

HOMOLOGICAL DIMENSIONS IN COTORSION PAIRS

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ABSTRACT. Given a ring R , two classes \mathcal{A} and \mathcal{B} of R -modules are said to form a cotorsion pair $(\mathcal{A}, \mathcal{B})$ in $\text{Mod } R$ if $\mathcal{A} = \text{Ker Ext}_R^1(-, \mathcal{B})$ and $\mathcal{B} = \text{Ker Ext}_R^1(\mathcal{A}, -)$. We investigate relative homological dimensions in cotorsion pairs. This can be applied to study the big and the little finitistic dimension of R . We show that $\text{Findim } R < \infty$ if and only if the following dimensions are finite for some cotorsion pair $(\mathcal{A}, \mathcal{B})$ in $\text{Mod } R$: the relative projective dimension of \mathcal{A} with respect to itself, and the \mathcal{A} -resolution dimension of the category \mathcal{P} of all R -modules of finite projective dimension. Moreover, we obtain an analogous result for $\text{findim } R$, and we characterize when $\text{Findim } R = \text{findim } R$.

Introduction

The study of homological dimensions which are obtained by replacing the projective or injective modules by certain subcategories was initiated by Auslander and Buchweitz in their seminal paper [5], which was one of the starting points for what is now called relative homological algebra.

Of course, the existence of approximations is the prerequisite for computing relative dimensions. In recent years, a powerful machinery for producing approximations was developed by using the notion of a cotorsion pair, see [9], [7], [8]. So it is not surprising that cotorsion pairs provide a good setting for investigating relative homological dimensions.

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The main purpose of this note is to use cotorsion pairs in order to obtain estimates for the finitistic dimensions. Recall that the (right) *big finitistic dimension* of R is defined as $\text{Findim}(R_R) = \text{pd } \mathcal{P}$, the supremum of the projective dimensions attained on the subcategory \mathcal{P} of all modules of finite projective dimension. By restricting to the subcategory $\mathcal{P}^{<\infty}$ of all modules in \mathcal{P} that admit a projective resolution consisting of finitely generated projective modules, one obtains $\text{findim}(R_R) = \text{pd } \mathcal{P}^{<\infty}$, the (right) *little finitistic dimension* of R . The Finitistic Dimension Conjecture is one of the main open problems in the representation theory of algebras. It asserts that $\text{findim}(R_R) < \infty$ for any artin algebra R .

Let now $(\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pair (see the definitions at the end of Section 1), and assume that the relative projective dimension

$$\text{pd}_{\mathcal{A}}(\mathcal{A}) = \min\{n \geq 0 : \text{Ext}_R^j(-, -)|_{\mathcal{A}} = 0 \text{ for any } j > n\}$$

is finite. Denote by $\text{resdim}_{\mathcal{A}}$ the resolution dimension with respect to \mathcal{A} . We prove the following inequality.

THEOREM 4.3.

$$\text{resdim}_{\mathcal{A}}(\mathcal{P}^{<\infty}) \leq \text{findim}(R_R) \leq \text{pd}_{\mathcal{A}}(\mathcal{A}) + \text{resdim}_{\mathcal{A}}(\mathcal{P}^{<\infty}).$$

As a consequence, we obtain a criterion for validity of the Finitistic Dimension Conjecture.

COROLLARY 4.4. *The following conditions are equivalent.*

- (a) $\text{findim}(R_R) < \infty$.
- (b) *There is a hereditary complete cotorsion pair $(\mathcal{A}, \mathcal{B})$ in $\text{Mod } R$ such that $\text{pd}_{\mathcal{A}}(\mathcal{A}) < \infty$ and $\text{resdim}_{\mathcal{A}}(\mathcal{P}^{<\infty}) < \infty$.*
- (c) *Every hereditary complete cotorsion pair $(\mathcal{A}, \mathcal{B})$ in $\text{Mod } R$ with $\text{pd}_{\mathcal{A}}(\mathcal{A})$ being finite satisfies that $\text{resdim}_{\mathcal{A}}(\mathcal{P}^{<\infty})$ is finite.*

An analogous result characterizes finiteness of $\text{Findim}(R_R)$. Moreover, we characterize equality of the big and the little finitistic dimension in terms of the kernel $\omega = \mathcal{A} \cap \mathcal{B}$ of a certain cotorsion pair $(\mathcal{A}, \mathcal{B})$, see Theorem 4.5. This generalizes a result from [2].

Finally, we prove that the finitistic dimension is bounded by the homological dimensions of tilting modules.

COROLLARY 4.7. *Let T be a tilting module in $\text{Mod } R$. If R is right noetherian then*

$$\text{Findim}(R_R) \leq \text{pd } T + \text{id } T.$$

As an application, we improve a result of Mazorchuk and Ovsienko [12] on properly stratified algebras having a simple preserving duality. More precisely, let R be such an algebra, and assume that every classical tilting right R -module is also cotilting. Denote by T the characteristic tilting module associated to R . Then it follows from Corollary 4.7 that $\text{Findim}(R_R) \leq 2 \text{pd } T$,

see Remark 4.8. Combining this with [12, Theorem 1], we conclude that $\text{Findim}(R_R) = \text{findim}(R_R) = 2 \text{pd} T$.

This paper is organized as follows. Section 1 is devoted to some preliminaries. In Section 2, we use results from [11] in order to obtain formulae that relate relative projective dimensions and resolution dimensions in cotorsion pairs. In Section 3, we apply these formulae to cotorsion pairs arising from tilting theory. Hereby, we refine some results from [1] and [4]. The finitistic dimensions are the topic of Section 4.

1. Preliminaries

We start this section by collecting all the background material that will be necessary in the sequel. First, we introduce some general notation. Next, we recall the notion of relative projective dimension and resolution dimension of a given class of modules. Finally, we also recall definition and basic properties of cotorsion pairs.

