(\mathbf{X}) > 0$ and $\mathbf{X}$ is not in the boundary of any dyadic cube. Let $R_{\mathbf{X}}$ be the largest parabolic dyadic cube in P containing $\mathbf{X}$ and such that $\operatorname{diam} R_{\mathbf{X}} \le \inf\{D(\mathbf{Z}) : \mathbf{Z} \in R_{\mathbf{X}}\}/20$.

Let $\{R_i\}$, $i \in I$, be a list of these cubes without repetition. Then the R_j have pairwise disjoint interiors, they cover $P \setminus \pi(\mathcal{Z})$ and do not intersect $\pi(\mathcal{Z})$. Moreover [2, Lemma 8.7], if $10R_i \cap 10R_j \ne \emptyset$ then

$$C^{-1} \operatorname{diam} R_j \le C \operatorname{diam} R_i \le C \operatorname{diam} R_j, \tag{14}$$

which defines some sort of closeness between R_j and R_i. We are now ready to define ψ off of $\pi(\mathcal{Z})$, but still in a neighborhood of $\pi(Q(S))$.

Let $U_0 = P \cap \mathcal{C}_{2L}(\pi(\mathbf{X}_0))$, where $\mathbf{X}_0 \in Q(S)$ is fixed and from now on $L = \operatorname{diam} Q(S)$. Also define $I_0 = \{i \in I : R_i \cap U_0 \ne \emptyset\}$. For $i \in I_0$, choose $Q_i \in S$ such that

$$C_1^{-1} \operatorname{diam} R_i \le \operatorname{diam} Q_i \le C_1 \operatorname{diam} R_i, \tag{15}$$

$$d(\pi(Q_i), R_i) \le C_1 \operatorname{diam} R_i \tag{16}$$

for certain constant $C_1 > 0$. This can be done by definition of $D(p)$ for $p \in R_i$. Note that there is only a finite number of R_i for which one particular Q may correspond to, and that there is a uniform bound for this finite number, not depending on Q. For a chosen Q_i, we define $H(Q_i)$, the *average height of P_{Q_i} with respect to P over R_i* as

$$H(Q_i) = \frac{1}{2}\Big(\sup_{\substack{\mathbf{X} \in P_{Q_i} \\ \pi(\mathbf{X}) \in 2R_i}} \pi^{\perp}(\mathbf{X}) + \inf_{\substack{\mathbf{X} \in P_{Q_i} \\ \pi(\mathbf{X}) \in 2R_i}} \pi^{\perp}(\mathbf{X}) \Big).$$

Now choose a *parabolic partition of the unity* $\{v_i\}$ adapted to $\{2R_i : i \in I_0\}$. Namely, we choose the v_i with the following properties: $0 \le v_i \le 1$, $v_i \in C_0^\infty(3R_i)$, $v_i \equiv 1$ on $2R_i$ and

$$(\operatorname{diam} R_i)^\ell \left|\frac{\partial^\ell}{\partial x^\ell} v_i\right| + (\operatorname{diam} R_i)^{2\ell}\left|\frac{\partial^\ell}{\partial t^\ell} v_i\right| \le C_2, \quad \ell = 1,2,3,\ldots, \tag{17}$$

with an absolute constant $C_2 > 0$. Then we normalize the v_i so that

$$\sum v_i \equiv 1 \quad \text{on } V \equiv \bigcup_{i\in I_0} 2R_i.$$

Let $B_i : P \to P^\perp$ be the constant function defined to take the value $H(Q_i)$. Define now for $\mathbf{Y} \in V$

$$\psi(\mathbf{Y}) = \sum v_i(\mathbf{Y}) B_i(\mathbf{Y}).$$

Note that $V \cap \pi(\mathcal{Z}) = \emptyset$ and $U_0 \setminus \pi(\mathcal{Z}) \subseteq V$, and so ψ is well defined on U_0.

LEMMA 2.5. *The function ψ is of* $\mathrm{Lip}(1,1/2)$ *type on U_0 with constant of the order of δ.*

Proof. Given a pair of points in U_0, we consider two cases:
Case 1: $\mathbf{Z}_1, \mathbf{Z}_2 \in 2R_j$. In this case,

$$\begin{aligned}
&|\psi(\mathbf{Z}_1) - \psi(\mathbf{Z}_2)| \\
&\quad \le \left|\sum_i v_i(\mathbf{Z}_1)[B_i(\mathbf{Z}_1) - B_i(\mathbf{Z}_2)]\right| + \left|\sum_i [v_i(\mathbf{Z}_2) - v_i(\mathbf{Z}_1)] B_i(\mathbf{Z}_2)\right| \\
&\quad = \left|\sum_i [v_i(\mathbf{Z}_2) - v_i(\mathbf{Z}_1)][B_i(\mathbf{Z}_1) - B_j(\mathbf{Z}_2)]\right|,
\end{aligned}$$

since $B_i(\mathbf{Z}_1) - B_i(\mathbf{Z}_2) = 0$ and $\sum_i [v_i(\mathbf{Z}_2) - v_i(\mathbf{Z}_1)] = 0$. If either $v_i(\mathbf{X}) \ne 0$ or $v_i(\mathbf{Y}) \ne 0$, then we can apply (14) and obtain $\operatorname{diam} R_i \approx \operatorname{diam} R_j$. Using the notations $\mathbf{Z}_1 = (x,t)$, $\mathbf{Z}_2 = (y,s)$ we obtain by (17)

$$\begin{aligned}
|v_i(\mathbf{Z}_1) - v_i(\mathbf{Z}_2)| &\le \frac{C}{(\operatorname{diam} R_j)^2}|s-t| + \frac{C}{\operatorname{diam} R_j}|x-y| \\
&\le \frac{C}{\operatorname{diam} R_j}\left[\frac{|s-t|}{\|\mathbf{Z}_1 - \mathbf{Z}_2\|} + |x-y|\right] \\
&\le C(\operatorname{diam} R_j)^{-1}\|\mathbf{Z}_1 - \mathbf{Z}_2\|.
\end{aligned}$$

CLAIM. *If $10R_i \cap 10R_j \ne \emptyset$, then $d(Q_i,Q_j) \lesssim \operatorname{diam} R_j$, and in consequence $|B_i(\mathbf{Z}) - B_j(\mathbf{Z})| \lesssim \varepsilon \operatorname{diam} R_j$ for $\mathbf{Z} \in 2R_j$.*

To prove this claim, we first pick $\mathbf{X} \in Q_j$, $\mathbf{Y} \in Q_i$ and assume $\|\mathbf{X} - \mathbf{Y}\| \ge 3^{-1} \operatorname{diam} Q_j$. Since $d(\mathbf{X}) \le \operatorname{diam} Q_j$ by definition, then Lemma 2.4 may be applied and we get $|\pi^\perp(\mathbf{X}) - \pi^\perp(\mathbf{Y})| \le \|\pi(\mathbf{X}) - \pi(\mathbf{Y})\| \lesssim \operatorname{diam} R_j$. Hence, $d(Q_i,Q_j) \le \|\mathbf{X} - \mathbf{Y}\| \lesssim \operatorname{diam} R_j$. This already implies the first part of the claim.

For the second conclusion of the claim, we take $\mathbf{Z} \in 2R_j$ and denote by $\mathbf{Z}_j$ and $\mathbf{Z}_i$ the points in P_{Q_j} and P_{Q_i} such that $\pi(\mathbf{Z}_j) = \mathbf{Z} = \pi(\mathbf{Z}_i)$. Then by Lemma 2.1, (15), (16) and (14)

$$|\mathbf{Z}_j - \mathbf{Z}_i| \lesssim \varepsilon[\operatorname{diam} Q_j + \operatorname{diam} Q_i] \lesssim \varepsilon \operatorname{diam} R_j.$$

Now with the Claim at hand, we conclude: If $\mathbf{Z}_1, \mathbf{Z}_2 \in 2R_j$, then $|\psi(\mathbf{Z}_1) - \psi(\mathbf{Z}_2)| \leq 2\delta\|\mathbf{Z}_1 - \mathbf{Z}_2\|$ whenever $\varepsilon \ll \delta$. Of course at some point, we used the fact that there is a bounded number of i such that $v_i(\mathbf{Z}_1) - v_i(\mathbf{Z}_2) \neq 0$.

