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AN ECKMANN-HILTON DUAL TO THE Π-ALGEBRAS OF HOMOTOPY THEORY

ANDREW PERCY

Abstract. We define an H-algebra to be an Eckmann-Hilton dual to the Π-algebras of homotopy theory. These cohomology algebras are an abstraction of the structure of cohomology groups together with the stable and unstable primary cohomology operations on them. We give sufficient properties of H-algebras to provide examples demonstrating the complexity of their structure and rôle in the homotopy classification problem. We reveal the generators of the integral cohomology operations and discuss relations between them. We also provide a series of restrictions leading from H-algebras to unstable algebras over the Steenrod algebra.

1. Introduction

Cohomology operations give an additional structure to the cohomology groups of a space and have been studied since the 1950's. Particularly useful for the classification of topological spaces have been the stable operations with $\mathbb{Z}/p$ coefficients which together with the cohomology groups give rise to unstable algebras over the Steenrod algebra. The effectiveness of these "cohomology algebras" is largely because the algebraic structure is so well understood. Thus we can, for instance, determine a complete list of polynomial algebras on two generators that can be realized as the $\mathbb{Z}/p$-cohomology of a space by determining their compatibility with the Steenrod algebra (see [1], §7.5).

In this paper we give an algebraic object modelling the integral cohomology groups together with all primary integral operations, stable and unstable. This is achieved by considering an Eckmann-Hilton dual to the Π-algebras of homotopy theory. We call these dual objects H-algebras though we may well have called them coΠ-algebras. We do not need to make any restrictions on the operations (they may be unstable or of higher torsion) however at the price of not, at this stage, being able to give an explicit description of

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the universal examples for cross-cap products nor all relations between the generating operations, although some are known. However, the generators and relations on the algebra of primary cohomology operations can be given by the internal structure of a certain "cohomology operation category". This category is denoted $\mathcal{H}$ and consists of finite products of integral Eilenberg-Mac Lane spaces. An H-algebra is a functor from $\mathcal{H}$ to the category of pointed sets sending products to products.

Π-algebras first appear in the literature around 1990 (see [8], [17] or [2]) and have been a useful algebraic invariant of homotopy type, especially in the problem of homotopy classification (see [3], [4] and [6]). In [5] Blanc has dualized his methods of [4] to give a partial classification of p-type using unstable coalgebras over the mod p Steenrod algebra as his algebraic foundation. His technique used these homology-like objects rather than cohomology-like objects because they were more natural for the dual method he was using. It is well known that we have some choice when determining which properties of the original concept we will Eckmann-Hilton dualize.

We make the definition of H-algebras in §2 and justify their duality with Π-algebras (compare with [16]) and also show that they model the "cohomology algebra" of a space. We address the problem of not explicitly knowing all relations on the set of generators for the k-ary cohomology operations and show that despite this, H-algebras form a category of universal graded algebras (CUGA). This allows the application of various techniques (see [7]) which we do not pursue here.

We look at a useful property by defining the product of H-algebras and determining its relation to the H-algebra of a wedge of spaces. The result allows us to give some examples, in §5, demonstrating the rôle of H-algebras in homotopy classification. These examples are limited to known operations but nevertheless demonstrate interesting phenomena such as the need for additional invariants for classification and the effect of changing coefficients.

We conclude §2 by using the Yoneda lemma to give a global description of the full subcategory of H-algebras of generalized Eilenberg-Mac Lane spaces. As a corollary we see that all morphisms in this category are induced by maps between the generalized Eilenberg-Mac Lane spaces.

In §3 we define cross-cap products which are known for homology [9] but may not appear in the literature for cohomology. We show that all primary operations are generated (under composition of universal examples) by compositions and cup products and give the relations between these operations known to the author and identify relations that remain to be determined.

In §4, we determine explicitly the duality between Π-algebras and unstable algebras over the Steenrod algebra. We first extend the definition of H-algebras to having coefficients in an arbitrary abelian group by taking the domain category to be products of Eilenberg-Mac Lane spaces over that group. If we then restrict this category to maps generated by stable operations and cup product then the image of a functor sending products to products will be an unstable algebra over the Steenrod algebra.

1.1. Notation and preliminaries. We will assume that all spaces are pointed connected CW-complexes which we write as $X \in \mathcal{CW}_*$. The n-th integral Eilenberg-Mac Lane space, $K(\mathbb{Z}, n)$, will be denoted K^n. A space with one non-trivial cohomology group will be called a co-Moore space. Examples of co-Moore spaces are the spheres.

We will refer to the Eckmann-Hilton dual of a concept as simply the dual of that concept. For the purposes of displaying the duality of homotopy and cohomology theories we look at reduced spectral cohomology associated to Eilenberg-Mac Lane spectra. Then, dual to the definition of homotopy groups, $\pi_n(X) = [S^n, X]$, we can define the ordinary, reduced cohomology groups by $\tilde{H}^n(X) = [X, K^n]$.

Given maps $f_i : Z \longrightarrow G_i$, the unique map into the product that they determine is denoted $\{f\}_{i\in I} : Z \longrightarrow \prod_{i\in I} G_i$. The composition of two maps f and g is given by juxtaposition gf which may also be denoted $g_*(f)$ or $f^*(g)$.

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2. H-algebras

The fact that primary homotopy operations are in bijective correspondence with elements of the homotopy groups of a wedge of spheres allows us to encode the Π-algebra structure as a contravariant functor from the category of finite wedges of spheres to pointed sets. The proof that cohomology operations are in bijective correspondence with elements of the cohomology groups of a product of Eilenberg-Mac Lane spaces is dual to that for homotopy operations (see [11], Theorem 12.1) provided we define cohomology operations to be k-ary operations over a fixed coefficient group rather than as unary operations involving changes of coefficient group, which is more customary. The duality of Property 2.2 and the dual property for homotopy operations justifies the claim that H-algebras are an Eckmann-Hilton dual to Π-algebras.

