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QUASICONFORMAL HARMONIC MAPS INTO NEGATIVELY CURVED MANIFOLDS

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ABSTRACT. Let $F : M \to N$ be a harmonic map between complete Riemannian manifolds. Assume that N is simply connected with sectional curvature bounded between two negative constants. If F is a quasiconformal harmonic diffeomorphism, then M supports an infinite dimensional space of bounded harmonic functions. On the other hand, if M supports no non-constant bounded harmonic functions, then any harmonic map of bounded dilation is constant.

1. Introduction

Let $F\colon M \to N$ be a differentiable map between manifolds M and N. Suppose that M is endowed with a complete Riemannian metric g and that N is endowed with a complete Riemannian metric h. If $x \in M$, then there is a self adjoint operator $(dF)^* dF\colon T_xM \to T_xM$, where the adjoint $(dF)^*$ is taken with respect to the metrics g on T_xM and h on $T_{F(x)}N$. Assume that $n = \dim M$ and denote the eigenvalues of $(dF)^*dF$ by $\lambda_1(x) \geq \lambda_2(x) \geq \ldots \geq \lambda_n(x) \geq 0$. These eigenvalues are all non–negative, but some of them may be zero. If F is a diffeomorphism and $\lambda_1(x) \leq c\lambda_n(x)$ for some constant c, then we say that F is a quasiconformal diffeomorphism.

Suppose that $\overline{M}$ is a compactification of M. Given $f \in C(\overline{M} - M)$, we would like to extend f to $C^2(M) \cap C(\overline{M})$, so that the extended f is a harmonic function on M. Our first main result, which also appears as Theorem 3.2 below, is as follows.

THEOREM 1.1. *Suppose that (N, h) is complete and simply connected with sectional curvature bounded between two negative constants. Let (M, g) be any complete Riemannian manifold and $F\colon (M, g) \to (N, h)$ a quasiconformal harmonic diffeomorphism. Then M admits a natural compactification $\overline{M}$ so that the Dirichlet problem at infinity is solvable.*

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If $(M, g) = (N, h)$ and F is the identity map, then this theorem is well known [1]. So the improvement here is that the result is invariant under quasiconformal harmonic diffeomorphisms.

One notable feature of Theorem 1.1 is that no curvature hypotheses are imposed on (M, g). The same feature is present in the theorem of [6] concerning harmonic maps of bounded dilatation. In the notation of the first paragraph above, a differentiable map F has bounded dilatation when $\lambda_1(x) \leq c\lambda_2(x)$, for some constant c. In particular, the rank of dF is either zero or greater than one, at each $x \in M$. We will prove the following theorem, which also appears as Theorem 4.3 below.

Theorem 1.2. *Suppose that (M, g) supports no nonconstant bounded harmonic functions. Assume that (N, h) is complete and simply connected with sectional curvature bounded between two negative constants. Then any harmonic map $F\colon M \to N$ of bounded dilatation is constant.*

Theorem 1.2 was proved earlier in [6] via stochastic methods. The new contribution here is to give an alternative proof using classical analysis and geometry.

There are some previous publications concerning harmonic maps of bounded dilatation and quasiconformal harmonic diffeomorphisms. Typically, the authors of these works rely on a Bochner formula, and (M, g) is assumed to have non–negative Ricci curvature. If (M, g) is only assumed to be a complete Riemannian manifold, the Bochner method seems difficult to apply. The techniques used below are therefore quite different from the techniques of [3], [8], and [9].

2. Harmonicity of the identity map

Suppose that (M, g) and $(N, \hat{g})$ are Riemannian manifolds and $F\colon M \to N$ is a harmonic diffeomorphism. We define another metric h on M by setting $h = F^*\hat{g}$. It follows from the definitions that the identity map $I\colon (M, g) \to (M, h)$ is a harmonic map. In particular, since F is a diffeomorphism, it can be used to define local charts on M and, in the corresponding coordinates, I is represented by F. However, the notion of harmonic map is independent of the choice of coordinates.