Throughout the paper, R will be an associative ring with unit, and $\text{Mod } R$ the category of all right R -modules. The subcategory of all modules possessing a projective resolution consisting of finitely generated modules is denoted by $\text{mod } R$. By a subcategory of $\text{Mod } R$ we always mean a full subcategory.

We denote by $\text{pd } X$ the *projective dimension* of X . Similarly, $\text{id } X$ is the *injective dimension* of X . For any nonnegative integer n , we set

$$\mathcal{P}_n := \{X \in \text{Mod } R : \text{pd } X \leq n\}$$

and $\mathcal{P}_n^{<\infty} := \mathcal{P}_n \cap \text{mod } R$; moreover,

$$\mathcal{P} := \{X \in \text{Mod } R : \text{pd } X < \infty\}$$

and $\mathcal{P}^{<\infty} := \mathcal{P} \cap \text{mod } R$. In particular, \mathcal{P}_0 consists of the projective R -modules.

Let now \mathcal{X} be a subcategory of $\text{Mod } R$. We denote by $\text{Add}(\mathcal{X})$ the class of all R -modules isomorphic to direct summands of direct sums of elements in \mathcal{X} . Moreover, for each positive integer i , we denote

$$\mathcal{X}^{\perp_i} := \{M \in \text{Mod } R : \text{Ext}_R^i(-, M)|_{\mathcal{X}} = 0\} \quad \text{and} \quad \mathcal{X}^{\perp} := \bigcap_{i>0} \mathcal{X}^{\perp_i}.$$

Dually, we have the classes ${}^{\perp_i} \mathcal{X}$ and ${}^{\perp} \mathcal{X}$.

A subcategory $\mathcal{X} \subseteq \text{Mod } R$ (respectively, $\mathcal{X} \subseteq \text{mod } R$) is said to be *resolving* if it is closed under extensions and kernels of surjections, and it contains all (respectively, all finitely generated) projective modules. If the dual properties hold true, then \mathcal{X} is a *coresolving* subcategory. For example, \mathcal{P} and ${}^{\perp} \mathcal{X}$ are resolving subcategories of $\text{Mod } R$, while \mathcal{X}^{\perp} is a coresolving subcategory of $\text{Mod } R$.

RELATIVE HOMOLOGICAL DIMENSIONS. Given a class of R -modules \mathcal{X} and an R -module M , the *relative projective dimension* of M with respect to \mathcal{X} is

defined as

$$\text{pd}_{\mathcal{X}}(M) := \min\{n \geq 0 : \text{Ext}_R^j(M, -)|_{\mathcal{X}} = 0 \text{ for any } j > n\}.$$

Dually, we denote by $\text{id}_{\mathcal{X}}(M)$ the *relative injective dimension* of M with respect to \mathcal{X} . Furthermore, for any class $\mathcal{Y} \subseteq \text{Mod } R$, we set

$$\text{pd}_{\mathcal{X}}(\mathcal{Y}) := \sup\{\text{pd}_{\mathcal{X}} Y : Y \in \mathcal{Y}\} \quad \text{and} \quad \text{id}_{\mathcal{X}}(\mathcal{Y}) := \sup\{\text{id}_{\mathcal{X}} Y : Y \in \mathcal{Y}\}.$$

If $\mathcal{X} = \text{Mod } R$, we just write $\text{pd}(\mathcal{Y})$ and $\text{id}(\mathcal{Y})$.

The following basic properties are straightforward.

LEMMA 1.1. *Let \mathcal{X} and \mathcal{Y} be classes in $\text{Mod } R$. Then*

$$\text{pd}_{\mathcal{X}}(\mathcal{Y}) = \text{id}_{\mathcal{Y}}(\mathcal{X}).$$

Furthermore, for any exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ in $\text{Mod } R$ we have

- (a) $\text{pd}_{\mathcal{X}}(M) \leq \max\{\text{pd}_{\mathcal{X}}(M'), \text{pd}_{\mathcal{X}}(M'')\};$
- (b) $\text{pd}_{\mathcal{X}}(M') \leq \max\{\text{pd}_{\mathcal{X}}(M), \text{pd}_{\mathcal{X}}(M'') - 1\};$
- (c) $\text{pd}_{\mathcal{X}}(M'') \leq \max\{\text{pd}_{\mathcal{X}}(M), \text{pd}_{\mathcal{X}}(M') + 1\}.$

RESOLUTION AND CORESOLUTION DIMENSION. Let $M \in \text{Mod } R$ and \mathcal{X} be a class of R -modules. The \mathcal{X} -coresolution dimension $\text{coresdim}_{\mathcal{X}}(M)$ of M is the minimal nonnegative integer n such that there is an exact sequence

$$0 \rightarrow M \rightarrow X_0 \rightarrow X_1 \rightarrow \dots \rightarrow X_n \rightarrow 0$$

with $\mathcal{X}_i \in \mathcal{X}$ for $0 \leq i \leq n$. If such n does not exist, we set $\text{coresdim}_{\mathcal{X}}(M) := \infty$. Also, we denote by \mathcal{X}^{\vee} the class of R -modules having a finite \mathcal{X} -coresolution.

Dually, we have the \mathcal{X} -resolution dimension $\text{resdim}_{\mathcal{X}}(M)$ of M , and the class \mathcal{X}^{\wedge} of R -modules having a finite \mathcal{X} -resolution.

Given a class $\mathcal{Y} \subseteq \text{Mod } R$, we set

$$\text{coresdim}_{\mathcal{X}}(\mathcal{Y}) := \sup\{\text{coresdim}_{\mathcal{X}}(Y) : Y \in \mathcal{Y}\},$$

and $\text{resdim}_{\mathcal{X}}(\mathcal{Y})$ is defined dually.

APPROXIMATIONS. Let \mathcal{X} be a class of R -modules. A morphism $f : X \rightarrow M$ is said to be an \mathcal{X} -precover if $X \in \mathcal{X}$ and $\text{Hom}_R(Z, f) : \text{Hom}_R(Z, X) \rightarrow \text{Hom}_R(Z, M)$ is surjective for any $Z \in \mathcal{X}$. Furthermore, an \mathcal{X} -precover $f : X \rightarrow M$ is *special* if $\text{Ker}(f) = 0$ and $\text{Coker}(f) \in {}^{\perp_1}\mathcal{X}$. We will freely use also the dual notion of (special) \mathcal{X} -preenvelope.