Definition 2.1. A *primary k-ary cohomology operation* of type $(n_1, n_2, \ldots, n_k; m)$ is a function of the underlying pointed sets of the cohomology groups,

$$\theta : \tilde{H}^{n_1}(X) \times \tilde{H}^{n_2}(X) \times \cdots \times \tilde{H}^{n_k}(X) \longrightarrow \tilde{H}^m(X), \quad m \geq \max\{n_i\},$$

which is natural in $X \in \mathcal{CW}_*$.

Property 2.2. The set of all primary cohomology operations of type $(n_1, n_2, \ldots, n_k; m)$ is in bijection with the elements of $\tilde{H}^m(\prod_{i=1}^k K^{n_i})$.

For any cohomology operation θ the bijection is given by $\theta(x_1, \ldots, x_k) = \theta(pr_1, \ldots, pr_k)\{x_1, \ldots, x_k\}$, where pr_i are the canonical projections associated to the product $\prod_{i=1}^{k} K^{n_i}$. Thus, $\theta(pr_1, \ldots, pr_k)$ is a *universal example* for the cohomology operation θ (see [14], Chapter III).

Example 2.3. For each $\gamma \in \tilde{H}^m(K^n)$ we have a *composition operation* $\gamma_* : \tilde{H}^n(X) \longrightarrow \tilde{H}^m(X), \quad x \longmapsto \gamma x$. The group object multiplication, $\mu_n : K^n \times K^n \longrightarrow K^n$, is the universal example for the *cohomology group addition*. We denote the universal example for *cup product* by $h_\cup$.

Definition 2.4. The *cohomology operation category*, $\mathcal{H}$, is the category of finite products of Eilenberg-Mac Lane spaces over $\mathbb{Z}$ and the point, with morphisms the homotopy classes of maps between them.

Definition 2.5. An *H-algebra* is a functor $Z : \mathcal{H} \longrightarrow \mathcal{SET}_*$ sending products to products and the point to 0.

Remark 2.6. For any H-algebra, Z, we write $Z^n = Z(K^n)$, and note that $Z^0 = Z(*) = 0$, where we use 0 for the pointed set with one element. We restrict the H-algebras by insisting that $Z(*) = 0$ so that we are modelling the cohomology of connected spaces.

Example 2.7. A *cohomology H-algebra* is a functor

$$[X, \ \ ] : \mathcal{H} \longrightarrow \mathcal{SET}_*, \quad X \in \mathcal{CW}_*$$

This functor, $[X, \ \ ]$, is the covariant hom functor with domain restricted to the full subcategory $\mathcal{H}$. We denote the cohomology H-algebra of a pointed space X by $\tilde{H}^*(X)$.

If we identify the isomorphic images in $\mathcal{SET}_*$ of all copies of $K^n \in \mathcal{H}$, then the image of the functor $[X, \ \ ]$ is the collection of cohomology groups with the primary cohomology operations on them.

The splitting of the long exact cohomology sequence of the pair $(K^p \times K^q, K^p \vee K^q)$ and the Künneth formula give us the following proposition.

Property 2.8.

$$\tilde{H}^m(K^p \times K^q) \cong \tilde{H}^m(K^p) \oplus \tilde{H}^m(K^q) \bigoplus_{i+j=m} \tilde{H}^i(K^p) \otimes \tilde{H}^j(K^q) \bigoplus_{i+j=m+1} \mathrm{Tor}(\tilde{H}^i(K^p), \tilde{H}^j(K^q))$$

We now use this result to justify the restriction to $m \geq \max\{n_i\}$ in Definition 2.1.

Given a morphism $\phi : \prod_j K^{n_j} \longrightarrow K^m$ with $\min\{n_j\} \leq m < \max\{n_j\}$, repeated application of Property 2.8 tells us that $\tilde{H}^m(\prod_{n_j>m} K^{n_j}) \cong 0$ since $\tilde{H}^m(K^{n_j}) = 0$ for $n_j > m$. Then, again by Property 2.8,

$$\tilde{H}^m(\prod_j K^{n_j}) = \tilde{H}^m(\prod_{n_j \leq m} K^{n_j} \times \prod_{n_j > m} K^{n_j}) \cong \tilde{H}^m(\prod_{n_j \leq m} K^{n_j})$$

so that we can consider the Eilenberg Mac Lane spaces, K^{n_j} with $n_j > m$, as not involved in the operation, $\prod_{n_j \leq m} \tilde{H}^{n_j}(X) \longrightarrow \tilde{H}^m(X)$, that is induced by ϕ.

Example 2.9. A *trivial H-algebra* is a functor sending each $K^n \in \mathcal{H}$ to a set $Z^n \in \mathcal{SET}_*$ in such a way that the only non-trivial operations are of the form $Z^n \times Z^n \times \cdots \times Z^n \longrightarrow Z^n$. In particular, this allows the group operation on each Z^n. By Remark 4.1, a trivial H-algebra will be a graded abelian group ($\mathbb{Z}$-module). In Corollary 2.17, we show that examples of trivial H-algebras are given by the cohomology H-algebra of a wedge of co-Moore spaces. Specifically, $\tilde{H}^*(S^2 \vee S^3)$ is a graded abelian group with copies of $\mathbb{Z}$ in degrees 2 and 3.