Case 2: $\mathbf{Z}_0 \in \pi(\mathcal{Z})$, $\mathbf{Z}_1 \in \bigcup_{j \in I_0} R_j$. Choose j such that $\mathbf{Z}_1 \in R_j$ and fix $\mathbf{Y} \in Q_j$. We have

$$\begin{aligned} |\psi(\mathbf{Z}_1) - \psi(\mathbf{Z}_2)| &\leq |\psi(\mathbf{Z}_1) - B_j(\mathbf{Z}_1)| + |B_j(\mathbf{Z}_1) - B_j(\pi(\mathbf{Y}))| \\ &\quad + |B_j(\pi(\mathbf{Y})) - \pi^\perp(\mathbf{Y})| + |\pi^\perp(\mathbf{Y}) - \psi(\mathbf{Z}_0)| \\ &\equiv L_1 + L_2 + L_3 + L_4. \end{aligned}$$

Observe that since B_j is a constant function then $L_1 = L_2 = 0$. Now by definition of ψ and the construction of the stopping-time regions $L_3 \lesssim \varepsilon \operatorname{diam} Q_j \lesssim \varepsilon\|\mathbf{Z}_1 - \mathbf{Z}_0\| \lesssim \delta\|\mathbf{Z}_1 - \mathbf{Z}_0\|$. Finally, by Lemma 2.4, letting $\mathbf{X} = (\mathbf{Z}_0, \psi(\mathbf{Z}_0))$ we obtain $L_4 = |\pi^\perp(\mathbf{Y}) - \pi^\perp(\mathbf{X})| \lesssim \delta|\pi(\mathbf{Y}) - \mathbf{Z}_0| \lesssim \delta|\mathbf{Z}_1 - \mathbf{Z}_0|$ by the very definition of Q_i. The lemma is proved. □

Again by a Whitney type argument, ψ can be extended to P as a $\operatorname{Lip}(1, 1/2)$ function with constant of the order of δ off of U_0. The remaining properties of ψ are established in the next subsection.

2.3. Verification of the properties of the graph. First, we prove the estimate (13). If $\mathbf{Z} \in \mathcal{Z}$, then (13) is clear. To continue the proof, we need the following lemma, which can be proved as in [2, p. 48] with small variations.

LEMMA 2.6. *Let $\mathbf{Y} \in U_0$ and $r > 0$ be such that $D(\mathbf{Y}) \leq r \leq L$. Suppose $Q \in S$ is such that $d(\mathbf{Y}, \pi(Q)) \leq Cr$ and $C^{-1}r \leq \operatorname{diam} Q \leq Cr$. Then $\pi^{-1}(B_r(\mathbf{Y})) \cap Q(S)$ is contained in $C_0 Q$ for certain C_0 that may depend on C. Also there is a constant $\widetilde{C}$ that depends only on k such that $\widetilde{C}^{-1} d(\mathbf{X}) \leq D(\pi(\mathbf{X})) \leq \widetilde{C} d(\mathbf{X})$ for all $\mathbf{X} \in Q(S)$*

With the lemma at hand, let $\mathbf{X} \in Q(S)$ be such that $d(\mathbf{X}) > 0$, and let $\mathbf{X}_0 = \pi(\mathbf{X})$. From the Lemma 2.6, we know $D(\mathbf{X}_0) > 0$ and hence $\mathbf{X}_0 \in R_i$ for some i. We apply Lemma 2.6 with $r = D(\mathbf{X}_0)$ and $Q = Q_i$, thus obtaining $\mathbf{X} \in C_0 Q_i$. Choosing $k \gg C_0$ one gets $|\pi^\perp(\mathbf{X}) - B_i(\pi(\mathbf{X}))| \leq 2\varepsilon \operatorname{diam} Q_i \leq 2C\varepsilon D(\mathbf{X}_0) \leq 2C^2\varepsilon d(\mathbf{X})$. Also note that $|B_i(\pi(\mathbf{X})) - \psi(\pi(\mathbf{X}))| \lesssim \varepsilon D(\mathbf{X}_0)$. This implies the estimate (13) by definition of D and R_i, and by the Claim above.

To finish the proof of Proposition 2.2, it remains to prove that ψ is a parabolic Lipschitz function. Denote the graph of ψ by Ψ, and let γ', ν' be the objects corresponding to Ψ as in Definition 1.1. We first point out another consequence of the construction made so far.

LEMMA 2.7. *Let* $\mathbf{X} \in E$. *If* $\mathbf{Y} \in \Psi \cap \mathcal{C}_{10L}(\mathbf{X})$ *then for certain* $\hat{c} > 0$

$$d(\mathbf{Y}; E) \leq \hat{c} d(\pi(\mathbf{Y}); \pi(\mathcal{Z})).$$

Proof. The result is trivial if $\mathbf{Y} \in \mathcal{Z}$. Now from the definitions if $\mathbf{Y} = (\psi(y,s), y, s)$, with $(y,s) \in R_i$ then $d(\mathbf{Y}; E) \lesssim \varepsilon \operatorname{diam} Q_i \lesssim \operatorname{diam} R_i \lesssim d(\pi(\mathbf{Y}); \pi(\mathcal{Z}))$ □

If we fix $(\hat{Z}, \hat{\tau}) \in \Psi$, with $\hat{\rho} > 0$, we have to obtain the estimate

$$\nu'\big(\mathcal{C}_{\hat{\rho}}(\hat{Z}, \hat{\tau}) \cap \Psi \times (0, \hat{\rho})\big) \lesssim \hat{\rho}^{n+1}. \tag{18}$$

Recall now that all of the above construction was made within a fixed rectangle $R_0 \subset E$. Let ρ denote the diameter of R_0. If $\hat{\rho} > \rho$ then

$$\gamma'\big(\hat{Z}, \hat{\tau}; \hat{\rho}\big) \lesssim \left(\frac{\rho}{\hat{\rho}}\right)^{n+3}$$

and so the Ahlfors–David condition (3) implies (18).

Now for $\hat{\rho} \leq \rho$, we separate the proof into the two possible scenarios (compare to [8, pp. 365–368]) which will give as a result the estimate

$$\nu'\big(\mathcal{Z} \cap \mathcal{C}_{\hat{\rho}}(\hat{Z}, \hat{\tau}) \times (0, \hat{\rho})\big) \lesssim (1 + \|\nu\|_+)\hat{\rho}^{n+1}. \tag{19}$$

2.3.1. *Estimate in* $\mathcal{Z}$. Given a fixed portion of the graph

$$\Psi_i = \{(\psi(y,s), y, s) \in \Psi : (y,s) \in R_i\},$$

we would like to find a decomposition of it by using surface balls centered at points in E. Set $\tilde{c} = 2\hat{c}$, where $\hat{c}$ is the constant of Lemma 2.7, and let $\rho_i = \operatorname{diam} R_i$. By the Lemma 2.7, we can find cylinders $\{\mathcal{C}_{\tilde{c}\rho_i}(Z_j, \tau_j)\}_j$, with $(Z_j, \tau_j) \in E$, such that they form a covering of $\overline{\Psi}_i$, and $\{\mathcal{C}_{\tilde{c}\rho_i/5}(Z_j, \tau_j\}_j$ are pairwise disjoint. Observe that every point in $\bigcup_j \{\mathcal{C}_{\tilde{c}\rho_i}(Z_j, \tau_j\}$ lies in no more than a uniform number of cylinders.