Finally, we recall the notion of cotorsion pair which was introduced by Salce in [13]. It is the analog of a torsion pair where the functor $\text{Hom}_R(-, -)$ is replaced by $\text{Ext}_R^1(-, -)$.

DEFINITION 1.2 ([13]). Let \mathcal{A} and \mathcal{B} be classes in $\text{Mod } R$. The pair $(\mathcal{A}, \mathcal{B})$ is said to be a *cotorsion pair* if $\mathcal{A} = {}^{\perp_1}\mathcal{B}$ and $\mathcal{A}^{\perp_1} = \mathcal{B}$. The class $\mathcal{A} \cap \mathcal{B}$ is called the *kernel* of the cotorsion pair $(\mathcal{A}, \mathcal{B})$.

LEMMA 1.3 ([13, Corollary 2.4]). *For a cotorsion pair $(\mathcal{A}, \mathcal{B})$ in $\text{Mod } R$, the following conditions are equivalent.*

- (a) *Every R -module has a special \mathcal{A} -precover.*
- (b) *Every R -module has a special \mathcal{B} -preenvelope.*

In this case, the cotorsion pair $(\mathcal{A}, \mathcal{B})$ is called complete.

LEMMA 1.4. *For a cotorsion pair $(\mathcal{A}, \mathcal{B})$ in $\text{Mod } R$, the following conditions are equivalent.*

- (a) *\mathcal{A} is resolving.*
- (b) *\mathcal{B} is coresolving.*
- (c) *$\text{id}_{\mathcal{A}}(\mathcal{B}) = 0$.*

In this case, the cotorsion pair $(\mathcal{A}, \mathcal{B})$ is called hereditary.

2. General results

In this section, we obtain general results for complete cotorsion pairs. They will later be applied, on one hand, to tilting theory, and on the other hand, to the big and the little finitistic dimension of the ring R .

Let us start by collecting some preliminary results. They are stated in [11] under stronger assumptions, but the proofs there also work in the present context.

PROPOSITION 2.1 ([11, Theorem 2.1 and Lemma 3.3]). *Let \mathcal{X} and \mathcal{Y} be classes in $\text{Mod } R$. The following statements hold true.*

- (a) *$\text{id}_{\mathcal{X}}(L) \leq \text{id}_{\mathcal{X}}(\mathcal{Y}) + \text{coresdim}_{\mathcal{Y}}(L)$ for every R -module L .*
- (b) *Assume that $\mathcal{Y} = \mathcal{X}^{\perp 1}$, or that \mathcal{Y} is a subcategory of \mathcal{X} which is closed under direct summands. Suppose further that $\text{id}_{\mathcal{X}}(\mathcal{Y}) = 0$. Then we have $\text{id}_{\mathcal{X}}(L) = \text{coresdim}_{\mathcal{Y}}(L)$ for any $L \in \mathcal{Y}^{\vee}$.*
- (c) *Let $\mathcal{Y} = \mathcal{X}^{\perp 1}$. Then $\text{coresdim}_{\mathcal{Y}}(M) \leq \text{id}_{\mathcal{X}}(M)$ for any $M \in \text{Mod } R$. In particular, $\text{coresdim}_{\mathcal{Y}}(\text{Mod } R) \leq \text{pd } \mathcal{X}$.*

Of course, these results also have dual versions which we will freely use in the sequel. We now turn to complete cotorsion pairs.

THEOREM 2.2. *Let $(\mathcal{A}, \mathcal{B})$ be a complete cotorsion pair in $\text{Mod } R$.*

- (a) *For any R -module M , we have*

$$\begin{aligned} \text{id } M &= \max\{\text{id}_{\mathcal{A}}(M), \text{id}_{\mathcal{B}}(M)\}, \\ \text{pd } M &= \max\{\text{pd}_{\mathcal{A}}(M), \text{pd}_{\mathcal{B}}(M)\}. \end{aligned}$$

- (b) *$\text{coresdim}_{\mathcal{B}}(\text{Mod } R) \leq \text{pd } \mathcal{A} \leq \text{id}_{\mathcal{A}}(\mathcal{B}) + \text{coresdim}_{\mathcal{B}}(\mathcal{A}) + 1$.*
- (c) *$\text{resdim}_{\mathcal{A}}(\text{Mod } R) \leq \text{id } \mathcal{B} \leq \text{pd}_{\mathcal{B}}(\mathcal{A}) + \text{resdim}_{\mathcal{A}}(\mathcal{B}) + 1$.*

Proof. (a) We only prove the first equality, the second one follows similarly. It is clear that $\max\{\text{id}_{\mathcal{A}}(M), \text{id}_{\mathcal{B}}(M)\} \leq \text{id } M$.

Take an R -module N . Since $(\mathcal{A}, \mathcal{B})$ is complete, we have an exact sequence

$$0 \rightarrow N \rightarrow B \rightarrow A \rightarrow 0$$

with $B \in \mathcal{B}$ and $A \in \mathcal{A}$. By Lemma 1.1(b), we infer that

$$\text{pd}_{\{M\}}(N) \leq \max\{\text{pd}_{\{M\}}(\mathcal{B}), \text{pd}_{\{M\}}(\mathcal{A})\}$$

for any $N \in \text{Mod } R$, which proves the result since $\text{pd}_{\{M\}}(\text{Mod } R) = \text{id } M$.