In this section, we will give various characterizations for the harmonicity of the identity map $I\colon (M, g) \to (N, h)$. The observations of the previous paragraph can often be invoked to reduce questions about harmonic diffeomorphisms to the study of the harmonicity of I. Let ∇^g and ∇^h denote the Levi–Civita connections of g and h. Define $E = \nabla^g - \nabla^h$, that is, $E(X, Y) = \nabla^g_X Y - \nabla^h_X Y$, for vector fields X and Y. One verifies that E is a tensor field. Our first characterization of the harmonicity of the identity map is given by the following result:

LEMMA 2.1. *$I\colon (M,g) \to (M,h)$ is harmonic if and only if $Tr_g E = 0$. In coordinate notation, this reads $g^{ij} E^k_{ij} = 0$.*

Proof. In local coordinates, the tension field of a map u is given by $\tau^\alpha(u) = \Delta u^\alpha + (\overline{\Gamma}^\alpha_{\beta\gamma} \circ u) u^\beta_i u^\gamma_j g^{ij}$. Here Δ is the invariant Laplacian of the metric g on the domain. Moreover, $\overline{\Gamma}^\alpha_{\beta\gamma}$ are the Christoffel symbols on the target. Since $\Delta f = Tr(\nabla df)$, for any function $f\colon M \to R$, one has

$$\Delta f = g^{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} - g^{ij} \Gamma^k_{ij} \frac{\partial f}{\partial x_k}.$$

Therefore,

$$\tau^\alpha = g^{ij} \frac{\partial^2 u^\alpha}{\partial x_i \partial x_j} - g^{ij} \Gamma^k_{ij} \frac{\partial u^\alpha}{\partial x_k} + (\overline{\Gamma}^\alpha_{\beta\gamma} \circ u) u^\beta_i u^\gamma_j g^{ij}.$$

Here Γ^k_{ij} are the Christoffel symbols of the metric g. If u is the identity map, then $u^\alpha(x) = x^\alpha$, and we find $\tau^\alpha(I) = g^{ij}(\overline{\Gamma}^\alpha_{ij} - \Gamma^\alpha_{ij}) = -\operatorname{Tr}_g E^\alpha$. A map is harmonic exactly when its tension field vanishes. □

Recall that the invariant Laplacian of a metric g is given by $\Delta_g = \operatorname{Tr}_g \operatorname{Hess}_g$. If two metrics g and h are given on the same manifold, then one may define the second order differential operator $\operatorname{Tr}_g \operatorname{Hess}_h$. In general, this is an elliptic operator, but not of divergence form. Another characterization of the harmonicity of the identity map is the following:

LEMMA 2.2. *Suppose that $I\colon (M,g) \to (M,h)$ is harmonic. Then $\Delta_g = \operatorname{Tr}_g \operatorname{Hess}_h$.*

Proof. In general, $\operatorname{Hess} f = \nabla df$, so that

$$\operatorname{Hess}_g f = \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \otimes dx_j - \Gamma^k_{ij} \frac{\partial f}{\partial x_k} dx_i \otimes dx_j,$$
$$\operatorname{Hess}_h f = \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \otimes dx_j - \overline{\Gamma}^k_{ij} \frac{\partial f}{\partial x_k} dx_i \otimes dx_j.$$

Taking the trace of the difference gives $\Delta_g f - \operatorname{Tr}_g \operatorname{Hess}_h f = \operatorname{Tr}_g(\operatorname{Hess}_g f - \operatorname{Hess}_h f) = -df(\operatorname{Tr}_g E) = 0$, by Lemma 2.1. □

In this paper, we are concerned with harmonic functions, that is, with $\operatorname{Ker}(\Delta_g)$ and $\operatorname{Ker}(\Delta_h)$. The goal is to show that certain existence results for bounded harmonic functions are preserved under appropriate types of harmonic diffeomorphisms. Naively, the most immediate idea is to consider conformal harmonic diffeomorphisms. If the identity map is both conformal and harmonic, then Lemma 2.2 gives $\Delta_g = \operatorname{Tr}_g \operatorname{Hess}_h = \phi \operatorname{Tr}_h \operatorname{Hess}_h = \phi \Delta_h$, where $h = \phi g$. However, this condition is too restrictive in dimensions $n = \dim M \geq 3$. For conformal maps and $n = 2$, most of our results follow easily from the uniformization theorem. The following lemma describes when the identity map is both conformal and harmonic.