Define now $\Psi_{i,j} = \Psi_i \cap \mathcal{C}_{\tilde{c}\rho_i}(Z_j, \tau_j)$ and $E_i = \{(Y,s) \in E : d(Y,s;\Psi_i) < 2\tilde{c}\rho_i\}$. Let $\mathcal{P}$ be any t-plane. Note that if $(\hat{X}, \hat{t}) \in \mathcal{Z}$ and $r > 0$ is such that $\mathcal{C}_r(\hat{X}, \hat{t}) \subset \mathcal{C}_{\hat{\rho}}(\hat{Z}, \hat{\tau})$ then we have

$$\int_{\Psi \cap \mathcal{C}_r(\hat{X}, \hat{t})} d(Y,s;\mathcal{P})^2 \, d\sigma(Y,s) \lesssim \sum_i \sum_j \int_{\Psi_{i,j} \cap \mathcal{C}_r(\hat{X}, \hat{t})} d(Y,s;\mathcal{P})^2 \, d\sigma(Y,s),$$

where the sum on i runs over $\mathfrak{I}(\hat{X}, \hat{t}; r) = \{i : \Psi_i \cap \mathcal{C}_r(\hat{X}, \hat{t}) \neq \emptyset\}$.

For $(Y,s) \in \overline{\Psi}_{i,j}$, we have $d(Y,s;\mathcal{P}) \lesssim \rho_i + \inf\{d(Z,\tau;\mathcal{P}) : (Z,\tau) \in \Delta_{\tilde{c}\rho_i}(Z_j,\tau_j)\}$, hence

$$\begin{aligned}
&\sum_j \int_{\Psi_{i,j}\cap\mathcal{C}_r(\hat{X},\hat{t})} d(Y,s;\mathcal{P})^2 \, d\sigma(Y,s) \\
&\quad\lesssim \rho_i \sigma\big(\Psi_i \cap \mathcal{C}_r(\hat{X},\hat{t})\big) + \sum_j \int_{\mathcal{C}_{5r/4}(\hat{X},\hat{t})\cap\Delta_{\tilde{c}\rho_i}(Z_j,\tau_j)} d(Y,s;\mathcal{P})^2 \, d\sigma(Y,s) \\
&\quad\lesssim \left[\rho_i^{n+3} + \int_{\mathcal{C}_{5r/4}(\hat{X},\hat{t})\cap E_i} d(Y,s;\mathcal{P})^2 \, d\sigma(Y,s)\right].
\end{aligned}$$

Observe that the E_i have uniformly bounded overlap, and so summing over i we obtain

$$\gamma'(\hat{X},\hat{t};r) \lesssim \left[\sum_{i\in\mathfrak{I}(\hat{X},\hat{t};r)} \left(\frac{\rho_i}{r}\right)^{n+3} + \gamma(\hat{X},\hat{t};5r/4)\right]. \tag{20}$$

Integrating over $\mathcal{C}_{\hat{\rho}}(\hat{Z},\hat{\tau}) \cap \mathcal{Z}$, and with a change of order of the integrals

$$\begin{aligned}
&\nu'\big(\mathcal{Z} \cap \mathcal{C}_{\hat{\rho}}(\hat{Z},\hat{\tau}) \times (0,\hat{\rho})\big) \\
&\quad= \int_0^{\hat{\rho}} \int_{\mathcal{Z}\cap\mathcal{C}_{\hat{\rho}}(\hat{Z},\hat{\tau})} \gamma'(\hat{X},\hat{t};r) \, d\sigma(\hat{X},\hat{t}) \frac{dr}{r} \\
&\quad\approx \int_{\mathcal{Z}\cap\mathcal{C}_{\hat{\rho}}(\hat{Z},\hat{\tau})} \left(\sum_{i\in\mathfrak{I}(\hat{X},\hat{t};r)} \int_{r_i(\hat{X},\hat{t})}^{\hat{\rho}} \left(\frac{\rho_i}{r}\right)^{n+3} \frac{dr}{r}\right) d\sigma(\hat{X},\hat{t}) \\
&\qquad + \nu\big(\mathcal{Z} \cap \mathcal{C}_{\hat{\rho}}(\hat{Z},\hat{\tau}) \times (0,\hat{\rho})\big),
\end{aligned}$$

where $r_i(\hat{X},\hat{t}) = d(\hat{X},\hat{t};\Psi_i)$. The last quantity is estimated as follows:

$$\begin{aligned}
&\lesssim \sum_{i\in(\hat{Z},\hat{\tau};\hat{\rho})} \int_{\mathcal{Z}\cap\mathcal{C}_{\hat{\rho}}(\hat{Z},\hat{\tau})} \left(\frac{\rho_i}{r_i(\hat{X},\hat{t})}\right)^{n+3} d\sigma(\hat{X},\hat{t}) + C\|\nu\|_+ \hat{\rho}^{n+1} \\
&\lesssim \sum_{i\in\mathfrak{I}(\hat{Z},\hat{\tau};\hat{\rho})} \rho_i^{n+1} + C\|\nu\|_+ \hat{\rho}^{n+1} \\
&\lesssim (1+\|\nu\|_+)\hat{\rho}^{n+1}.
\end{aligned}$$

2.3.2. *Estimate in $E \setminus \mathcal{Z}$.* For $r > 0$ and $(\hat{x},\hat{t}) \in P$ define

$$\kappa(\hat{x},\hat{t};r) = \frac{1}{r^{n+3}} \left[\inf_L \int_{Q_r(\hat{x},\hat{t})} |\psi(y,s) - L(y)|^2 \, dy \, ds\right], \tag{21}$$

where the infimum is taken over linear functions L of the y variable only. Also recall that $Q_r(\hat{x},\hat{t})$ denotes a cube on P centered at $(\hat{x},\hat{t})$ with radius $r > 0$. We clearly have

$$c^{-1}\kappa(\hat{x},\hat{t};r) \le \gamma'(\psi(\hat{x},\hat{t}),\hat{x},\hat{t};r) \le c\kappa(\hat{x},\hat{t};r) \tag{22}$$

for certain constant c that may depend on the Ahlfors–David constant M of E, and the Lip$(1,1/2)$ constant of ψ. This, along with Taylor's theorem, imply that for $0<r\le 2\rho_i/3$ and $(\hat{x},\hat{t})\in \overline{R}_i$ one has $\kappa(\hat{x},\hat{t};r)\lesssim r^2/\rho_i^2$. Next, observe that if $2\rho_i/3<r\le 8\rho_i$ then the inequality $\kappa(\hat{x},\hat{t};r)\lesssim r^2/\rho_i^2$ still holds. Altogether, we obtain

$$\nu'\big(\overline{\Psi}_i\times(0,8\rho_i)\big)\lesssim \rho_i^{n+1}. \tag{23}$$

We still need to consider the case $r>8\rho_i$. By the Whitney property of R_i and (22), we can choose $(x_i',t_i')\in\pi(\mathcal{Z})$ such that $\operatorname{diam} R_i\approx d(R_i;x_i',t_i')$, and also $\gamma'(\hat{X},\hat{t};r)\lesssim \gamma'(X_i',t_i';3r/2)$ for $(\hat{x},\hat{t})\in R_i$. Let $\hat{X}=\psi(\hat{x},\hat{t})$, $X_i'=\psi(x_i',t_i')$. Then for $(\hat{X},\hat{t})\in\overline{\Psi}_i$, after integrating we obtain

$$\begin{aligned}\int_{8\rho_i}^{\hat{\rho}}\gamma'(\hat{X},\hat{t};r)\frac{dr}{r} &\lesssim \int_{8\rho_i}^{\hat{\rho}}\gamma'(X_i',t_i':3r/2)\frac{dr}{r}+\int_{2\rho_i}^{\hat{\rho}}\sum_{j\in\Im(X_i',t_i';2r)}\left(\frac{\rho_j}{r}\right)^{n+3}\frac{dr}{r}\\ &\lesssim \|\nu\|_+ + \sum_{j\in\Im(X_i',t_i';2r)}\left(\frac{\rho_j}{\rho_i+\rho_j+d(R_i,R_j)}\right)^{n+3},\end{aligned}$$

where in the first inequality we have used the argument that yields (20).