(b) The first inequality follows from Proposition 2.1(c) since $\mathcal{B} = \mathcal{A}^{\perp 1}$. To prove the second one, let M be an R -module. Since $(\mathcal{A}, \mathcal{B})$ is complete, there is an exact sequence $0 \rightarrow M \rightarrow B \rightarrow A \rightarrow 0$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Thus, by the dual of Lemma 1.1(c), we get that

$$\text{id}_{\mathcal{A}}(M) \leq \max\{\text{id}_{\mathcal{A}}(\mathcal{B}), \text{id}_{\mathcal{A}}(\mathcal{A}) + 1\} \quad \text{for any } M \in \text{Mod } R.$$

Furthermore, we know from Proposition 2.1(a) that $\text{id}_{\mathcal{A}}(\mathcal{A}) \leq \text{id}_{\mathcal{A}}(\mathcal{B}) + \text{coresdim}_{\mathcal{B}}(\mathcal{A})$, so the proof is complete.

(c) It is shown similarly. □

COROLLARY 2.3. *Let $(\mathcal{A}, \mathcal{B})$ be a hereditary complete cotorsion pair in $\text{Mod } R$. Then*

- (a) $\text{pd}_{\mathcal{A}}(\mathcal{A}) = \text{pd } \mathcal{A}$ and $\text{id}_{\mathcal{B}}(\mathcal{B}) = \text{id } \mathcal{B}$.
- (b) $\text{coresdim}_{\mathcal{B}}(\text{Mod } R) \leq \text{pd } \mathcal{A} \leq \text{coresdim}_{\mathcal{B}}(\mathcal{A}) + 1$.
- (c) $\text{resdim}_{\mathcal{A}}(\text{Mod } R) \leq \text{id } \mathcal{B} \leq \text{resdim}_{\mathcal{A}}(\mathcal{B}) + 1$.

Proof. This follows easily from Theorem 2.2 since $\text{pd}_{\mathcal{B}}(\mathcal{A}) = \text{id}_{\mathcal{A}}(\mathcal{B}) = 0$. □

The following result will be useful in investigating cotorsion pairs where $\text{pd } \mathcal{A}$ is finite. It relies on work of Auslander and Buchweitz.

LEMMA 2.4. *Let $(\mathcal{A}, \mathcal{B})$ be a hereditary complete cotorsion pair in $\text{Mod } R$ with kernel ω . The following statements hold true.*

- (a) $\mathcal{A} \cap \omega^{\vee} = \{X \in \mathcal{A} : \text{id}_{\mathcal{A}}(X) < \infty\}$.
- (b) $\text{pd}_{\mathcal{B}}(M) = \text{resdim}_{\omega}(M)$ for any $M \in \omega^{\wedge}$.
- (c) $\text{pd}_{\omega}(M) = \text{resdim}_{\mathcal{A}}(M)$ for any $M \in \mathcal{A}^{\wedge}$.

Proof. First of all, we observe that ω is an injective cogenerator for \mathcal{A} in the sense of Auslander–Buchweitz [5, p. 17], since $(\mathcal{A}, \mathcal{B})$ is hereditary and complete. Thus, statements (a) and (c) follow immediately from [5, Lemma 4.3 and Proposition 2.1]. As for (b), this is a consequence of the result dual to Proposition 2.1(b), because ω is a subcategory of \mathcal{B} closed under direct summands with $\text{pd}_{\mathcal{B}}(\omega) = 0$. □

Now, we are ready to state one of our key results.

THEOREM 2.5. *Let $(\mathcal{A}, \mathcal{B})$ be a hereditary complete cotorsion pair in $\text{Mod } R$ with kernel ω . The following statements hold true.*

- (a) $\text{pd } \mathcal{A} = \text{pd}_{\mathcal{A}}(\mathcal{A}) = \text{coresdim}_{\mathcal{B}}(\mathcal{A}) = \text{coresdim}_{\omega}(\mathcal{A}) = \text{coresdim}_{\mathcal{B}}(\text{Mod } R)$.

(b) $\text{pd}_{\mathcal{A}}(\mathcal{A}) < \infty$ if and only if $\mathcal{A} \subseteq \omega^\vee$ and $\text{pd}\omega < \infty$. In this case, we have

$$\text{pd}_{\mathcal{A}}(\mathcal{A}) = \text{pd}\omega = \text{id}_\omega(\mathcal{A}).$$

Proof. First of all, recall that $\text{pd}_{\mathcal{A}}(\mathcal{A}) = \text{pd}\mathcal{A}$ by Corollary 2.3(a).

If $\text{pd}_{\mathcal{A}}(\mathcal{A}) = \infty$, then by Corollary 2.3(b) we get that $\text{coresdim}_{\mathcal{B}}(\mathcal{A}) = \infty$, thus $\text{coresdim}_{\mathcal{B}}(\text{Mod } R) = \infty$, and since $\omega \subseteq \mathcal{B}$, also that $\text{coresdim}_\omega(\mathcal{A}) = \infty$.

If $\text{pd}_{\mathcal{A}}(\mathcal{A}) < \infty$, then Corollary 2.3(b) yields $\text{Mod } R = \mathcal{B}^\vee$, and we can apply Proposition 2.1(b) since $\text{id}_{\mathcal{A}}(\mathcal{B}) = 0$. So $\text{coresdim}_{\mathcal{B}}(\text{Mod } R) = \text{id}_{\mathcal{A}}(\text{Mod } R) = \text{pd}\mathcal{A}$. Further, $\mathcal{A} \subseteq \omega^\vee$ by Lemma 2.4(a). So, as in [11, Corollary 2.3], we infer from Proposition 2.1 that $\text{coresdim}_{\mathcal{B}}(\mathcal{A}) = \text{coresdim}_\omega(\mathcal{A}) = \text{id}_\omega(\mathcal{A}) = \text{id}_{\mathcal{A}}\mathcal{A} = \text{pd}\mathcal{A}$.

(b) \Rightarrow : We have seen above that $\mathcal{A} \subseteq \omega^\vee$ and $\text{Mod } R = \mathcal{B}^\vee$. Then the result dual to Lemma 2.4(c) yields that $\text{coresdim}_{\mathcal{B}}(\text{Mod } R) = \text{id}_\omega(\text{Mod } R) = \text{pd}\omega$. Hence, we conclude as in (a) that $\text{pd}\omega = \text{id}_\omega(\mathcal{A}) = \text{pd}_{\mathcal{A}}(\mathcal{A}) < \infty$.