Lemma 2.3. *Assume that $h = \phi g$ is conformal to g.*

(i) *If $n = 2$, then $I\colon (M, g) \to (M, h)$ is harmonic.*

(ii) *If $n \geq 3$ and $I\colon (M, g) \to (M, h)$ is harmonic, then ϕ is constant.*

Proof. The standard local formula for the Christoffel symbols is

$$\Gamma_{ij}^{k} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{il}}{\partial x_j} + \frac{\partial g_{jl}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_l} \right).$$

If $\overline{\Gamma}_{ij}^{k}$ denote the Christoffel symbols for h, then one computes

$$g^{ij}(\overline{\Gamma}_{ij}^{k} - \Gamma_{ij}^{k}) = \frac{1}{2} \phi^{-1}(2-n) g^{ki} \frac{\partial \phi}{\partial x_i}.$$

The result now follows from Lemma 2.1. □

If $n \geq 3$, it follows that conformal harmonic diffeomorphisms are isometries, after rescaling the metric on the domain by a constant factor. The paucity of conformal harmonic diffeomorphisms suggests the study of quasiconformal harmonic diffeomorphisms. If $I\colon (M, g) \to (M, h)$ is harmonic, then $\Delta_g = \mathrm{Tr}_g \mathrm{Hess}_h$, by Lemma 2.2. If I is also quasiconformal, then there exists a third metric $\tilde{g} = \phi g$ and $\tilde{g}$ is quasi–isometric to h. Since $\mathrm{Ker}(\Delta_g) = \mathrm{Ker}(\mathrm{Tr}_{\tilde{g}} \mathrm{Hess}_h)$, one may hope to prove that bounded harmonic functions for Δ_g exist when there are bounded harmonic functions for Δ_h. The idea is that the operator $\mathrm{Tr}_{\tilde{g}} \mathrm{Hess}_h$ is closely related to Δ_h, since $\tilde{g}$ and h are quasi–isometric and $\Delta_h = \mathrm{Tr}_h \mathrm{Hess}_h$. One serious difficulty is that the operator $\mathrm{Tr}_{\tilde{g}} \mathrm{Hess}\, h$ need not be of divergence form. Many results about the Laplacian Δ_h are strongly dependent upon its divergence form structure.

3. Quasiconformal harmonic diffeomorphisms

Let (N, h) be a complete, simply connected, negatively curved Riemannian manifold, with sectional curvature bounded between two negative constants, $-b^2 \leq K \leq -a^2 < 0$. The Riemannian manifold N admits a natural compactification $\overline{N} = N \cup S(\infty)$, where each point in $S(\infty)$ represents a class of asymptotic geodesic rays. Given a continuous function $f\colon S(\infty) \to R$, the Dirichlet problem at infinity is to extend f continuously to $\overline{N}$, so that the extended f is harmonic on N, i.e., $\Delta_h f = 0$. The solvability of the Dirichlet problem at infinity was established in [1].

Suppose that $\hat{h}$ is a second metric on N, which is quasi–isometric to h, i.e., $c_1 h \leq \hat{h} \leq c_2 h$. We will need to solve the Dirichlet problem at infinity for the second order elliptic operator defined by $\mathrm{Tr}_{\hat{h}} \mathrm{Hess}\, h$. The method of [1] is remarkably robust and can be extended in a straightforward way. This is noteworthy since $\mathrm{Tr}_{\hat{h}} \mathrm{Hess}\, h$ may not be of divergence form. Many analytic methods do not generalize readily from the divergence form to other elliptic operators. We have the following result.

Proposition 3.1. *Suppose that* $f\colon S(\infty) \to R$ *is a given continuous function. Then* f *extends uniquely to* $C^2(N) \cap C(\overline{N})$*, so as to satisfy* $\mathrm{Tr}_{\hat{h}} \operatorname{Hess}_h f = 0$ *on* N*.*

Proof. Uniqueness follows from the maximum principle applied on sufficiently large geodesic balls in N. Existence is the more difficult issue. Here we follow the arguments of [1]. For the convenience of the reader and for future reference, we summarize the main ideas.

Choose a fixed basepoint $0 \in N$ and identify $S(\infty)$ with the collection of geodesic rays starting at 0. This provides a homeomorphism between $S(\infty)$ and the unit sphere $S_0(1)$, centered at 0. Using this identification, $S(\infty)$ is endowed with a noncanonical differentiable structure. By the maximum principle and elliptic regularity, we reduce the existence question to the case where $f\colon S(\infty) \to R$ is Lipschitz continuous.