It remains to integrate over $\overline{\Psi}_i$ and sum over i:

$$\begin{aligned}&\sum_{i\in\Im(\hat{Z},\hat{\tau};\hat{\rho})}\int_{\overline{\Psi}_i}\int_{8\rho_i}^{\hat{\rho}}\gamma'(\hat{X},\hat{t};r)\frac{dr}{r}\,d\sigma(\hat{X},\hat{t})\\ &\quad\lesssim \|\nu\|_+\hat{\rho}^{n+1}+\sum_{j\in\Im(\hat{Z},\hat{t};2\hat{\rho})}\rho_j^{n+3}\int_P\big(\rho_j+d(y,s;x_j't_j')\big)^{-(n+3)}\,dy\,ds\\ &\quad\lesssim\Bigg[\|\nu\|_+\hat{\rho}^{n+1}+\sum_{j\in\Im(\hat{Z},\hat{t};2\hat{\rho})}\rho_j^{n+1}\Bigg]\lesssim(1+\|\nu\|_+)\hat{\rho}^{n+1}.\end{aligned}$$

This along with (23) yields (19), which is the desired estimate.

2.4. Carleson packing condition for the stopping-time regions. To prove (CD5) in the definition of the parabolic Corona decomposition it suffices to prove the following lemma.

LEMMA 2.8. *For every $R\in\Delta$*

$$\sum_{\substack{S\in\mathcal{F}\\ Q(S)\subseteq R}}\sigma(Q(S))\le C\sigma(R). \tag{24}$$

The next paragraphs contain the proof of this lemma.

Fixing $S=S(Q_0)$, let $m(S)$ denote the set of minimal cubes of S and let $\mathcal{U}$ denote the union of all the cubes in $m(S)$. We separate $m(S)$ into two families, according to the options that any minimal cube R has: either $R\in K(S)$ and at least one descendant of R is in $\mathcal{B}(R_0)$; or $R\in S\setminus K(S)$.

Let $m_1(S)$ denote the set of minimal cubes with at least one descendant in $\mathcal{B}(R_0)$, and let $m_2(S)$ denote the set of minimal cubes $R \in S \setminus K(S)$ with $d(P_R \cap P_0(R); P_{Q_0} \cap P_0(kR)) > \delta \operatorname{diam} R$ (recall definitions in page 540). Also we set

$$\mathcal{U}_i = \mathcal{U}_i(S) \equiv \bigcup_{Q \in m_i(S)} Q, \quad i = 1, 2.$$

Accordingly, we have three types of regions:

$$\mathcal{I} = \{S \in \mathcal{F} : \sigma\big(Q(S) \setminus \mathcal{U}\big) \geq \theta\sigma(Q(S))\},$$
$$\mathcal{II} = \{S \in \mathcal{F} : \sigma(\mathcal{U}_1) \geq \theta\sigma(Q(S))\}$$

and

$$\mathcal{III} = \{S \in \mathcal{F} : \sigma(\mathcal{U}_2) \geq \theta\sigma(Q(S))\}.$$

Condition (24) for the class $\mathcal{I}$ is essentially due to the disjointneess of the different S in $\mathcal{F}$. For the class $\mathcal{II}$, one can use the Carleson packing condition for $\mathcal{B}(R_0)$ to obtain (24). Indeed,

$$\sum_{\substack{S \in \mathcal{II} \\ Q(S) \subseteq R}} \sigma(Q(S)) \leq \frac{1}{\theta} \sum_{\substack{S \in \mathcal{II} \\ Q(S) \subseteq R}} \sum_{Q \in m_1(S)} \sigma(Q) \leq \frac{1}{\theta} \sum_{\substack{S \in \mathcal{II} \\ Q(S) \subseteq R}} \sum_{Q \in m_1(S)} \sigma(B_Q) \lesssim \sigma(R),$$

where B_Q denotes one of the descendants of Q in the class $\mathcal{B}(R_0)$, and in the last inequality we use (CD1).

In order to prove (24) for class $\mathcal{III}$, we use the approximating graphs, as we now describe. Let S be one of the stopping time regions in $\mathcal{III}$, and set

$$\mathcal{X} = \{(Z, \tau; r) \in E \times \mathbb{R}_+ : (Z, \tau) \in k_0 Q(S), k_0^{-1}\delta(Z, \tau) \leq r \leq k_0 L\},$$

where k_0 is to be chosen. Define $\delta(Z, \tau) = \inf_{Q \in S}\{d(Z, \tau; Q) + \operatorname{diam} Q\}$ and as before let $L = \operatorname{diam} Q(S)$.

Observe that if we prove that for some $\eta > 0$ that may depend on appropriately chosen parameters ϵ, δ and k_0, but not on S, one has

$$\iint_{\mathcal{X}} \gamma(Z, \tau; k_0 r)\, d\sigma(Z, \tau) \frac{dr}{r} \geq \eta\sigma(Q(S)), \tag{25}$$

then we can obtain

$$\begin{aligned}\sum_{\substack{S \in \mathcal{III} \\ Q(S) \subseteq R}} \sigma(Q(S)) &\leq \frac{1}{\eta} \sum_{S \in \mathcal{III}} \iint_{\mathcal{X}} \gamma(Z, \tau; k_0 r)\, d\sigma(Z, \tau) \frac{dr}{r} \\ &\lesssim \int_0^{k_0 \operatorname{diam} R} \int_{k_0 R} \gamma(Z, \tau; k_0 r)\, d\sigma(Z, \tau) \frac{dr}{r} \leq C\sigma(R)\end{aligned}$$

by the parabolic uniform rectifiability, and with $C > 0$ depending on k_0. This is the desired estimate.

Now in order to prove (25) it suffices to prove that if the estimate

$$\iint_{\mathcal{X}} \gamma(Z,\tau;k_0 r)\, d\sigma(Z,\tau)\frac{dr}{r} \leq \eta\sigma(Q(S)) \tag{26}$$

holds for every $\eta > 0$ then $S \notin \mathcal{III}$, which of course is a contradiction.

Suppose then that (26) holds. Recall that $\{R_i : i \in I\}$ is an enumeration of the rectangles in Section 2.2, and that $U_0 = P \cap \mathcal{C}_L(\pi(\mathbf{X}_0))$ for certain $\mathbf{X}_0 \in Q(S)$. Define now $U_2 = P \cap \mathcal{C}_{L/2}(\pi(\mathbf{X}_0))$, and $I_2 = \{i \in I : R_i \cap U_2\}$. In the next lemma, we establish the conditions for the choice of the constant k_0.

Lemma 2.9. *If k_0 is large enough and κ is as defined in (21), then*

$$\frac{1}{L^{n+1}}\int_0^L \int_{U_2} \kappa(p;k_0 r)\, dp\frac{dr}{r} \leq C\left[\varepsilon + \frac{1}{L^{n+1}}\iint_{\mathcal{X}} \gamma(Z,\tau;k_0 r)\, d\sigma(Z,\tau)\frac{dr}{r}\right],$$

where C does not depend on ε or δ.

Once Lemma 2.9 is established, under the assumption (26) we obtain for every $\eta > 0$

$$\int_0^L \int_{U_2} \kappa(p;k_0 r)\, dp\frac{dr}{r} \leq C\eta^2\sigma(Q(S)). \tag{27}$$

Lemma 2.10. *Suppose that for certain $S \in \mathcal{F}$ the estimate (27) holds. Then $S \notin \mathcal{III}$.*

After proving the technical Lemmata 2.9 and 2.10, the proof of all the steps that establish that E has a parabolic Corona decomposition will be finished.

2.5. Proof of the technical Lemmata 2.9 and 2.10. We keep notations and constructions introduced in the previous paragraph. For the proof of Lemma 2.9, we note that the arguments in [2, Lemma 13.4] are adaptable with small changes due to the definition of ψ and of the functional γ. While we will omit these details, we provide a fairly complete argument that yields Lemma 2.10. This proof is an adaptation from the original arguments in [2, Chapters 10, 11 and 14].