Definition 2.10. A *morphism of H-algebras* $\phi : Z \longrightarrow W$ is a natural transformation.

That is, ϕ is a collection of functions ϕ_C with $\phi_C : Z(C) \longrightarrow W(C)$, for each $C \in \mathcal{H}$, such that, given any map $\alpha : C \longrightarrow C'$ we have $W(\alpha)\phi_C = \phi_{C'} Z(\alpha)$.

Property 2.11. A weak homotopy equivalence $f : X \longrightarrow Y$ induces an isomorphism of H-algebras $f^* : \tilde{H}^*(Y) \cong \tilde{H}^*(X)$.

In the dual case for Π-algebras the Hilton theorem tells us that all primary homotopy operations are generated (by composition of universal examples) by Whitehead products and compositions, as well as the action of the fundamental group, and all relations between these operations are known. In the decomposition of Property 2.8, with coefficients in $\mathbb{Z}$, we need an interpretation of the cross-cap products in the Tor groups as maps between Eilenberg-Mac Lane spaces before we can give an explicit set of universal examples that generate all primary cohomology operations. We discuss the universal examples generating all cohomology operations and what is known of the relations between them in §3 but for now we point out that, by Property 2.2, any relation between cohomology operations can be expressed in terms of the universal examples corresponding to each side of the equation being in the same homotopy class of maps. In this way, any relation between operations is implicit in the category $\mathcal{H}$.

EXAMPLE 2.12. If $\gamma, \delta \in \tilde{H}^m(K^n)$ and $x \in \tilde{H}^n(X)$ then $(\gamma+\delta)x = \gamma x + \delta x$ and this relation between composition and group addition is a statement that $\mu_m\{\gamma, \delta\}x \sim \mu_m\{\gamma x, \delta x\}$.

The fact that the set of generating homotopy operations and all relations between them are known allows us an alternative definition of Π-algebras as a graded group with three operations satisfying the relations between Whitehead product, composition and action of the fundamental group. The alternative definition of H-algebras is not as satisfying but will still be useful in defining products, free objects and providing examples. The alternative definition of an H-algebra, Z, is given by its image in $\mathcal{SET}_*$.

DEFINITION 2.13. An *H-algebra* is a graded abelian group together with operations that are in bijective correspondence with elements of the groups $\tilde{H}^m(\prod_{i=1}^{j} K^{n_i})$ and that obey the set of compatibility identities implicit in the category $\mathcal{H}$.

REMARK 2.14. Definition 2.13 tells us that H-algebras, like Π-algebras, form a CUGA or category of universal algebras with underlying graded group. All CUGA's are equipped with adjoint free and underlying functors and also have all limits and colimits, hence products and coproducts ([7], §2.1.2 or [14]).

DEFINITION 2.15. The *product of H-algebras* $\prod_{i\in I} J_i$ is given by the graded abelian group $\{\prod_{i\in I}(J_i)^m \mid m \in \mathbb{N}\}$ with primary cohomology operations acting componentwise.

Here $(J_i)^m = J_i(K^m)$ is the group in degree m of the H-algebra J_i and $\prod_{i\in I}(J_i)^m$ is their (direct) product. The projections are given by $pr_l : \prod_{i\in I} J_i \longrightarrow J_l,\ \ (j_i)_{i\in I} \longmapsto j_l$. Given H-algebra morphisms $x_i : Z \longrightarrow J_i$, the unique map into the product that they determine is given by $\{x_i\}_{i\in I} : Z \longrightarrow \prod_{i\in I} J_i,\ z \longmapsto (x_i(z))_{i\in I}$.

THEOREM 2.16. *Given a collection of spaces $\{X_i \mid i \in I\}$,*

$$\tilde{H}^*(\vee_{i\in I} X_i) \cong \prod_{i\in I} \tilde{H}^*(X_i) \qquad \text{as } H\text{-algebras.}$$

The proof of Theorem 2.16 is a straightforward consequence of the universal properties of products and coproducts and the componentwise action of operations on a product of H-algebras.

Since any co-Moore space has only one non-trivial cohomology group we have

COROLLARY 2.17. *Let $\{X_i \mid i \in I\}$ be a collection of co-Moore spaces. Then $\tilde{H}^*(\vee_{i\in I} X_i)$ is a trivial H-algebra.*

Dual to the fact that the homotopy Π-algebras of wedges of spheres are the free objects in the category of Π-algebras ([17], §4.5), the cohomology H-algebra of a product of Eilenberg-Mac Lane spaces is a free H-algebra. However, the construction is not strictly dual.

Let $U : H - \mathcal{ALG} \longrightarrow Gr\mathcal{SET}_*$ be the underlying functor from H-algebras to the category of graded, pointed sets. We will construct the free functor $F : Gr\mathcal{SET}_* \longrightarrow H - \mathcal{ALG}$ as a left adjoint to the underlying functor.

Given $G_\bullet, H_\bullet \in Gr\mathcal{SET}_*$ and $f : G_\bullet \longrightarrow H_\bullet$, let

$$FG_\bullet = [\prod_{n\in\mathbb{N}} \prod_{g\in G_n^-} K_g^n,\ ] \qquad \text{where } G_n^- = G_n/*$$

and $Ff : FG_\bullet \longrightarrow FH_\bullet$ be induced by the map

$$\hat{f} : \prod_{n\in\mathbb{N}} \prod_{h\in H_n^-} K_h^n \longrightarrow \prod_{n\in\mathbb{N}} \prod_{g\in G_n^-} K_g^n$$

described as follows.