The next step is to construct some approximate solutions $\zeta(\hat{f})$. We extend f radially from $S(\infty)$ to a function $\hat{f}$ on $\overline{N}$. By using a cut–off function near the basepoint 0, we may assume that $\hat{f}$ is continuous. Let $\chi\colon R \to R$ be a fixed C^2 approximation of the characteristic function of $[0,1]$, with χ having support in $[-2,2]$. If $\rho(x,y)$ denotes the geodesic distance, corresponding to the metric h, then we define

$$\zeta(\hat{f}) = \frac{\int\limits_N \chi(\rho^2(x,y))\hat{f}(y)dy}{\int_N \chi(\rho^2(x,y))dy}.$$

The integration uses the volume element induced by the metric h. Applying the Hessian comparison theorem [4], it follows that

$$\|d\zeta(\hat{f})\| + \|\operatorname{Hess}_h \zeta(\hat{f})\| \le c_1 e^{-ar},$$

where the norms are relative to h and $r(x) = \rho(0,x)$ is the geodesic distance from the basepoint, again relative to h. The exponential decay rate is e^{-ar}, since this corresponds to an upper bound of the angle subtended in $S_0(1)$, by a ball of radius two at distance r from 0. Note that all of the above constructions and estimates, concerning $\zeta(\hat{f})$, depend only upon the metric h.

It is straightforward to calculate

$$\operatorname{Hess}_h e^{-\delta r} = e^{-\delta r}[\delta^2 dr \otimes dr - \delta \operatorname{Hess}_h r].$$

Applying the Hessian comparison theorem [4] to h and the fact that $\hat{h}$ is quasi–isometric to h, we find

$$\mathrm{Tr}_{\hat{h}} \operatorname{Hess}_h e^{-\delta r} \le -c_2 \delta e^{-\delta r}$$

for δ sufficiently small. Consequently, $f_+ = \zeta(\hat{f}) + c_3 e^{-\delta r}$ is a supersolution to the Dirichlet problem at infinity for $\mathrm{Tr}_{\hat{h}} \operatorname{Hess} h$. Similarly, $f_- = \zeta(\hat{f}) - c_3 e^{-\delta r}$ is a subsolution to the same problem. Solving the Dirichlet problem on an exhaustion of M by compact sets, and using f_+ and f_- as barrier functions at infinity, we solve the Dirichlet problem at infinity for the operator $\mathrm{Tr}_{\hat{h}} \operatorname{Hess}_h$. □

Assume now that (M, g) is another complete Riemannian manifold and $F\colon M \to N$ is a quasiconformal harmonic diffeomorphism, relative to the Riemannian structure (N, h) on the range. If $\lambda_1(x) \geq \lambda_2(x) \geq \ldots \geq \lambda_n(x)$ are the eigenvalues of $(dF)^* dF$, at $x \in M$, then quasiconformal means that $\lambda_1(x) \leq c_4 \lambda_n(x)$. Since F is a homeomorphism, the compactification $\overline{N}$ of N induces a compactification $\overline{M}$ of M, so that $\overline{M} = M \cup S(\infty)$. The main result of this section is the following theorem.

THEOREM 3.2. *Suppose that $f\colon S(\infty) \to R$ is a given continuous function. Then f extends uniquely to $C^2(M) \cap C(\overline{M})$ as a harmonic function for the metric g, $\Delta_g f = 0$.*

Proof. Pulling back the metric h by the diffeomorphism F, we reduce the problem to the case where $M = N$ and F is the identity map. So assume that $I\colon (M, g) \to (N, h)$ is harmonic and quasiconformal. By Lemma 2.2, the harmonicity of I gives $\Delta_g = \mathrm{Tr}_g \operatorname{Hess} h$. Of course, g need not be quasi–isometric to h. However, we may define a third metric $\hat{h} = \lambda_1(x) g$. Then $\mathrm{Ker}(\Delta_g) = \mathrm{Ker}(\mathrm{Tr}_g \operatorname{Hess} h) = \mathrm{Ker}(\mathrm{Tr}_{\hat{h}} \operatorname{Hess} h)$. Since $\hat{h}$ is quasi–isometric to h, the result follows from Proposition 3.1. □

The methods of this section can also be used to prove the following more elementary result.