Let P_0 and $P_0^\perp$ denote the translates of P and $P^\perp$ (resp.) to the origin. We denote the variables and points on the plane P (or P_0) with lowercase letters p, q, etc. Choose $\varphi \in C_0^\infty$, defined on P_0, satisfying the following:

- φ is supported in $\mathcal{C}_{1/20}(\vec{0})$ and $\varphi \equiv 0$ on $\mathcal{C}_{1/100}(\vec{0})$,
- $\varphi(\lambda x, \lambda^2 t) = \varphi(x,t)$ for $(x,t) \in P_0$,
- the following *spatial moment condition* holds: for every polynomial F, in space variable only, and of degree at most one, one has

$$\int_P \varphi F\, dp = 0.$$

We denote by $\varphi * \psi$ the function defined on P as

$$\varphi * \psi(p) = \int_P \varphi(p-q)\psi(q)\,dq.$$

By the moment condition, for any linear function G of the y variable only the following holds:

$$(\varphi_\lambda * \psi)^2(p_0) = [\varphi_\lambda * (\psi - G)]^2(p_0) \lesssim \|\varphi\|_\infty \lambda^{-(n+1)} \int_{Q_\lambda(p_0)} |\psi - G|^2\,dp,$$

where as before $Q_\lambda(p_0)$ denotes a cube centered at p_0 of radius λ, over the plane P. Hence, by (21) and (22) and the assumption (27)

$$\int_V \int_0^L |\varphi_\lambda * \psi(p)|^2 \frac{d\lambda}{\lambda^3}\,dp \lesssim \eta^2 \sigma(Q(S)), \tag{28}$$

where $V = \{\mathbf{Z} \in P : d(\mathbf{Z}; \pi(Q(S))) \leq 2L\}$.

Now write $\psi = \psi_1 + \psi_2$, where

$$\psi_1(p) = \int_L^\infty \varphi_\lambda * \varphi_\lambda * \psi(p) \frac{d\lambda}{\lambda} + \int_0^L \int_{P \setminus V} \varphi_\lambda(p-q)\varphi_\lambda * \psi(q)\,dq \frac{d\lambda}{\lambda},$$

$$\psi_2(p) = \int_0^L \int_V \varphi_\lambda(p-q)\varphi_\lambda * \psi(q)\,dq \frac{d\lambda}{\lambda}.$$

For further future analysis of ψ_2, we define

$$m_{\mathcal{Q}}(\psi_2) = \frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}} \psi_2(p)\,dp$$

and the maximal operator

$$\mathcal{M}\psi_2(p) = \sup_{\mathcal{Q}} \frac{1}{|\mathcal{Q}|^{1/(n+1)}} \left(\frac{1}{|\mathcal{Q}|} \int_{\mathcal{Q}} |\psi_2 - m_{\mathcal{Q}}(\psi_2)| \right),$$

where the supremum is taken over cubes $\mathcal{Q}$ in P containing p and of the form $Q_\rho(x,t) \equiv \{(z,\tau) \in P : |z-x| + |\tau - t|^{1/2} < \rho\}$, for certain $(x,t) \in P$ and $\rho > 0$.

Set $V_j = \{p \in P : d(p; Q(S)) \leq 30L/2^j\}$. Let $\widetilde{\mathcal{Q}} \equiv Q_r(p_0)$ be a cube contained in V_1, and suppose that $r < r_0 L$, where r_0 is small and to be chosen later. Let $H_{\widetilde{\mathcal{Q}}}$ be the graph of the function $\psi_{\widetilde{\mathcal{Q}}}(p) = \psi(p_0) + \nabla\psi_1(p_0) \cdot (p - p_0)$. Our ultimate goal is now to prove

$$\sup_{\mathbf{X} \in \Psi \cap \pi^{-1}(\widetilde{\mathcal{Q}})} \left[\frac{1}{r} d(\mathbf{X}; H_{\widetilde{\mathcal{Q}}}) \right] \leq (1 + r_0) r\delta. \tag{29}$$

Define $F = \{p \in V_2 : \mathcal{M}\psi_2(p) \leq \delta\}$ and let $(1+r_0)r\delta \lesssim \xi r\delta$, where ξ is now the one to be chosen.

Claim. *If $Q \in m_2(S)$ and ξ is small enough, then $d(\pi(Q); F) > \operatorname{diam} Q$.*

Assuming this claim momentarily, we proceed with the following lemma.

Proof of Lemma 2.10. Let $Q\in m_2(S)$ and pick $x_Q\in Q$. Consider $\mathcal{C}_Q\equiv \mathcal{C}_{k_2 \operatorname{diam} Q}(x_Q)$. If k_2 is appropriately large, then certain family $\{\mathcal{C}_Q : Q\in T\}$ of these cubes cover $\mathcal{U}_2$ and $\{\mathcal{C}_{3\operatorname{diam} Q}(x_Q) : Q\in T\}$ are pairwise disjoint. Hence,

$$\sigma(\mathcal{U}_2)\leq \sum_{Q\in T}\sigma(E\cap \mathcal{C}_Q)\lesssim \sum(\operatorname{diam} Q)^{n+1}.$$

Now by Lemma 2.4, the sets $D_Q=P\cap \mathcal{C}_{\operatorname{diam} Q}(\pi(x_Q))$, for $Q\in T$, are pairwise disjoint. In particular, $|\bigcup_{Q\in T} D_Q|\geq \mathcal{U}_2/k_2$. But $D_Q\subset V_2$ and $D_Q\cap F=\emptyset$ for all $Q\in T$ by the claim, and therefore from (33) we obtain

$$\sigma(\mathcal{U}_2)\leq c\left|\bigcup_{Q\in T} D_Q\right|\leq c|V_2\setminus F|\leq \frac{\tilde{c}}{\delta}\sigma(Q(S)).$$

This yields $\sigma(\mathcal{U}_2)\leq \sigma(Q(S))/3$, and so $S\notin \mathcal{III}$. $\square$

In order to prove the claim (using estimate (29)), take $\mathbf{X}_Q\in Q$, and set $p_0=\pi(\mathbf{X}_Q)$ and $\widetilde{\mathcal{Q}}\equiv Q_r(p_0)$, with $r=10\operatorname{diam} Q$. Take now $\mathbf{X}\in Q$ and note that the previous constructions imply

$$\|\mathbf{X}-(\pi(\mathbf{X}),\psi(\pi(\mathbf{X})))\|\leq c\varepsilon \operatorname{diam} Q.$$

Since $\widetilde{\mathcal{Q}}\cap F\neq\emptyset$, by (29), we obtain for ε small $d(\mathbf{X};H_{\widetilde{\mathcal{Q}}})<c\xi\delta r$. Thus, we have two t-planes close to Q, and so by Lemma 2.1 we have $d(H_{\widetilde{\mathcal{Q}}};P_Q)\lesssim (\varepsilon+\xi\delta)\operatorname{diam} Q$. In either case, by choosing ξ and ε appropriately such that $(\varepsilon+\xi\delta)\ll\delta$, we obtain $d(H_{\widetilde{\mathcal{Q}}};P)\gtrsim \delta\operatorname{diam} Q$.

LEMMA 2.11. *Given $\rho>0$ and $M>0$ there exists $\varepsilon_0>0$ such that if $\varepsilon<\varepsilon_0$ then for every $Q\in S$ with $\operatorname{diam} Q\geq \rho L/M$ one has $d(P_Q\cap P_0(Q);P\cap P_0(Q))<\delta\operatorname{diam} Q$.*

Proof. Consider the chain $Q<Q_1<Q_2<\cdots<Q_T=Q(S)$, where $Q_j<Q_{j+1}$ means that $Q_j\subset Q_{j+1}$ and $\operatorname{diam} Q_{j+1}\approx 2\operatorname{diam} Q_j$. Since $Q_j\in\mathcal{G}$ then $d(P_{Q_j};P_{j+1})\leq C\varepsilon\operatorname{diam} Q$ and so $d(P_Q;P)\leq CT\varepsilon\operatorname{diam} Q$. Since

$$T\leq \log\left(\frac{2\operatorname{diam} Q(S)}{\operatorname{diam} Q}\right)\leq \log\left(\frac{2M}{r_0}\right)$$

then $d(P_Q\cap P_0(Q);P_{Q_0}\cap P_0(Q))\leq C\varepsilon\operatorname{diam} Q\log(2M/r_0)$, and if ε is suitable small we obtain $d(P_Q\cap P_0(Q);P_{Q_0}\cap P_0(Q))\leq C\delta\operatorname{diam} Q$. $\square$

Now we can finish the proof of the claim. Indeed, suppose that the claim is false, that is, suppose $\operatorname{diam} Q\geq d(\pi(Q);F)$ for $Q\in m_2(S)$. Observe that for ρ and ε small enough, we must have $\operatorname{diam} Q\leq \rho L$, since otherwise the lemma would imply $Q\notin m_2(S)$.