Since f is a graded function we write $f = \{f_n : G_n \longrightarrow H_n \mid n \in \mathbb{N}\}$. Let $j \in J_h$ index the elements $g_j \in G_n^-$ whose image under f_n is $h \in H_n^-$. Let $\{1_{K^n}\}_{j\in J_h} : K_h^n \longrightarrow \prod_{j\in J_h} K_{g_j}^n$ be the unique product map determined by $pr_j\{1_{K^n}\}_{j\in J_h} = 1_{K^n}$ We define a map $\tilde{f} : \prod_{n\in\mathbb{N}} \prod_{h\in H_n^-} K_h^n \longrightarrow (\prod_{n\in\mathbb{N}} \prod_{j\in J_h} K_{g_j}^n) \times *$ by the diagram

$$\begin{tikzcd}
\prod\limits_{n\in\mathbb{N}} \prod\limits_{h\in \mathrm{Im}\, f_n^-} K_h^n \arrow[rr, "\prod\limits_{n\in\mathbb{N}} \prod\limits_{h\in \mathrm{Im}\, f_n^-} \{1_{K^n}\}_{j\in J_h}"] & & \prod\limits_{n\in\mathbb{N}} \prod\limits_{j\in J_h} K_{g_j}^n \\
\prod\limits_{n\in\mathbb{N}} \prod\limits_{h\in H_n^-} K_h^n \arrow[u, "pr"] \arrow[d, "pr"'] \arrow[rr, "\tilde{f}"] & & (\prod\limits_{n\in\mathbb{N}} \prod\limits_{j\in J_h} K_{g_j}^n) \times * \arrow[u, "pr"'] \arrow[d, "pr"] \\
\prod\limits_{n\in\mathbb{N}} \prod\limits_{\substack{h\notin \mathrm{Im}\, f_n^- \\ h\neq *}} K_h^n \arrow[rr] & & *
\end{tikzcd}$$

Let $\phi : (\prod_{n\in\mathbb{N}} \prod_{j\in J_h} K_{g_j}^n)\times * \xrightarrow{\cong} \prod_{n\in\mathbb{N}} \prod_{j\in J_h} K_{g_j}^n$ and $i : \prod_{n\in\mathbb{N}} \prod_{j\in J_h} K_{g_j}^n \hookrightarrow \prod_{n\in\mathbb{N}} \prod_{g\in G_n^-} K_g^n$. We define

$$\hat{f} = i\phi\tilde{f} : \prod_{n\in\mathbb{N}} \prod_{h\in H_n^-} K_h^n \longrightarrow \prod_{n\in\mathbb{N}} \prod_{g\in G_n^-} K_g^n$$

and let $Ff = \hat{f}^* : FG_\bullet \longrightarrow FH_\bullet$.

The unit of adjunction is the natural transformation $\varepsilon = \{\varepsilon_{G_\bullet} \mid G_\bullet \in Gr\mathcal{SET}_*\}$ with $\varepsilon_{G_\bullet} : G_\bullet \longrightarrow UFG_\bullet = \{H^m(\prod_{n\in\mathbb{N}}\prod_{g\in G_n^-} K_g^n) \mid m \in \mathbb{N}\}$ given by identifying $g \in G_n^-$ with $pr_g : \prod_{n\in\mathbb{N}}\prod_{g\in G_n^-} K_g^n \longrightarrow K_g^n$ and $* \in G_n$ with $0 \in H^n(\prod_{n\in\mathbb{N}}\prod_{g\in G_n^-} K_g^n)$.

We define the counit of adjunction $\delta : FU \longrightarrow 1$ as follows. Given $\alpha \in FUZ$ we have $\alpha : \prod_{n\in\mathbb{N}}\prod_{u\in Z^{n-}} K_u^n \longrightarrow K^p$ for some $p \in \mathbb{N}$ inducing

$$Z(\alpha) : Z(\prod_{n\in\mathbb{N}}\prod_{u\in Z^{n-}} K_u^n) \longrightarrow Z(K^p)$$

We select the element $(u)_{u\in Z^n, n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}}\prod_{u\in Z^{n-}} Z_u^n$ having u in the factor of Z_u^n. Since Z sends products to products we can identify $(u)_{u\in Z^n,n\in\mathbb{N}}$ with a certain element $w \in Z(\prod_{n\in\mathbb{N}}\prod_{u\in Z^{n-}} K_u^n)$. Then we have $Zpr_u(w) = u$. For every $p \in \mathbb{N}$ we define

$$\delta_Z : FUZ(K^p) \longrightarrow Z(K^p), \quad \alpha \longmapsto Z\alpha(w).$$

Then $\delta : FU \longrightarrow 1$ is the natural transformation given by $\{\delta_Z \mid Z \in H-\mathcal{ALG}\}$.

As a consequence of the definition of H-algebras we can provide an alternative description of the subcategory of cohomology H-algebras of generalized Eilenberg-Mac Lane spaces.