PROPOSITION 3.3. *Suppose that (M, g) is a complete Riemannian manifold with no Greens function. Assume that (N, h) is complete and simply connected with strictly negative sectional curvature $K_N \leq -a^2$. Then there exists no quasiconformal harmonic diffeomorphism $F\colon M \to N$.*

Proof. We apply the argument and the notation of the proof of Theorem 3.2. This reduces us to the case where $M = N$ and F is the identity map, $I\colon (M, g) \to (N, h)$. Recall that $\hat{h}$ is quasi–isometric to h, but $\hat{h}$ is also conformal to g. The Hessian comparison theorem implies that $\exp(-\delta r)$ is a positive supersolution of $\mathrm{Tr}_{\hat{h}} \operatorname{Hess}_h f = 0$. So, Δ_g admits a positive superharmonic function, using the identification $\Delta_g = \mathrm{Tr}_g \operatorname{Hess} h$, of Lemma 2.2. This contradicts the hypothesis that (M, g) admits no Greens function. □

4. Harmonic maps of bounded dilatation

Suppose that (M, g) and (N, h) are complete Riemannian manifolds. We assume that (N, h) is simply connected and negatively curved, with sectional curvatures bounded between two negative constants, $-b^2 \leq K_N \leq -a^2 < 0$. One has the geometric compactification, $\overline{N} = N \cup S(\infty)$, constructed using asymptotic classes of geodesic rays. No curvature conditions will be imposed upon (M, g). We study harmonic maps $F\colon M \to N$. In contrast to Section 3, one no longer assumes that F is a diffeomorphism.

Our first result concerns bounded harmonic maps. Proposition 4.1 was proved in [6] via both probabilistic and classical analysis methods. The proof given below is a small modification of Kendall's classical method. One step in [7] used the heat equation on M, but the proof below employs only the theory of elliptic equations. This modification is more consistent with the other arguments in this paper and suggests some of the developments below.

PROPOSITION 4.1. *Suppose that (M, g) supports no non–constant bounded harmonic functions. Then any bounded harmonic map, $F\colon M \to N$, is constant. We need only assume that N is complete and simply connected with $K_N \leq 0$.*

Proof. Since F is bounded, and N is complete, the closure of the image $\overline{F(M)}$ is compact. If $y \in N$, let $d_y(x) = d(x, y)$ denote the geodesic distance from x to y. For each fixed y, suppose that c_y is the infimum of all harmonic functions $h\colon M \to R$ satisfying $h(m) \geq d_y(F(m))$ for all $m \in M$. Apriori, c_y is a harmonic function on M. Since (M, g) supports no non–constant bounded harmonic functions, c_y is actually constant.

Let $\mathcal{D}_j$ be an exhaustion of M by open sets with regular boundaries and compact closures. Suppose that $v_{y,j}\colon \mathcal{D}_j \to R$ is a harmonic function with boundary values $c_y - d_y \circ F$, on $\partial\mathcal{D}_j$. Passing to a subsequence in j, $v_{y,j}$ converges to a bounded harmonic function a_y, which is necessarily constant. Invoking the definition of c_y, one deduces that $a_y = 0$. Suppose that one applies the same argument to a finite sum $s = \sum_{i=1}^{k} c_{y_i} - d_{y_i} \circ F$. If $w_j | \partial\mathcal{D}_j = s$ solves the Dirichlet problem, then $w_j \to \sum_{i=1}^{k} a_{y_i} = 0$. Consequently, for any $r > 0$, we have

$$\bigcap_{i=1}^{k} \{m | c_{y_i} - d_{y_i}(F(m)) \leq r\} \neq \emptyset.$$

Otherwise $s \geq r$, and thus $w_j \geq r$, by the maximum principle. However, this contradicts $w_j \to 0$.

Suppose that

$$T = \bigcap_{r>0} \bigcap_{y \in \overline{F(M)}} \{x \in \overline{F(M)} | c_y - d_y(x) \leq r\}.$$

By the compactness of $\overline{F(M)}$ and the finite intersection property of compact sets, T is not the empty set. If $z \in T$, then $c_y = d_y(z)$ for all $y \in \overline{F(M)}$. In particular, $c_z = d_z(z) = 0$. Therefore $d_z(F(m)) = 0$ for all $m \in M$, and $F(M) = z$. □

We now record a lemma concerning bounded harmonic functions. The techniques used in proving Lemma 4.2 are related to the modifications made in Kendall's proof of Proposition 4.1. Similar arguments have been used by Grigor'yan [5] in his study of bounded harmonic functions and massive sets.