Now choose Q^* the largest ancestor of Q such that $10\operatorname{diam} Q^*\leq\rho$. Let $\widetilde{\mathcal{Q}}^*=Q_{10\operatorname{diam} Q^*}(p_0)$. By repeating the constructions above, we can obtain $d(P_{Q^*},H_{\widetilde{\mathcal{Q}}^*})<C\delta\operatorname{diam} Q^*$. Note also that $H_{\widetilde{\mathcal{Q}}^*}=H_{\widetilde{\mathcal{Q}}}$, since their definitions

depend only on p_0, and so $d(P_{Q^*}; P_Q) < C\delta \operatorname{diam} Q$. This already implies $d(P_Q \cap P_0(Q); P \cap P_0(Q)) < \delta \operatorname{diam} Q$, with ε even smaller. This again implies $Q \notin m_2(S)$, hence obtaining a contradiction that proves the Claim.

From now on, we focus on the proof of (29), which is the only remaining estimate to be proved. First, observe that for $f \in L^2(P)$ the following *reproducing formula* holds

$$f \approx \int_0^\infty \varphi_\lambda * \varphi_\lambda * f \frac{d\lambda}{\lambda}$$

in a distributional sense, and with constant independent of f. Moreover, φ can be chosen to be equal to $\lambda \partial W_\lambda / \partial \lambda$ for $W \in C_0^\infty(P)$, nonnegative, supported in $Q_1(\vec{0})$, and with $\int W = 1$. In this case, we even have $\|\varphi\|_\infty < \omega$ for an absolute constant ω.

Choose Φ such that

$$\Phi_h = \int_{\lambda > L} \varphi_\lambda * \varphi_\lambda \frac{d\lambda}{\lambda}.$$

Then with an appropriate choice of h (by the reproducing formula)

$$\Phi = \int_{\lambda > 1} \varphi_\lambda * \varphi_\lambda \frac{d\lambda}{\lambda} = \delta_0 - \int_{\lambda \leq 1} \varphi_\lambda * \varphi_\lambda \frac{d\lambda}{\lambda}, \tag{30}$$

where δ_0 denotes the Dirac δ mass at 0.

Estimates for ψ_1. Set $\psi_1 = \psi_{11} + \psi_{12}$, where

$$\begin{aligned}
\psi_{11}(p) &\equiv \int_L^\infty \varphi_\lambda * \varphi_\lambda * \psi(p) \frac{d\lambda}{\lambda}, \\
\psi_{12}(p) &= \int_0^L \int_{P \setminus V} \varphi_\lambda(p - q) \varphi_\lambda * \psi(q)\, dq \frac{d\lambda}{\lambda}.
\end{aligned}$$

By properties of ψ and φ, one may obtain

$$|\nabla \psi_{12}| \leq C, \qquad |\nabla^2 \psi_{12}| \leq C/L \quad \text{on } V_1. \tag{31}$$

Good estimates for Φ and formula (30) imply that $\psi_{11} = \Phi_h * \psi$ satisfies (31). All in all ψ_1 itself satisfies (31).

Estimates for ψ_2. Choose $F \in L^2(P)$. Then by Cauchy's inequality,

$$\begin{aligned}
&\left| \int_P F |\nabla \psi_2| \right| \\
&\leq \left(\int_0^L \int_V |\nabla \varphi_\lambda| * \big|F|(q) |\varphi_\lambda * \psi(q)\big|\, dq \frac{d\lambda}{\lambda} \right) \\
&\leq \left(\int_0^L \int_V |\varphi_\lambda * \psi(q)|^2\, dq \frac{d\lambda}{\lambda^3} \right)^{1/2} \left(\int_0^L \int_V [|\nabla \varphi_\lambda| * |F|(q)]^2\, dq \lambda\, d\lambda \right)^{1/2} \\
&\lesssim \eta [\sigma(Q(S))]^{1/2} \left(\int_P |F|^2 \right)^{1/2}
\end{aligned}$$

by Plancherel's inequality, (31) and (28). In conclusion,

$$\int_P |\nabla\psi_2|^2 \lesssim \eta\sigma(Q(S)). \tag{32}$$

Observe that the following Poincaré-type inequality holds for ψ_2 and any $\mathcal{Q}$:

$$|\mathcal{Q}|^{-(n+2)/(n+1)} \int_{\mathcal{Q}} |\psi_2 - m_{\mathcal{Q}}\psi_2|\, dp \lesssim \frac{1}{|\mathcal{Q}|} \int_{\mathcal{Q}} |\nabla\psi_2|\, dp.$$

Now estimates for Hardy–Littlewood maximal function of $|\nabla\psi_2|$ and (32) yield

$$\int_P [\mathcal{M}\psi_2]^2 \lesssim \int |\nabla\psi_2|^2 \lesssim \eta\sigma(Q(S)). \tag{33}$$

We are ready now to give an estimate for all of ψ. Recall that $F = \{p \in V_2 : \mathcal{M}\psi_2(p) \leq \delta\}$ and suppose that $\widetilde{\mathcal{Q}} \cap F \neq \emptyset$. For $p \in \widetilde{\mathcal{Q}}$, one clearly has

$$\begin{aligned} &|\psi(p) - \psi(p_0) - \nabla\psi_1(p_0)\cdot(p-p_0)| \\ &\quad\leq |\psi_2(p) - \psi_2(p_0)| + |\psi_1(p) - \psi_2(p_0) - \nabla\psi_1(p_0)\cdot(p-p_0)| \\ &\quad\equiv I + II. \end{aligned}$$

To control I, we define $\lambda = \operatorname{osc}_{p\in\widetilde{\mathcal{Q}}} \psi_2 \equiv \sup_{p\in\widetilde{\mathcal{Q}}} |\psi_2(p) - m_{\widetilde{\mathcal{Q}}}(\psi_2)|$. Choose $q \in \mathcal{C}$ such that this supremum is attained, that is, $\lambda = |\psi_2(q) - m_{\widetilde{\mathcal{Q}}}(\psi_2)|$.

Since ψ is Lipschitz in space variables, by (31) applied to ψ_1 we get $\|\nabla\psi_2\|_{L^\infty(\mathcal{C})} \lesssim \delta$. But if $p \in \widetilde{\mathcal{Q}}$ satisfies $\|p-q\| \leq \lambda/2C\delta$, then $|\psi_2(p)-\psi_2(q)| \leq C\delta\|p-q\| \leq \lambda/2$, and since $\lambda = |\psi_2(q) - m_{\widetilde{\mathcal{Q}}}\psi_2| \leq |\psi_2(p) - \psi_2(q)| + |\psi_2(p) - m_{\widetilde{\mathcal{Q}}}\psi_2| \leq \lambda/2 + |\psi_2(p) - m_{\widetilde{\mathcal{Q}}}\psi_2|$ then

$$|\psi_2(p) - m_{\widetilde{\mathcal{Q}}}\psi_2| \geq \frac{\lambda}{2}. \tag{34}$$

Suppose $r \geq \lambda/2C\delta$. Then integrating (34)

$$\int_{\widetilde{\mathcal{Q}}} |\psi_2 - m_{\widetilde{\mathcal{Q}}}\psi_2| \geq \frac{\lambda}{2C}\left(\frac{\lambda}{2C\delta}\right)^{n+1}$$

and so

$$\lambda \lesssim \delta^{(n+1)/(n+2)} \left(\int_{\widetilde{\mathcal{Q}}} |\psi_2 - m_{\widetilde{\mathcal{Q}}}\psi_2|\right)^{1/(n+2)}. \tag{35}$$