Theorem 2.18. *The subcategory of H-algebras given by the cohomology H-algebras $[X,\ ]$, with $X \in \mathcal{H}$, is equivalent to the category $\mathcal{H}^{op}$.*

Proof. The result follows from the Yoneda lemma which states for $K \in \mathcal{H}$,

$$Nat([K,\ ],[X,\ ]) \cong \mathcal{H}[X,K] \cong \mathcal{H}^{op}[K,X]$$

and the equivalence is given by identifying $[X,\ ]$ with $X \in \mathcal{H}^{op}$ and the morphism of H-algebras $\phi : [K,\ ] \longrightarrow [X,\ ]$ with the morphism given by

$$\phi_K(1_K) : X \longrightarrow K \in \mathcal{H}^{op}[K,X]. \qquad \square$$

Corollary 2.19. *In the full subcategory of H-algebras given by $[X,\ ]$, $X \in \mathcal{H}$, all morphisms are induced by maps in $\mathcal{H}$.*

Proof. Every map $f : X \longrightarrow Y$ in $\mathcal{H}$ induces an H-algebra morphism $f^* : [Y,\ ] \longrightarrow [X,\ ]$. But the Yoneda bijection gives

$$y(f^*) = f_Y^*(1_Y) = 1_Y f = f$$

hence, the induced morphisms are the only morphisms. $\square$

3. Generators of cohomology operations

In this section we give details about the generating primary cohomology operations. It is convenient to consider cellular cochain complexes since, as yet, an equivalent description as maps between Eilenberg-Mac Lane spaces has not been found for the cross-cap products.

3.1. Cross products. Given spaces X, Y and cohomology classes $< x >\in H^i(X)$, $< y >\in H^j(Y)$ we define the *cohomology cross product* ([10], p. 278) $\bar{\times} : H^i(X)\times H^j(Y) \longrightarrow H^{i+j}(X\times Y)$ as being induced in cellular cohomology by the cochain map

$$\bullet : C^i(X) \times C^j(Y) \longrightarrow C^{i+j}(X \times Y), \ \ (x, y) \longmapsto x(e^i_\alpha)y(e^j_\beta)$$

for each generator $e^i_\alpha \in C_i(X)$ and $e^j_\beta \in C_j(Y)$. For any other (i+j)-cell $e^{i-n}_\alpha \times e^{j+n}_\beta \in C_{i+j}(X \times Y)$, we set $(x \bullet y)(e^{i-n}_\alpha \times e^{j+n}_\beta) = 0$. The cohomology cross product is also known as the *external cup product* ([10], p. 210 and p. 278) since

$$< x > \bar{\times} < y >= pr^*_1 < x > \cup\, pr^*_2 < y >,$$

where pr^*_1, pr^*_2 are induced by composition with the projections associated to $X \times Y$. This product is bilinear ([10], p. 218) so we have a homomorphism

$$\bar{\times} : H^i(X) \otimes H^j(Y) \longrightarrow H^{i+j}(X \times Y).$$

The coboundary map satisfies ([10], p. 279)

$$\delta(x \bullet y) = \delta x \bullet y + (-1)^i x \bullet \delta y.$$

3.2. Cross-cap products. Given a class $< x >\in H^i(X)$ which has n-torsion, $< nx >= 0 \in H^i(X)$ so nx is a coboundary, hence there is $a \in C^{i-1}(X)$ with $\delta a = nx$. Similarly, given a class $< y >\in H^j(Y)$ which has m-torsion, my is a coboundary, hence there is $b \in C^{j-1}(Y)$ with $\delta b = my$. Let $l = (n, m)$ be the highest common divisor of n and m and let $n = lg$ and $m = lh$.

Consider the cochain $gx \bullet b + (-1)^{i+1} a \bullet hy \in C^{i+j-1}(X \times Y)$. We have

$$\begin{aligned}
\delta(gx \bullet b + (-1)^{i+1} a \bullet hy) &= \delta(gx \bullet b) + (-1)^{i+1}\delta(a \bullet hy)\\
&= g\delta x \bullet b + (-1)^i gx \bullet \delta b\\
&\qquad + (-1)^{i+1}(\delta a \bullet hy + (-1)^{i+1} a \bullet h\delta y)\\
&= (-1)^i gx \bullet \delta b + (-1)^{i+1}\delta a \bullet hy\\
&= (-1)^i gx \bullet my + (-1)^{i+1} nx \bullet hy\\
&= (-1)^i gx \bullet lhy + (-1)^{i+1} lgx \bullet hy\\
&= (-1)^i lgh(x \bullet y) + (-1)^{i+1} lgh(x \bullet y)\\
&= 0.
\end{aligned}$$

Hence, $gx \bullet b + (-1)^{i+1} a \bullet hy$ is a cocycle and we define the cross-cap product of $< x >\in H^i(X)$ and $< y >\in H^j(Y)$ as

$$< x > \bullet_{n,m} < y >=< gx \bullet b + (-1)^{i+1} a \bullet hy >\in H^{i+j-1}(X \times Y).$$

$< x > \bullet_{n,m} < y >$ has l-torsion since

$$\begin{aligned}\delta(a \bullet b) &= \delta a \bullet b + (-1)^{i+1} a \bullet \delta b \\ &= lgx \bullet b + (-1)^{i+1} a \bullet lhy \\ &= l(gx \bullet b + (-1)^{i+1} a \bullet hy)\end{aligned}$$

so that $l(gx \bullet b + (-1)^{i+1} a \bullet hy)$ is a coboundary.

3.3. Cross-cap products and the Künneth Formula. The Künneth Formula is derived ([10], p. 274) in the form of a split short exact sequence

$$\bigoplus_{i+j=n} H^i(X) \otimes H^j(Y) \xrightarrow{\bar{\times}} H^n(X \times Y) \longrightarrow \bigoplus_{i+j=n+1} \mathrm{Tor}(H^i(X), H^j(Y))$$

which gives us

$$\begin{aligned}H^n(K^p \times K^q) \cong \quad & H^n(K^p) \oplus H^n(K^q) \bigoplus_{\substack{i+j=n \\ i,j \neq 0}} H^i(K^p) \otimes H^j(K^q) \\ & \bigoplus_{i+j=n+1} \mathrm{Tor}(H^i(K^p), H^j(K^q)).\end{aligned} \tag{3.1}$$

The Tor terms correspond to cross-cap products. The proof follows that of Eilenberg and Mac Lane ([9], §11 and §12) who give a set of generators and relations for the Tor groups and show that the cross cap products of homology form a group also given by these generators and relations. The proof is valid for cochain complexes as well as chain complexes. An indication of how the generators and relations correspond is given by comparing the facts that $\mathrm{Tor}(\mathbb{Z}/n, \mathbb{Z}/m) = \mathbb{Z}/l$ for $l = (n, m)$ and the cross-cap product of an element of n-torsion and an element of m-torsion has l-torsion.