\Leftarrow : Since $\omega \subseteq \mathcal{A} \subseteq \omega^\vee$, we deduce from Theorem 1.1(b) that $\text{pd}\mathcal{A} = \text{pd}\omega < \infty$. □

3. Applications to tilting theory

We now apply our previous results to cotorsion pairs arising from tilting theory.

DEFINITION 3.1. [1] A module $T \in \text{Mod } R$ is called *tilting module* provided:

- (T1) $\text{pd}T < \infty$,
- (T2) $\text{Ext}_R^i(T, T^{(I)}) = 0$ for any $i > 0$ and all sets I ,
- (T3) $\text{coresdim}_{\text{Add } T}(R_R) < \infty$.

Every tilting module T induces a complete hereditary cotorsion pair $(\mathcal{A}, \mathcal{B})$ in $\text{Mod } R$ where $\mathcal{B} = T^\perp$. It is called the *tilting cotorsion pair* induced by T . Tilting cotorsion pairs are characterized in [1] by the property that the kernel is closed under coproducts and $\text{pd}\mathcal{A} < \infty$. We can now refine this characterization as follows.

THEOREM 3.2. Let $(\mathcal{A}, \mathcal{B})$ be a hereditary complete cotorsion pair in $\text{Mod } R$ with kernel ω .

- (a) The following conditions are equivalent:
 - (a1) $(\mathcal{A}, \mathcal{B})$ is a tilting cotorsion pair.
 - (a2) $\text{pd}_{\mathcal{A}}(\mathcal{A}) < \infty$ and ω is closed under coproducts.
 - (a3) ω is closed under coproducts, $\text{pd}\omega < \infty$ and $\mathcal{A} \subseteq \omega^\vee$.
- (b) If $(\mathcal{A}, \mathcal{B})$ is a tilting cotorsion pair induced by a tilting R -module T , then $\text{pd}T = \text{pd}_{\mathcal{A}}(\mathcal{A}) = \text{coresdim}_{\mathcal{B}}(\mathcal{A}) = \text{coresdim}_{\text{Add } T}(\mathcal{A}) = \text{coresdim}_{\mathcal{B}}(\text{Mod } R)$.

Proof. By [1, Theorem 4.1], we know that $(\mathcal{A}, \mathcal{B})$ is a tilting cotorsion pair if and only if $\text{pd}\mathcal{A} < \infty$, and ω is closed under coproducts. So, the result follows from Theorem 2.5 using that $\omega = \text{Add } T$, cf. [1, Lemma 2.4]. □

An important result of Eklöf and Trlifaj [9, Theorem 10] states that any set \mathcal{X} of R -modules gives rise to a complete cotorsion pair $({}^{\perp 1}(\mathcal{X}^{\perp 1}), \mathcal{X}^{\perp 1})$, known as the cotorsion pair *generated* by \mathcal{X} . As a consequence, every resolving subcategory \mathcal{X} of $\text{mod } R$ consisting of modules of bounded projective dimension generates a tilting cotorsion pair, see [4, Theorem 2.2]. Also this result can be reduced to the relative projective dimension.

COROLLARY 3.3. *Let \mathcal{D} be a resolving subcategory of $\text{mod } R$ such that $\text{pd}_{\mathcal{D}}(\mathcal{D})$ is finite. Then the cotorsion pair $(\mathcal{A}, \mathcal{B})$ generated by \mathcal{D} is a tilting cotorsion pair.*

Proof. We know that $(\mathcal{A}, \mathcal{B})$ is hereditary, complete, and \mathcal{B} is closed under coproducts. Then also the kernel ω is closed under coproducts. Moreover, $\mathcal{A} \subseteq \varinjlim \mathcal{D}$ by [3, Theorem 2.3], so every $A \in \mathcal{A}$ has the form $A = \varinjlim D_i$. Hence, for any $D \in \mathcal{D}$ and $j > n$, we have

$$\text{Ext}_R^j(D, A) = \text{Ext}_R^j(D, \varinjlim D_i) \simeq \varinjlim \text{Ext}_R^j(D, D_i) = 0$$

since $D \in \text{mod } R$ and $\text{pd}_{\mathcal{D}}(\mathcal{D}) = n$. In other words, $\mathcal{D} \subseteq {}^{\perp j} \mathcal{A}$ for any $j > n$. Note that \mathcal{A} consists of all direct summands of \mathcal{D} -filtered modules, see [8, Corollary 3.2.3]. So, it follows from [7, Theorem 7.3.4] that $\mathcal{A} \subseteq {}^{\perp j} \mathcal{A}$ for $j > n$, proving that $\text{pd}_{\mathcal{A}}(\mathcal{A}) \leq n$. Now the statement follows immediately from Theorem 3.2. □

4. Finitistic dimensions

Let us now consider a slightly more general situation. We assume that our cotorsion pair $(\mathcal{A}, \mathcal{B})$ satisfies $\text{pd}_{\mathcal{A}}(\mathcal{A}) < \infty$, but we drop the assumption that ω is closed under coproducts. It turns out that this is the setup needed for studying the finitistic dimensions.

We start with some general properties.

PROPOSITION 4.1. *Let $(\mathcal{A}, \mathcal{B})$ be a hereditary complete cotorsion pair in $\text{Mod } R$ with kernel ω . Assume $\text{pd}_{\mathcal{A}}(\mathcal{A}) < \infty$. The following statements hold true.*

- (a) $\mathcal{A}^\wedge = \{M \in \text{Mod } R : \text{pd}_{\mathcal{B}}(M) < \infty\} = \mathcal{P}$.
- (b) $\text{resdim}_{\mathcal{A}}(M) = \text{pd}_{\mathcal{B}}(M)$ for any $M \in \mathcal{P}$.
- (c) $\text{pd } M \leq \text{pd}_{\mathcal{A}}(\mathcal{A}) + \text{resdim}_{\mathcal{A}}(M)$ for any $M \in \mathcal{P}$.
- (d) $\mathcal{P} \cap \mathcal{B} = \omega^\wedge$.