Suppose now that $r \leq \lambda/2C\delta$. Then for $p \in \widetilde{\mathcal{Q}}$ one has $\|p-q\| < r < \lambda/2C\delta$ and by (34) $|\psi_2(p) - m_{\widetilde{\mathcal{Q}}}\psi_2| \geq \lambda/2$. Integrating we obtain $m_{\widetilde{\mathcal{Q}}}(|\psi_2 - m_{\widetilde{\mathcal{Q}}}\psi_2|) \geq C\lambda$. On the other hand, since $\|\nabla\psi_2\|_{L^\infty(\mathcal{C})} \leq C\delta$ then $m_{\widetilde{\mathcal{Q}}}(|\psi_2 - m_{\widetilde{\mathcal{Q}}}\psi_2|)/r \leq C\delta$, and in conclusion

$$\begin{aligned} C\lambda &\leq m_{\widetilde{\mathcal{Q}}}(|\psi_2 - m_{\widetilde{\mathcal{Q}}}\psi_2|) = r\left[\frac{1}{r} m_{\widetilde{\mathcal{Q}}}(|\psi_2 - m_{\widetilde{\mathcal{Q}}}\psi_2|)\right]^{\frac{1}{n+2}+\frac{n+1}{n+2}} \\ &\leq Cr\left[\frac{1}{r} m_{\widetilde{\mathcal{Q}}}(|\psi_2 - m_{\widetilde{\mathcal{Q}}}\psi_2|)\right]^{\frac{1}{n+2}} \delta^{\frac{n+1}{n+2}}. \end{aligned} \tag{36}$$

Combining (35) and (36), we obtain

$$I \le 2\lambda \lesssim r\left[\frac{1}{r} m_{\widetilde{\mathcal{Q}}}(|\psi_2 - m_{\widetilde{\mathcal{Q}}}\psi_2|)\right]^{\frac{1}{n+2}} \delta^{\frac{n+1}{n+2}}. \tag{37}$$

On the other hand, note that by Taylor's theorem and (31)

$$II \lesssim \frac{\delta r^2}{L} \lesssim \delta r_0 r. \tag{38}$$

If we now choose $\mathbf{X} \in \widetilde{\mathcal{Q}} \cap F$, we obtain

$$|\psi(p) - \psi(p_0) - \nabla\psi_1(p_0)\cdot(p-p_0)| \le [\mathcal{M}\psi_2(\mathbf{X})]^{1/(n+2)}\delta^{(n+1)/(n+2)} + C\delta r_0 r.$$

By (33), the estimate for ψ we were seeking is

$$\sup_{p\in\widetilde{\mathcal{Q}}} |\psi(p) - \psi(p_0) - \nabla\psi_1(p_0)\cdot(p-p_0)| \lesssim (1+r_0)r\delta. \tag{39}$$

With essentially the same reasoning leading to (39), we now obtain

$$\sup_{\mathbf{X}\in\Psi\cap\pi^{-1}(\widetilde{\mathcal{Q}})} \left[\frac{1}{r} d(\mathbf{X}; H_{\widetilde{\mathcal{Q}}})\right] \le (1+r_0)r\delta$$

which was our ultimate goal, namely, estimate (29).

3. Proof of Theorem 1.5

This proof combines arguments in [2, Chapters 3–5] and [3, II Chapter 2]. We include a sketch of the adaptations of ideas and constructions therein, for completeness and in order to state properly the adaptation of the result from [4] that mentioned in the Introduction.

Let $\psi(X,t)$ be a smooth function on $(X,t) \in \mathbb{R}^n \times \mathbb{R}$ which is odd in the X variable, and with compact support. In order to define a kernel to which we can apply the hypothesis, consider the set Ω of sequences $\omega = \{\omega_j\}$, with $w_j \in \{-1,1\}$, endowed with the product topology. The measure Π on Ω assigns equal probability to the values ± 1. Consider projections $\epsilon_j : \Omega \longrightarrow \{-1,1\}$ given by $\epsilon_j(\omega) = \omega_j$. Recall that it is well known that

$$\begin{aligned}\sum_{j=-m}^{m} &\left|\int_E \psi_j(X-Y,t-s) f(Y,s)\, d\sigma(Y,s)\right|^2 \\ &= \int_\Omega \left|\sum_{j=-m}^{m} \int_E \epsilon_j(\omega)\psi_j(X-Y,t-s) f(Y,s)\, d\sigma(Y,s)\right|^2 d\Pi(\omega),\end{aligned}$$

where $\psi_j(X,t) = 2^{-j(n+1)}\psi(2^{-j}X, 2^{-2j}t)$. Define

$$K_m(X,t;\omega) = \sum_{j=-m}^{m} \epsilon_j(\omega)\psi_j(X,t).$$

Observe that this kernel is still odd in the X variable and it satisfies the other assumptions for good kernels. By our assumptions,

$$\begin{aligned}\int_E \left| \int_E K_m(X-Y,t-s;\omega) f(Y,s)\, d\sigma(Y,s) \right|^2 d\sigma(X,t) \\ \leq C(m,\omega) \int_E |f(X,t)|^2\, d\sigma(Y,s)\end{aligned}$$

with a constant $C(m,\omega)$ that depends on m and ω. After the "completeness argument" of [2, p. 22], one gets

$$\sum_{j=-m}^{m} \left| \int_E \psi_j(X-Y,t-s) f(Y,s)\, d\sigma(Y,s) \right|^2 \leq C \int_E |f(X,t)|^2\, d\sigma(Y,s), \tag{40}$$

this time with a constant independent of m and ω. Applying (40) to characteristic functions of balls, it is not hard to prove that

$$\left(\sum_{j=-\infty}^{\infty} \left| \int_E \psi_j(X-Y,t-s)\, d\sigma(Y,s) \right|^2 \right) d\sigma(X,t)\, d\delta_{2^k}(u)$$

is a Carleson measure over $E \times (0,\infty)$.

Using this, we now construct a collection of cubes satisfying a Carleson measure type property. For $\tau > 0$, small let $R(\tau)$ denote the set of cubes $Q \in \Delta$ with the property that there exist $\mathbf{X}, \mathbf{Y} \in 2Q$ such that $d(2\mathbf{X} - \mathbf{Y}; E) \geq \tau \operatorname{diam} Q$.

Claim 1. *With the definitions and notations above, one has*

$$\sum_{\substack{Q \in R(\tau) \\ Q \subset R}} \sigma(Q) \leq C(\tau)\sigma(R).$$

The proof of this claim can be easily adapted from [2, p. 24].

Let $\varepsilon > 0$ be given and define $\mathcal{G}(\varepsilon)$ as the family of cubes $Q \in \Delta$ for which there is a plane P_Q satisfying the following properties:

$$d(\mathbf{X}; P_Q) \leq \varepsilon \operatorname{diam} Q \quad \text{for all } \mathbf{X} \in 2Q; \tag{41}$$

$$\text{if } \mathbf{Y} \in P_Q \text{ and } d(\mathbf{Y}; Q), \text{ then } \quad d(\mathbf{Y}; E) \leq \varepsilon \operatorname{diam} Q. \tag{42}$$

Claim 2. *Set $\mathcal{B}(\varepsilon) = \Delta \setminus \mathcal{G}(\varepsilon)$. Then*

$$\sum_{\substack{Q \in \mathcal{B}(\varepsilon) \\ Q \subset R}} \sigma(Q) \leq C(\varepsilon)\sigma(R) \tag{43}$$

for all $R \in \Delta$ and all $\varepsilon > 0$.

The proof can be adapted this time from [2, pp. 28–32]. The point is now that the estimates in the two claims above will lead us to a parabolic generalized Corona decomposition, defined in Proposition 1.3. Here is where we follow ideas in [3, II Chapter 2] and [4]

First, we record a useful adaptation of a theorem about coronizations of Ahlfors–David regular sets.