LEMMA 4.2. *Suppose that w_1 and w_2 are bounded subharmonic functions on (M, g). Assume that (i) $\sup w_1 = \sup w_2 = 1$ and (ii) $w_1 + w_2 \leq 2 - \epsilon$. Then (M, g) supports at least one nonconstant bounded harmonic function.*

Proof. Let $\mathcal{D}_j$ be an exhaustion of M by open sets with compact closures and regular boundaries. For $i = 1, 2$, we solve the Dirichlet problem on $\mathcal{D}_j$, to obtain bounded harmonic functions $v_{i,j}$, with boundary values w_i on $\partial\mathcal{D}_j$. Letting $j \to \infty$ and passing to a subsequence gives harmonic functions v_i with $w_i \leq v_i \leq 1$. Moreover, since harmonic functions assume their maximum values on the boundary, and $w_1 + w_2 \leq 2 - \epsilon$, we have $v_{1,j} + v_{2,j} \leq 2 - \epsilon$, for every j. Consequently, $v_1 + v_2 \leq 2 - \epsilon$. However, since $\sup v_i = \sup w_i = 1$, if v_i is constant, we must have $v_i \equiv 1$. The condition that both of the functions v_i are constant is thus inconsistent with $v_1 + v_2 \leq 2 - \epsilon$. □

Let $F\colon M \to N$ be a differentiable map. For each $x \in M$, let $\lambda_1(x) \geq \lambda_2(x) \geq \ldots \geq \lambda_r(x) > 0$, be the positive eigenvalues of the self–adjoint linear transformation $(dF)^* dF\colon T_xM \to T_xM$. We say that F has bounded dilatation if, for each $x \in M$, either (i) $dF(x) = 0$ or (ii) $\lambda_1(x) \leq c\lambda_2(x)$, $r \geq 2$, where c is a constant independent of x. A map of bounded dilatation cannot have rank one at any point. The rank is either zero or at least two.

Theorem 4.3 was proved in [6] using stochastic methods. The techniques of Section 3 lead naturally to a proof using classical analysis and geometry. The main result of this section is the following theorem.

THEOREM 4.3. *Suppose that (M, g) supports no nonconstant bounded harmonic function. Then any harmonic map $F\colon M \to N$ of bounded dilatation is constant.*

Proof. If F is bounded, then the result follows from Proposition 4.1. So we may assume that $\overline{F(M)} \cap S(\infty) \neq \phi$, where $\overline{F(M)}$ denotes the closure of the image of F in $\overline{N}$.

Suppose that $\overline{F(M)} \cap S(\infty)$ contains at least two distinct points p_1, p_2. Choose a basepoint $0 \in N$ and identify $S_0(1)$ and $S(\infty)$, as in the proof of Proposition 3.1. Since $p_1 \neq p_2$, we may choose two Lipschitz continuous functions ψ_1, ψ_2, defined on $S(\infty)$, and satisfying:

(i) $\psi_i \equiv 1$ on a neighborhood of p_i;
(ii) ψ_1 and ψ_2 have disjoint supports: $\operatorname{supp} \psi_1 \cap \operatorname{supp} \psi_2 = \emptyset$;
(iii) $\psi_i \leq 1$, for $i = 1, 2$.

We construct the approximate solutions $\zeta(\hat{\psi}_i)$, for the Dirichlet problem at infinity on N, as in Proposition 3.1 and its proof. One easily ensures that the properties (i), (ii), and (iii) of the ψ_i are also satisfied by the $\zeta(\hat{\psi}_i)$.

If $\phi\colon N \to R$ is a C^2–function, defined on N, then since F is harmonic, $\Delta(\phi \circ F) = D^2\phi(dF, dF) = \phi_{\alpha\beta} F_i^\alpha F_j^\beta g^{ij}$. Employing the hypothesis that F has bounded dilatation, we deduce $|\Delta(\phi \circ F)| \leq c_1 \|D^2\phi\| \lambda_2(x)$, when $dF\colon T_xM \to T_{F(x)}N$ is not identically zero. If $dF = 0$, at x, then $\Delta(\phi \circ F)(x) = 0$, for any ϕ. In particular, we may take $\phi = \zeta(\hat{\psi}_i)$, and deduce, from the estimate of $\|D^2\zeta(\hat{\psi}_i)\|$ appearing in the proof of Proposition 3.1,

$$|\Delta(\zeta(\hat{\psi}_i) \circ F)| \leq c_2 \lambda_2 e^{-ar\circ F},$$

where r denotes the geodesic distance from the basepoint 0 in N.