Proof. Since $\text{pd}_{\mathcal{B}}(\mathcal{A}) = 0$ and $\mathcal{A} = {}^{\perp 1} \mathcal{B}$, we get from the statement dual to Proposition 2.1(b) that $\text{pd}_{\mathcal{B}}(M) = \text{resdim}_{\mathcal{A}}(M)$ for any $M \in \mathcal{A}^\wedge$. This yields the inclusion $\mathcal{A}^\wedge \subseteq \{M \in \text{Mod } R : \text{pd}_{\mathcal{B}}(M) < \infty\}$. The reverse inclusion follows from the statement dual to Proposition 2.1(c). On the other hand, since $\mathcal{P}_0 \subseteq \mathcal{A}$, we have that $\text{resdim}_{\mathcal{A}}(M) \leq \text{resdim}_{\mathcal{P}_0}(M) = \text{pd } M$ for any $M \in \text{Mod } R$, implying $\mathcal{P} \subseteq \mathcal{A}^\wedge$. Finally, $\text{pd } \mathcal{A} < \infty$ by Corollary 2.3. Then

we infer from Lemma 1.1(c) that $\mathcal{A}^\wedge \subseteq \mathcal{P}$, which completes the proof of (a) and (b).

Dualizing Proposition 2.1(a), we obtain $\text{pd}_{\mathcal{A}}(M) \leq \text{pd}_{\mathcal{A}}(\mathcal{A}) + \text{resdim}_{\mathcal{A}}(M)$ for any $M \in \mathcal{A}^\wedge = \mathcal{P}$. Note moreover that $\mathcal{P} = \mathcal{P}_0^\wedge$ and $\text{pd } M = \text{resdim}_{\mathcal{P}_0}(M)$. Thus $\text{pd}_{\mathcal{A}}(M) = \text{pd } M$ by the dual of Proposition 2.1(b). This proves (c).

Finally, we prove (d). We have that $\omega^\wedge \subseteq \mathcal{P} \cap \mathcal{B}$ since \mathcal{P} and \mathcal{B} are closed under cokernels of monomorphisms, and $\text{pd } \omega < \infty$ by Theorem 2.5. For the reverse inclusion, let $X \in \mathcal{P} \cap \mathcal{B}$ and $r = \text{pd}_{\mathcal{B}}(X) \leq \text{pd } X$. Since $(\mathcal{A}, \mathcal{B})$ is complete and $\mathcal{A} \subseteq \mathcal{P}$, there is an exact sequence $0 \rightarrow K_r \rightarrow W_r \xrightarrow{f_r} W_{r-1} \rightarrow \dots \rightarrow W_1 \xrightarrow{f_1} W_0 \xrightarrow{f_0} X \rightarrow 0$ such that $W_i \in \omega$ and $K_i := \text{Ker}(f_i) \in \mathcal{P} \cap \mathcal{B}$ for each i . By assumption, $\text{Ext}_R^{r+1}(X, -)|_{\mathcal{B}} = 0$. Since $W_i \in {}^\perp K_{r+1}$, we infer by dimension shifting that $0 = \text{Ext}_R^{r+1}(X, K_r) \simeq \text{Ext}_R^1(K_{r-1}, K_r)$. Hence $K_{r-1} \in \omega$, and $X \in \omega^\wedge$. □

We now introduce two numbers $\alpha = \text{resdim}_{\mathcal{A}}(\mathcal{P})$ and $\beta = \text{resdim}_{\mathcal{A}}(\mathcal{P}^{<\infty})$ which allow to obtain interesting estimates for the finitistic dimensions.

LEMMA 4.2. *Let $(\mathcal{A}, \mathcal{B})$ be a hereditary complete cotorsion pair in $\text{Mod } R$ with kernel ω . Assume $\text{pd}_{\mathcal{A}}(\mathcal{A}) < \infty$, and set $\alpha = \text{resdim}_{\mathcal{A}}(\mathcal{P})$. The following statements hold true.*

- (a) $\alpha = \text{resdim}_{\mathcal{A}}(\omega^\wedge) = \text{resdim}_{\omega}(\omega^\wedge) = \text{pd}_{\omega}(\mathcal{P}) \leq \text{pd } \omega^\wedge$.
- (b) $\text{pd } \omega^\wedge = \text{Findim}(R_R)$.

Proof. (a) If $M \in \mathcal{P}$, then $\text{resdim}_{\mathcal{A}}(M) = \text{pd}_{\mathcal{B}}(M)$ by Proposition 4.1(b). Further, taking an exact sequence $0 \rightarrow M \rightarrow B \rightarrow A \rightarrow 0$ where $B \in \mathcal{B}$ and $A \in \mathcal{A}$, we have $B \in \mathcal{P} \cap \mathcal{B} = \omega^\wedge$. Thus, we infer from Lemma 1.1(b) that $\text{pd}_{\mathcal{B}}(M) \leq \text{pd}_{\mathcal{B}}(B) \leq \text{pd}_{\mathcal{B}}(\omega^\wedge)$. This proves $\alpha \leq \text{pd}_{\mathcal{B}}(\omega^\wedge)$. On the other hand, $\text{resdim}_{\mathcal{A}}(\omega^\wedge) \leq \alpha$ since $\omega^\wedge \subseteq \mathcal{P}$. Hence, $\alpha = \text{resdim}_{\mathcal{A}}(\omega^\wedge) = \text{pd}_{\mathcal{B}}(\omega^\wedge) \leq \text{pd}(\omega^\wedge)$ by Proposition 4.1.

Finally, by Lemma 2.4(b) and (c) we have $\alpha = \text{resdim}_{\omega}(\omega^\wedge) = \text{pd}_{\omega}(\mathcal{P})$.