We now look at the operations for which equation (3.1) provides universal examples. The summands $H^n(K^p)$ and $H^n(K^q)$ give composition operations on $H^p(X)$ and $H^q(X)$ resulting in an element of $H^n(X)$. The summands $H^i(K^p) \otimes H^j(K^q)$ with $i + j = n$ and $i, j \neq 0$ give operations $H^p(X) \times H^q(X) \longrightarrow H^n(X)$. For $\alpha : K^p \longrightarrow K^i$, $\beta : K^q \longrightarrow K^j$, $a \in H^p(X)$ and $b \in H^q(X)$ the operation is given by

$$(a, b) \longmapsto \alpha a \cup \beta b$$

as shown in the following diagram in which we have identified cross products with the external cup product and denoted the universal example for cup product by $h_\cup$.

$$\begin{tikzcd}
 & K^p \arrow[rr, "\alpha"] & & K^i & \\
X \arrow[ur, "a"] \arrow[r, "\{a,b\}"] \arrow[dr, "b"'] & K^p \times K^q \arrow[u, "pr_1"'] \arrow[d, "pr_2"] \arrow[rr, "\alpha\times\beta"] & & K^i \times K^j \arrow[u, "pr_1"'] \arrow[d, "pr_2"] \arrow[r, "h_\cup"] & K^{i+j} \\
 & K^q \arrow[rr, "\beta"'] & & K^j &
\end{tikzcd}$$

Though a little more complicated, given an element with n-torsion $\alpha : K^p \longrightarrow K^i$ with $\delta\epsilon = n\alpha$, an element of m-torsion $\beta : K^q \longrightarrow K^j$ with $\delta\gamma = m\beta$, $a \in H^p(X)$ and $b \in H^q(X)$, the cross-cap product of a and b is given by the composite

$$a \bullet_{n,m} b = \mu_{i+j-1}\{h_\cup\{g\alpha a, \gamma b\}, (-1)^{i+1} h_\cup\{\epsilon a, h\beta b\}\}$$

where the factor of $(-1)^{i+1}$ is replaced by the identity map if i is odd and the group inversion if i is even.

Thus, all primary operations are composites of compositions and cup products with the understanding that for cross-cap products we must make a selection of ϵ and γ such that $\delta\epsilon = n\alpha$ and $\delta\gamma = m\beta$. It seems necessary to find an explicit method of choosing these maps ϵ and γ given only a knowledge of the maps α and β but the author cannot provide such a method at this stage.

3.4. Relations between operations. To gain an explicit description of H-algebras as graded abelian groups together with the operations of composition and cup product, we need to know all relations between these operations. This was achieved for stable operations in integral cohomology by Kochman (see [13]) with the restriction to spaces having no higher torsion in their homology groups. Relations between unstable operations are known for some general cohomology theories (see [12]), but the results do not apply to integral cohomology.

We list the following relations which are straightforward consequences of universal examples for cohomology operations and the universal property of products.

Property 3.1. If $f, g \in \tilde{H}^n(K^p)$, $h \in \tilde{H}^m(K^p)$, $x \in \tilde{H}^n(X)$ and $y \in \tilde{H}^p(X)$ then

$$(f+g)_*(x) = f_*(x) + g_*(x), \tag{3.2}$$

$$(f \cup h)_*(y) = f_*(y) \cup h_*(y). \tag{3.3}$$

A decomposition for $f_*(x + x')$ is not known to the author although if a dual to the Hilton-Hopf invariants could be formulated then such a relation should be obtainable from Property 2.8 in a similar way to deriving the Hilton Formula from the Hilton Theorem in homotopy theory.

A decomposition for $f_*(x) \cup z$ would generalize the Cartan Formulas for stable operations over finite cyclic coefficient groups and would be dual to the Barcus-Barratt formula. To the author's knowledge, such a relation does not appear in the literature for integral coefficients and unstable compositions.

4. Arbitrary abelian group coefficients

In this section we generalize H-algebras to cohomology algebras over an arbitrary abelian group R. We will use the constructions given previously for H-algebras and simply reinterpret the symbols $\tilde{H}^n(X)$ and K^n as the cohomology group $\tilde{H}^n(X;R)$ and the Eilenberg-Mac Lane space $K(R,n)$. We note, though, in order to have cup products we require the coefficients to be in a commutative ring with unity.

In fact, if we consider functors from the category of finite products of all Eilenberg-Mac Lane spaces over abelian groups then the image of these functors in $\mathcal{SET}_*$ (identifying isomorphic images) will be an algebraic object consisting of all cohomology groups and primary operations including changes of coefficients. We could also consider cohomology operation categories for any spectral cohomology theory.

H-algebras are clearly a generalization of other known "primary cohomology algebras" and we now give some details relating H-algebras to unstable algebras over the Steenrod algebra.