For $\delta > 0$, we have $D^2(e^{-\delta r}) = e^{-\delta r}[\delta^2 dr \otimes dr - \delta D^2 r]$. Since F has bounded dilatation its rank is either zero or strictly greater than one. In particular, $\operatorname{im}(dF)$ cannot be contained in the kernel of D^2r, which is one dimensional. Applying the Hessian comparison theorem [4] and our curvature hypothesis on N, we deduce that, for δ sufficiently small,

$$\Delta(e^{-\delta r\circ F}) \leq -c_3 \lambda_2 \delta e^{-\delta r\circ F}$$

at points where dF is not identically zero.

For an appropriate constant c_4, we define subharmonic functions $w_i = \zeta(\hat{\psi}_i) \circ F - c_4 \exp(-\delta r \circ F)$, for $i = 1, 2$. The w_i satisfy the hypotheses of Lemma 4.2, since $\zeta(\hat{\psi}_i)$ have disjoint supports. Lemma 4.2 then asserts that M admits a nonconstant bounded harmonic function. This contradiction completes the proof in the case when $\overline{F(M)} \cap S(\infty)$ contains at least two points.

Suppose now that $\overline{F(M)} \cap S(\infty)$ consists of a single point p_1. Let p_2 be any point in $F(M)$, the image of M in N. There is a geodesic with $\gamma(0) = p_2$ and $\lim_{t\to\infty} \gamma(t) = p_1$. We choose our basepoint to be at $\gamma(t_0)$, where t_0 is sufficiently large. There will then be two points which are antipodal on a very large sphere centered at the origin $0 = \gamma(t_0)$. The method used when $\overline{F(M)} \cap S(\infty)$ has cardinality at least two then applies, with small modifications. □

Remark. The lower bound of the sectional curvature is needed for the application of the Hessian comparison theorem in the proof of Theorem 4.3. The stochastic approach requires such a bound for quite different reasons. However, no lower bound is needed if the Ricci curvature of M is non-negative.

Holomorphic maps F between Kaehler manifolds are harmonic. Moreover, since the eigenspaces of $(dF)^*dF$ have complex dimension at least one, holomorphic maps have bounded dilatation, $\lambda_1(x) = \lambda_2(x)$, in the real tangent space. Consequently, one has:

Corollary 4.4. *Suppose that $F\colon M \to N$ is a holomorphic map, between complete Kaehler manifolds, where (i) M supports no nonconstant bounded harmonic functions, and (ii) N is simply connected with sectional curvature bounded between two negative constants. Then F is the constant map.*

The proof of Theorem 4.3 generalizes readily to give the following more technical statement, which may nevertheless retain a certain interest.

Proposition 4.5. *Suppose that $F\colon M \to N$ is a harmonic map of bounded dilatation between complete Riemannian manifolds. Assume that N is simply connected with sectional curvature bounded between two negative constants. If $\overline{F(M)} \cap S(\infty)$ has cardinality at least $k \geq 1$, then the space of bounded harmonic functions on M has dimension at least $k+1$.*

Proof. Following the argument used to establish Theorem 4.3, we obtain $k+1$ nonnegative harmonic functions h_i, and $k+1$ points p_j, so that $|h_i(p_j) - \delta_{ij}| < \epsilon$, for any $\epsilon > 0$. If some linear combination $\sum_{i=1}^{k+1} c_i h_i$ is zero, then evaluating at p_j, we get $\sum_{i=1}^{k+1} h_i(p_j)c_i = 0$. But the square matrix $h_i(p_j)$ may be taken arbitrarily close to the identity matrix. In particular, $h_i(p_j)$ is invertible, and $c_i = 0,\ 1 \leq i \leq k+1$. $\square$

One also obtains the following improvement of Proposition 3.3.

Proposition 4.6. *Suppose that (M, g) is a complete Riemannian manifold with no Greens function. Assume that (N, h) is complete and simply connected with strictly negative sectional curvature $K_N \leq -a^2 < 0$. Then any harmonic map $F\colon M \to N$ of bounded dilatation is constant.*

Proof. If F is bounded, then Proposition 4.1 shows that F is constant. Otherwise, the proof of Theorem 4.3 shows that $\exp(-\delta r \circ F)$ is a bounded nonconstant superharmonic function on M. $\square$

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