(b) We know from Proposition 4.1 that $\omega^\wedge = \mathcal{P} \cap \mathcal{B} \subseteq \mathcal{P}$, so $\text{pd } \omega^\wedge \leq \text{Findim}(R_R)$. Now let $M \in \mathcal{P}$. As above, there is an exact sequence $0 \rightarrow M \rightarrow B \rightarrow A \rightarrow 0$ where $B \in \mathcal{P} \cap \mathcal{B} = \omega^\wedge$ and $A \in \mathcal{A}$. Then we know by Lemma 1.1(b) that $\text{pd } M \leq \max\{\text{pd } B, \text{pd } A - 1\}$. But $\text{pd } \mathcal{A} = \text{pd } \omega$ by Theorem 2.5, so $\text{pd } A \leq \text{pd } \omega^\wedge$. Thus, $\text{pd } M \leq \text{pd } \omega^\wedge$, and we conclude $\text{Findim}(R_R) = \text{pd } \omega^\wedge$. □

THEOREM 4.3. *Let $(\mathcal{A}, \mathcal{B})$ be a hereditary complete cotorsion pair in $\text{Mod } R$. Assume $\text{pd}_{\mathcal{A}}(\mathcal{A}) < \infty$, and set $\alpha = \text{resdim}_{\mathcal{A}}(\mathcal{P})$, and $\beta = \text{resdim}_{\mathcal{A}}(\mathcal{P}^{<\infty})$. The following statements hold true.*

- (a) $\alpha \leq \text{Findim}(R_R) \leq \text{pd}_{\mathcal{A}}(\mathcal{A}) + \alpha$.
- (b) $\beta \leq \text{findim}(R_R) \leq \text{pd}_{\mathcal{A}}(\mathcal{A}) + \beta$.

Proof. (a) $\alpha \leq \text{pd} \omega^\wedge = \text{Findim}(R_R)$ by Lemma 4.2, and $\text{Findim}(R_R) \leq \text{pd}_{\mathcal{A}}(\mathcal{A}) + \alpha$ by Proposition 4.1(c).

(b) We have that $\beta = \text{pd}_{\mathcal{B}}(\mathcal{P}^{<\infty}) \leq \text{pd} \mathcal{P}^{<\infty} = \text{findim}(R_R)$ by Proposition 4.1(b). Furthermore, $\text{findim}(R_R) \leq \text{pd}_{\mathcal{A}}(\mathcal{A}) + \beta$ by Proposition 4.1(c). □

We now obtain criteria for finiteness of the finitistic dimensions.

COROLLARY 4.4. *Let R be a ring.*

(I) *The following conditions are equivalent.*

(a) $\text{Findim}(R_R) < \infty$.

(b) *There is a hereditary complete cotorsion pair $(\mathcal{A}, \mathcal{B})$ in $\text{Mod } R$ such that $\text{pd}_{\mathcal{A}}(\mathcal{A}) < \infty$ and $\text{resdim}_{\mathcal{A}}(\mathcal{P}) < \infty$.*

(c) *Every hereditary complete cotorsion pair $(\mathcal{A}, \mathcal{B})$ in $\text{Mod } R$ such that $\text{pd}_{\mathcal{A}}(\mathcal{A}) < \infty$ satisfies $\text{resdim}_{\mathcal{A}}(\mathcal{P}) < \infty$.*

(II) *The following conditions are equivalent.*

(a) $\text{findim}(R_R) < \infty$.

(b) *There is a hereditary complete cotorsion pair $(\mathcal{A}, \mathcal{B})$ in $\text{Mod } R$ such that $\text{pd}_{\mathcal{A}}(\mathcal{A}) < \infty$ and $\text{resdim}_{\mathcal{A}}(\mathcal{P}^{<\infty}) < \infty$.*

(c) *Every hereditary complete cotorsion pair $(\mathcal{A}, \mathcal{B})$ in $\text{Mod } R$ such that $\text{pd}_{\mathcal{A}}(\mathcal{A}) < \infty$ satisfies $\text{resdim}_{\mathcal{A}}(\mathcal{P}^{<\infty}) < \infty$.*

Proof. (I) The implications (b) \Rightarrow (a) and (a) \Rightarrow (c) follow directly from Theorem 4.3. To prove (c) \Rightarrow (b), we choose the cotorsion pair $(\mathcal{P}_n, \mathcal{P}_n^\perp)$ for some natural number n . It is well known that $(\mathcal{P}_n, \mathcal{P}_n^\perp)$ is hereditary and complete, see [7, Theorem 7.4.6]. Furthermore, $\text{pd}_{\mathcal{P}_n}(\mathcal{P}_n) \leq n$ by Corollary 2.3, so $\text{resdim}_{\mathcal{A}}(\mathcal{P})$ is finite by the assumption in (c).

(II) It is proven similarly. □

Recall the the Second Finitistic Dimension Conjecture states that the little finitistic dimension $\text{findim}(R_R)$ is finite for every Artin algebra R . Corollary 4.4(II) gives criteria for the validity of this conjecture.

The First Finitistic Dimension Conjecture stated that $\text{Findim}(R_R)$ is equal to $\text{findim}(R_R)$. It is well known by now that this conjecture fails in general. The first example of this phenomenon was given by Huisgen–Zimmermann in 1992. Later, Smalø even showed that the difference between the big and the little finitistic dimension can be arbitrarily large, see [14]. However, in many interesting cases the two numbers do coincide. The following result characterizes this situation.

THEOREM 4.5. *Let R be a right noetherian ring with $\text{findim}(R_R) < \infty$. Let $(\mathcal{A}, \mathcal{B})$ be the cotorsion pair generated by $\mathcal{P}^{<\infty}$, and let ω be its kernel. The following statements hold true.*

(a) $\text{Findim}(R_R) - \text{findim}(R_R) \leq \text{resdim}_{\mathcal{A}}(\mathcal{P})$.

(b) $\text{Findim}(R_R) = \text{findim}(R_R)$ if and only if $\text{pd}\omega^\wedge = \text{pd}\omega$.