CLAIM 3. *Let E be a set in $\mathbb{R}^{n+1}$ satisfying a (M,R) David–Ahlfors condition. Let $\mathcal{A}\subset\Delta$ be a family of cubes that satisfies a packing condition of the form*

$$\sum_{\substack{Q\in\mathcal{A}\\ Q\subset R}}\sigma(Q)\le C\sigma(R)$$

for all $R\in\Delta$. Then there exist a coronization $(\mathcal{B},\mathcal{G},\mathcal{F})$ of E such that $\mathcal{A}\subset\mathcal{B}$. Moreover, for each region $S\in\mathcal{F}$ we can find an (M',R) Ahlfors–David regular set E_S that satisfies

$$d(\mathbf{X};E_S)\le 4\operatorname{diam}Q\quad\textit{whenever } \mathbf{X}\in 2Q \textit{ for some cube } Q\in S. \tag{44}$$

The constant M' depends only on the Ahlfors–David constant M of E.

The first conclusion is [3, I Lemma 3.22], while the second one is proved essentially in [3, pp. 99–100]. Applying this claim to $\mathcal{B}$ as defined above, we obtain a coronization that we denote $(\mathcal{B}',\mathcal{G}',\mathcal{F})$ and such that $\mathcal{B}(\varepsilon)\subseteq\mathcal{B}'$. According to Proposition 1.3, it remains to prove that $E_S\in\mathrm{BPPLG}(C(\eta))$, and for that purpose we can follow the plan of [3, p. 102ff]. Namely, we follow the next steps:

(a) From (43), which is an estimate for E, we obtain a similar estimate for each E_S and
(b) All E_S have big projections as we now define:

An (M,R) Ahlfors–David regular set E is said to *have big projections* if there exists $\theta>0$ such that for each $(X,t)\in E$ and $r>0$ there is a t-plane P such that, if π_P denotes the projection on P, then

$$\big|\pi_P\big(\mathcal{C}_r(X,t)\cap E\big)\big|\ge\theta r^{n+1}. \tag{45}$$

To finish the proof of Theorem 1.5 we use (a) and (b) and the following theorem, whose proof is in the next section.

THEOREM 3.1. *If E is an (M,R) Ahlfors–David regular set that has big projections and it satisfies (43) (considering the definitions in (41) and (42)), then $E\in\mathrm{BPPLG}(C(\tau))$.*

Observe that we can reduce the estimate (43) to another estimate involving a Carleson measure estimate as follows. For $(X,t)\in E$ and $r>0$, define

$$\beta(X,t;r)=\inf_P\left\{\sup\frac{d(Y,s;P)}{r}:(Y,s)\in E\cap\mathcal{C}_r(X,t)\right\}, \tag{46}$$

where the infimum is taken over all t-planes P. Then, in order to obtain (43) it suffices to prove that if $A \equiv A(\tau) = \{(X,t;\rho) \in E \times (0,\infty) : \beta(X,t;\rho) > \tau\}$ then

$$\chi_A(X,t;\rho)d\sigma(X,t)\frac{d\rho}{\rho} \text{ is a Carleson measure.} \tag{47}$$

The step (a) is now obtained as follows. Fixing $S \in \mathcal{F}$, we define for $(X,t) \in E_S$ and $r > 0$

$$\beta_S(X,t;r) = \inf_P \left\{ \sup \frac{d(Y,s;P)}{r} : (Y,s) \in E_s \cap \mathcal{C}_r(X,t) \right\},$$

where the infimum is taken over all t-planes P. In order to obtain

$$\sum_{\substack{Q \in \mathcal{B}' \\ Q \subset R}} \sigma(Q) \le C(\tau)\sigma(R) \quad \text{for all } R \in \Delta$$

or the equivalent formulation stated in (47), we can follow the argument in [3, pp. 103–104], where one essentially uses the corresponding estimate for E.

For the proof of step (b) one can follow the constructions and arguments in [3, pp. 104–110]. Although the planes in those constructions are not as our t-planes, the arguments can be applied verbatim.

4. Sketch of proof of Theorem 3.1

Throughout this section, we denote by $|F|$ the n-dimensional Hausdorff measure of a set F. Let $(X,t) \in E$ and $r > 0$. By the big projections property, there exists a t-plane P such that $|\pi_P(E \cap \mathcal{C}_r(X,t)| > \theta r^{n+1}$. Now choose $Q_0 \in \Delta$ and $\varepsilon > 0$ such that $Q_0 \subset \mathcal{C}_r(X,t) \cap E$, $\operatorname{diam} Q_0 \ge \varepsilon r$ and still $|\pi_P(Q_0)| \ge 2\varepsilon\sigma(Q_0)$ holds. Let $f : Q_0 \longrightarrow P$ be defined as $f \equiv \pi_P$. Then obviously f is an affine function (as a function taking values on $\mathbb{R}^{n+1}$) and it satisfies $\|f(X,t) - f(Y,s)\| \le \|X - Y, t - s\|$.

We now quote an adaptation from [4, Theorem 2.11]. The proof included therein (pp. 864–867) can be adapted with few easy adaptations. The only property that one requires is: if $f \in \operatorname{Lip}(1,1/2)$ and $\sigma(A) \le \tilde{c}\sigma(Q_0)$ then $|f(A)| \le \tilde{c}\sigma(Q_0)$. In our case actually $|f(A)| \le \sigma(A)$ holds.

Claim. *Suppose $K > 0$ and $\varepsilon > 0$ are given. Then there exists constants α, τ, N such that the following is true: There exists closed subsets F_j of Q_0, $1 \le j \le N$, such that*

$$\|f(X,t) - f(Y,s)\| \ge \tau\|X - Y, t - s\|, \quad (X,t),(Y,s) \in F_j; \tag{48}$$

$$\left| f\left(Q_0 \setminus \left(\bigcup_j F_j \right) \right) \right| \le \varepsilon|Q_0|. \tag{49}$$

The dependance of the constants is described as follows: $\alpha = \alpha(n,\varepsilon,M)$, $\tau = \tau(n,\varepsilon,M,K)$, $N = N(n,\varepsilon,M,K)$.

We now address a consequence of this claim. Observe that by (49), we have

$$|f(\cup F_j)| \geq |f(Q_0)| - \varepsilon\sigma(Q_0) \geq \varepsilon\sigma(Q_0) \tag{50}$$

which implies $\sigma(f(F_{j_0})) \geq \varepsilon\sigma(Q_0)/N$ for some j_0. Now by (48), we can write $F_{j_0} = \{p + \psi(p) : p \in f(F_{j_0})\}$ for some $\psi : f(F_{j_0}) \subset P \longrightarrow \mathbb{R}$ and $\psi \in \mathrm{Lip}(1, 1/2)$. But as we have done before a couple of times, we must prove that ψ is a parabolic Lipschitz function. For that purpose, we write a result in [8, pp. 362–363, 365–373] in such a way that it implies at once the conclusion of the proof of Theorem 3.1.

Proposition 4.1. *Let $E \subset \mathbb{R}^{n+1}$ be a uniformly rectifiable set in the parabolic sense. Suppose that there exist a t-plane P with a canonical projection $\pi : \mathbb{R}^{n+1} \longrightarrow P$ associated to it, and a set $F \subset E \cap \Delta_r(X,t)$, for some $0 < r < R$, $(X,t) \in E$, and such that the following conditions hold*:

(i) *There exists $g : \pi(F) \longrightarrow \mathbb{R}$ with $|g(y,s) - g(z,\tau)| \leq q_1\|y - z, s - \tau\|$, and such that $(\psi(y,s), y, s) \in F$ for every $(y,s) \in \pi(F)$;*

(ii) $H^{n+1}(\pi(F)) \geq q_2 r^{n+1}$.

Then there exists a parabolic Lipschitz function $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $g(y,s) = \psi(y,s)$ for $(y,s) \in \pi(F_1)$, for certain closed set $F_1 \subset F$ with $H^{n+1}(F_1) \geq q_3 r^{n+1}$, for certain constant q_3 depending on q_2 and the Carleson norm (4) of E.

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Jorge Rivera-Noriega, Facultad de Ciencias, Universidad Autónoma del Estado de Morelos, Av. Universidad 1001, Col Chamilpa, Cuernavaca Mor CP 62209, México

E-mail address: rnoriega@uaem.mx