Remark 4.1. The compositions of degree 0 give $H^n(X;R)$ a module structure over the coefficient group, R, whenever R is a cyclic abelian group. This follows from the fact that, for these coefficient groups, $\tilde{H}^n(K(R,n);R) \cong R$ and, using Example 2.12, defining the action of $r \in R$ on $x \in \tilde{H}^n(X;R)$ by

$$rx = \underbrace{(1_{K(R,n)} + \cdots + 1_{K(R,n)})}_{r \text{ summands}} x = \underbrace{x + \cdots + x}_{r \text{ summands}}.$$

This gives the standard module structure for these coefficient groups. With cohomology compositions over the integers, the abelian group structure is returned, however, for the integers modulo p the compositions of degree 0 give a $\mathbb{Z}/p$ -module structure on each cohomology group. It also follows from the definition of the ring action that the stable operations are R-module homomorphisms (see [15]).

We let $\mathcal{H}_S(R)$ be the category of finite products of Eilenberg-Mac Lane spaces over R with morphisms generated, under composition, by maps in the stable range of cohomology operations, projections and the group object multiplications. If R is a ring with underlying cyclic abelian group, by Remark 4.1, the image of any functor $Z : \mathcal{H}_S(R) \longrightarrow \mathcal{SET}_*$ is a graded R-module with stable operations (R-module homomorphisms) acting on it, provided we identify all isomorphic copies of Z^n, $n \geq 0$. Then we can consider functors $Z : \mathcal{H}_S(R) \longrightarrow R{-}\mathcal{MOD}$ sending products to products allowing for the group additions but also allowing for the existence of tensor products. Then the R-bilinear composition of stable operations gives rise to an R-algebra structure on the collection of stable operations and since cohomology R-modules form a graded module over the R-algebra of stable operations, the graded R-module $Z^* = \{(Z;R)^n \mid n \geq 0\}$ is also a graded module over this R-algebra. Then we have, for example, that the image of a functor $Z : \mathcal{H}_S(\mathbb{Z}) \longrightarrow \mathbb{Z} - \mathcal{MOD}$ sending products to products and the point to 0, is a module over the stable integral cohomology operations, as studied by Kochman (see [13]).

Similarly, we let $\mathcal{H}_S^{\cup}(R)$ be the category of finite products of Eilenberg-Mac Lane spaces over R with morphisms generated, under composition, by the universal examples for cup product as well as the maps generating the morphisms of $\mathcal{H}_S(R)$. By the same argument given above, the image of a functor $Z : \mathcal{H}_S^{\cup}(\mathbb{Z}/2) \longrightarrow \mathbb{Z}/2 - \mathcal{MOD}$ sending products to products and the point to 0, is an unstable algebra over the Steenrod algebra.

5. Examples

Example 5.1. $\mathbb{C}P^2$ and $S^2 \vee S^4$ have the same cohomology groups but different integral cohomology rings and hence, have different H-algebras. By Property 2.11, these spaces are not homotopy equivalent.

The cohomology ring of the complex projective spaces $\mathbb{C}P^n$ are known to be truncated polynomial algebras, $\mathbb{Z}[\alpha]/(\alpha_{n+1})$, on one generator of degree 2, ([10], Theorem 3.12, p. 212). Clearly then $\mathbb{C}P^2$ has a nontrivial cohomology ring. By Theorem 2.16 and Corollary 2.17, $S^2 \vee S^4$ has the same cohomology groups but a trivial H-algebra and hence trivial cohomology ring.

Coefficients in $\mathbb{Z}/2$ have the property that

$$\cup^2 : \tilde{H}^n(X;\mathbb{Z}/2) \longrightarrow \tilde{H}^{2n}(X;\mathbb{Z}/2), \quad x \longmapsto x \cup x$$

is a stable operation ([10], p. 489). Consequently, there is an element $f \in \tilde{H}^{2n+1}(K(\mathbb{Z}/2, n+1);\mathbb{Z}/2)$ such that $h_{\cup^2} \sim \Omega f$, where $h_{\cup^2}$ is the universal example for $\cup^2$. We use this fact and the adjunction of suspension and loop functors to find a non-trivial composition in Example 5.2.

Example 5.2. The spaces $\Sigma\mathbb{R}P^2$ and $\Sigma\mathbb{R}P \vee \Sigma^2\mathbb{R}P$ have the same trivial cohomology ring structure but different $H(\mathbb{Z}/2)$-algebras since there is a non-trivial composition on $\tilde{H}^*(\Sigma\mathbb{R}P^2;\mathbb{Z}/2)$.

We know ([10], Theorem 3.12, p. 212) that the cohomology ring of $\mathbb{R}P^n$, with $\mathbb{Z}/2$ coefficients, is a polynomial ring generated by an element of degree

1, truncated at degree $n+1$. Then, by the adjunction of Ω and Σ,

$$\tilde{H}^i(\Sigma\mathbb{R}P^2;\mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & i=2,3 \\ 0 & \text{otherwise} \end{cases}$$

and, by Theorem 2.16, $\Sigma\mathbb{R}P \vee \Sigma^2\mathbb{R}P$ has the same cohomology groups over $\mathbb{Z}/2$ and has trivial compositions since these act componentwise. Both spaces have trivial cohomology ring structure over $\mathbb{Z}/2$ because the cup product of any non-zero elements is in a trivial group. We will look for an operation on $\tilde{H}^*(\Sigma\mathbb{R}P^2;\mathbb{Z}/2)$ given by composition with an element of $\tilde{H}^3(K(\mathbb{Z}/2,2);\mathbb{Z}/2)$. However, by the adjunction of Ω and Σ, we will achieve this if we can find a non-trivial composition on $\tilde{H}^*(\mathbb{R}P^2;\mathbb{Z}/2)$ by a loop element of the group $\tilde{H}^2(K(\mathbb{Z}/2,1);\mathbb{Z}/2)$.