Proof. We know from [2, Theorem 2.6] and Theorem 3.2(b) that $(\mathcal{A}, \mathcal{B})$ is a tilting cotorsion pair with $\text{pd}_{\mathcal{A}}(\mathcal{A}) = \text{findim}(R_R)$. So, statement (a) follows from Theorem 4.3. On the other hand, by Proposition 2.5 and Lemma 4.2(b) we further have $\text{findim}(R_R) = \text{pd}\omega$, and $\text{Findim}(R_R) = \text{pd}\omega^\wedge$, proving statement (b). \square

REMARK 4.6. Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in $\text{Mod } R$. Set $\alpha = \text{resdim}_{\mathcal{A}}(\mathcal{P})$.

(1) Note that $\alpha < \infty$ does not imply that $\text{pd}_{\mathcal{A}}(\mathcal{A}) < \infty$. For example, if we take the trivial cotorsion pair given by $\mathcal{A} = \text{Mod } R$ and $\mathcal{B} = \mathcal{I}_0$, we always have $\alpha = 0$ since $\mathcal{P} \subseteq \mathcal{A}$, but $\text{pd}_{\mathcal{A}}(\mathcal{A})$ is the global dimension of R .

(2) Let $(\mathcal{A}, \mathcal{B})$ be hereditary and complete with $\text{pd}_{\mathcal{A}}(\mathcal{A}) < \infty$. Then $\alpha = 0$ if and only if $\mathcal{A} = \mathcal{P}$, and in this case $\text{Findim}(R_R) = \text{pd}_{\mathcal{A}}(\mathcal{A}) < \infty$.

In particular, if $(\mathcal{A}, \mathcal{B})$ is the cotorsion pair generated by $\mathcal{P}^{<\infty}$, then $\alpha = 0$ implies that $\text{Findim}(R_R) = \text{findim}(R_R) = \text{pd}_{\mathcal{A}}(\mathcal{A}) < \infty$.

So, our results generalize [2, Proposition 3.1 and Theorem 3.2].

(3) The inequality in Theorem 4.5(a) can be strict. For example, if R is the finite dimensional algebra from [10], then $\text{Findim}(R_R) = \text{findim}(R_R) = 1$, but the cotorsion pair $(\mathcal{A}, \mathcal{B})$ generated by $\mathcal{P}^{<\infty}$ satisfies $\alpha = 1$. In fact, it is shown in [3] that $\mathcal{A} \subsetneq \mathcal{P}$, thus $0 < \alpha \leq \text{Findim}(R_R) = 1$ by Theorem 4.3.

(4) Let R be a ring with $\text{findim}(R_R) < \infty$. If $(\mathcal{A}, \mathcal{B})$ is the cotorsion pair generated by $\mathcal{P}^{<\infty}$, then $\mathcal{P}^{<\infty} \subseteq \mathcal{A}$, and so $\beta = \text{resdim}_{\mathcal{A}}(\mathcal{P}^{<\infty}) = 0$.

We close the paper with a result concerning arbitrary tilting modules.

COROLLARY 4.7. *Let T be a tilting module in $\text{Mod } R$. If R is right noetherian then*

$$\text{Findim}(R_R) \leq \text{pd}T + \text{id}T.$$

Proof. We apply the first formula in Theorem 4.3 to the tilting cotorsion pair $(\mathcal{A}, \mathcal{B})$ induced by T . Recall that its kernel ω coincides with $\text{Add}T$, and $\text{pd}_{\mathcal{A}}(\mathcal{A}) = \text{pd}T$ by Theorem 3.2. So, we obtain $\text{Findim}(R_R) \leq \text{pd}T + \alpha$, where $\alpha = \text{pd}_{\text{Add}T}(\mathcal{P}) = \text{id}_{\mathcal{P}}(\text{Add}T) \leq \text{id}(\text{Add}T)$ by Lemma 4.2(a). Now, if R is right noetherian, then $\text{id}(\text{Add}T) = \text{id}T$. \square

REMARK 4.8. Let R be right noetherian.

(1) Note that there is a tilting module T in $\text{Mod } R$ with $\text{id}T < \infty$ if and only if $\text{id}R_R < \infty$. Indeed, the only-if-part holds true by condition (T3) in the definition of a tilting module, and for the if-part one chooses $T = R_R$.

Choosing $T = R_R$, Corollary 4.7 gives $\text{Findim}(R_R) \leq \text{id}(R_R)$. This was proved by Bass in [6].

(2) The inequality in Corollary 4.7 is sharp. To see this, let R be an Iwanaga–Gorenstein ring with $\text{id}(R_R) = n$. Consider the minimal injective coresolution $0 \rightarrow R_R \rightarrow I_0 \rightarrow I_1 \rightarrow \dots \rightarrow I_n \rightarrow 0$. By [4, Section 3], the injective module $T := \bigoplus_{i=0}^n I_i$ is tilting, and $\text{Findim}(R_R) = n = \text{pd}T = \text{pd}T + \text{id}T$.

We could also choose the tilting module $T = R_R$. In this case, $\text{Findim}(R_R) = \text{id}T = \text{pd}T + \text{id}T$.

(3) In case R is an artin algebra, $T = {}_R T$ is a classical tilting and cotilting left R -module and $A := \text{End}(T)$, it was proven in [15] that $\text{findim}({}_A A) \leq \text{pd}T + \text{id}T$.

(4) If R is a properly stratified algebra having a simple preserving duality, and such that every classical tilting right R -module is also cotilting (see [12]), then $\text{Findim}(R_R) = \text{findim}(R_R)$. Indeed, by [12, Theorem 1], we have that $\text{findim}(R_R) = 2\text{pd}T$ where T is the so called characteristic tilting module associated to the properly stratified algebra R . Moreover, as a consequence of the simple preserving duality, it can be seen that $\text{pd}T = \text{id}T$. Therefore, from Corollary 4.7 it follows that $2\text{pd}T = \text{findim}(R_R) \leq \text{Findim}(R_R) \leq 2\text{pd}T$, proving our assertion.

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