If α generates $\tilde{H}^1(\mathbb{R}P^2;\mathbb{Z}/2)$ then $\cup^2(\alpha)=\alpha\cup\alpha$ generates $\tilde{H}^2(\mathbb{R}P^2;\mathbb{Z}/2)$ showing that composition with $h_{\cup^2}\in\tilde{H}^2(K(\mathbb{Z}/2,1);\mathbb{Z}/2)$ is non-trivial. Since $h_{\cup^2}\sim\Omega f$ for some $f\in\tilde{H}^3(K(\mathbb{Z}/2,2);\mathbb{Z}/2)$, a non-trivial composition is given by $f_*:\tilde{H}^2(\Sigma\mathbb{R}P^2;\mathbb{Z}/2)\longrightarrow\tilde{H}^3(\Sigma\mathbb{R}P^2;\mathbb{Z}/2)$.

The next example we shall look at demonstrates that the H-algebra structure is insufficient, on its own, for homotopy classification.

Example 5.3. The spaces, $\mathbb{R}P^3$ and $S^3\vee\mathbb{R}P^2$ have the same H-algebras but are not homotopy equivalent.

We know that ([10], p. 144, and the Universal Coefficient Theorem)

$$\tilde{H}^i(\mathbb{R}P^3;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/2 & i=2, \\ \mathbb{Z} & i=3, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\tilde{H}^i(\mathbb{R}P^2) \cong \begin{cases} \mathbb{Z}/2 & i=2, \\ 0 & \text{otherwise.} \end{cases}$$

Then, by Theorem 2.16, $\mathbb{R}P^3$ and $S^3\vee\mathbb{R}P^2$ have isomorphic cohomology groups.

By Corollary 2.17, $S^3\vee\mathbb{R}P^2$ has a trivial H-algebra. Now, cup products are trivial on $\mathbb{R}P^3$ because the image groups are always trivial. Compositions are also trivial since $\tilde{H}^3(K^2)\cong 0$. There are, in fact, no non-trivial operations by an inductive argument using Property 2.8 and the facts that $\tilde{H}^2(K^2)\cong\mathbb{Z}$ and $\tilde{H}^1(K^2)\cong 0$. By Remark 4.1, a factor of K^3 in the domain space of a universal example returns the group structure on $\tilde{H}^3(\mathbb{R}P^3)$ and hence induces a trivial operation. Consequently, the spaces have the same H-algebra.

Since S^3 is the universal covering space for $\mathbb{R}P^3$, $\pi_2(\mathbb{R}P^3)\cong\pi_2(S^3)\cong 0$ ([10], Example 1.43, p. 74). Similarly, S^2 is the universal covering space for $\mathbb{R}P^2$ and hence $\pi_2(\mathbb{R}P^2)\cong\pi_2(S^2)\cong\mathbb{Z}$. Then

$$\pi_2(S^3\vee\mathbb{R}P^2)\cong\pi_2(S^3)\oplus\pi_2(\mathbb{R}P^2)\oplus\pi_3(S^3\times\mathbb{R}P^2,S^3\vee\mathbb{R}P^2)\ncong 0$$

so that $\mathbb{R}P^3$ and $S^3 \vee \mathbb{R}P^2$ have different homotopy groups and hence are not homotopy equivalent.

We saw in Example 5.3 that $\mathbb{R}P^3$ and $S^3 \vee \mathbb{R}P^2$ have identical H-algebras but are not homotopy equivalent. We now show that these spaces have different $H(\mathbb{Z}/2)$-algebras demonstrating that the coefficient ring is a significant detail in the "cohomology algebras" under discussion.

Example 5.4. $\mathbb{R}P^3$ and $S^3 \vee \mathbb{R}P^2$ have different cohomology rings over $\mathbb{Z}/2$ coefficients and hence, non-isomorphic $H(\mathbb{Z}/2)$-algebras.

Firstly, the cohomology ring over $\mathbb{Z}/2$ coefficients of $\mathbb{R}P^3$ is a polynomial algebra generated by an element of degree 1, and truncated at degree 3. Similarly, the cohomology ring over $\mathbb{Z}/2$ coefficients of $\mathbb{R}P^2$ is a polynomial algebra generated by an element of degree 1 and truncated at degree 2. By the universal coefficient theorem $\tilde{H}^3(S^3;\mathbb{Z}/2) \cong \mathbb{Z}/2$. Thus, by Property 2.16, $\mathbb{R}P^3$ and $S^3 \vee \mathbb{R}P^2$ have the same cohomology groups over $\mathbb{Z}/2$. Now, since operations act componentwise on a product of H-algebras, by Theorem 2.16, if α generates $\tilde{H}^1(\mathbb{R}P^2 \vee S^3;\mathbb{Z}/2) \cong \tilde{H}^1(\mathbb{R}P^2;\mathbb{Z}/2) \times \tilde{H}^1(S^3;\mathbb{Z}/2)$ we have

$$\alpha \cup \alpha \cup \alpha = 0 \in \tilde{H}^3(\mathbb{R}P^2 \vee S^3;\mathbb{Z}/2).$$

Thus the cohomology ring structures differ since, if γ generates $\tilde{H}^1(\mathbb{R}P^3;\mathbb{Z}/2)$, then $\gamma \cup \gamma \cup \gamma$ generates $\tilde{H}^3(\mathbb{R}P^3;\mathbb{Z}/2)$.

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School of Mathematics, Statistics and Computer Science, University of New England, Armidale, NSW 2350, Australia

E-mail address: `apercy@turing.une.edu.au`