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EISENSTEIN SERIES AND APPROXIMATIONS TO π

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Dedicated to K. Venkatachaliengar

1. Introduction

On page 211 in his lost notebook, in the pagination of [19], Ramanujan listed eight integers, 11, 19, 27, 43, 67, 163, 35, and 51 at the left margin. To the right of each integer, Ramanujan recorded a linear equation in Q^3 and R^2. Although Ramanujan did not indicate the definitions of Q and R, we can easily (and correctly) ascertain that Q and R are the Eisenstein series

$$Q(q) := 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n} \tag{1.1}$$

and

$$R(q) := 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1-q^n}, \tag{1.2}$$

where $|q| < 1$. To the right of each equation in Q^3 and R^2, Ramanujan entered an equality involving π and square roots. (For the integer 51, the linear equation and the equality involving π, in fact, are not recorded by Ramanujan.)

The equations in Q^3 and R^2 cannot possibly hold for all values of q with $|q| < 1$. Thus, the first task was to find the correct value of q for each equation. After trial and error we found that $q = -\exp(-\pi\sqrt{n})$, where n is the integer at the left margin. (We later read that K. Venkatachaliengar [20, p. 135] had also discovered that $q = -\exp(-\pi\sqrt{n})$.) The equalities in the third column lead to approximations to π that are remindful of approximations given by Ramanujan in his famous paper on modular equations and approximations to π [15], [18, p. 33] and studied extensively by J. M. and P. B. Borwein [6, Chap. 5]. As will be seen, this page in the lost notebook is closely connected with theorems connected with the modular j-invariant stated by Ramanujan

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on the last two pages of his third notebook [10] and proved by the authors [4], [3, pp. 309–322].

In Section 2, we prove a general theorem from which the linear equations in Q^3 and R^2 in the second column follow as corollaries. In Sections 3 and 4, we offer two methods for proving the equalities in the third column and show how they lead to approximations to π. Ramanujan's equations for π lead in Section 4 to certain numbers t_n (defined by (4.28)). The numbers t_n guide us in Section 5 to a proof of a general series formula for $1/\pi$, which is equivalent to a formula found by D. V. and G. V. Chudnovsky [13] and the Borweins [9]. The first series representations for $1/\pi$ of this type were found by Ramanujan [15], [18, pp. 23–39], but first proved by the Borweins [8]. The work of the Borweins [6], [7], [8] and Chudnovskys [13] significantly extends Ramanujan's work. One of Ramanujan's series for $1/\pi$ yields 8 digits of π per term, while one of the Borweins' [7] gives 50 digits of π per term. Our simpler method enables us in Section 5 to determine a series for $1/\pi$ which yields about 73 or 74 digits of π per term.

2. Eisenstein series and the modular j–invariant

Recall the definition of the modular j–invariant $j(\tau)$,

$$j(\tau) = 1728\frac{Q^3(q)}{Q^3(q) - R^2(q)}, \qquad q = e^{2\pi i\tau}, \qquad \operatorname{Im}\tau > 0. \tag{2.1}$$

In particular, if n is a positive integer,

$$j\left(\frac{3+\sqrt{-n}}{2}\right) = 1728\frac{Q_n^3}{Q_n^3 - R_n^2}, \tag{2.2}$$

where, for brevity, we set

$$Q_n := Q(-e^{-\pi\sqrt{n}}) \qquad \text{and} \qquad R_n := R(-e^{-\pi\sqrt{n}}). \tag{2.3}$$

In his third notebook, at the top of page 392 in the pagination of [17], Ramanujan defined a certain function J_n of singular moduli, which, as the authors [4] easily showed, has the representation

$$J_n = -\frac{1}{32}\sqrt[3]{j\left(\frac{3+\sqrt{-n}}{2}\right)}. \tag{2.4}$$

Hence, from (2.2) and (2.4),

$$(-32J_n)^3 = 1728\frac{Q_n^3}{Q_n^3 - R_n^2}. \tag{2.5}$$

After a simple manipulation of (2.5), we deduce the following theorem.

THEOREM 2.1. *For each positive integer n,*

$$\left(\left(\frac{8}{3}J_n\right)^3+1\right)Q_n^3-\left(\frac{8}{3}J_n\right)^3R_n^2=0, \tag{2.6}$$

where J_n is defined by (2.4), *and Q_n and R_n are defined by* (2.3).

EXAMPLES 2.2 [19, p. 211]. *We have*

$$\begin{aligned}
539Q_{11}^3-512R_{11}^2=&0,\\
(8^3+1)Q_{19}^3-8^3R_{19}^2=&0,\\
(40^3+9)Q_{27}^3-40^3R_{27}^2=&0,\\
(80^3+1)Q_{43}^3-80^3R_{43}^2=&0,\\
(440^3+1)Q_{67}^3-440^3R_{67}^2=&0,\\
(53360^3+1)Q_{163}^3-53360^3R_{163}^2=&0,\\
((60+28\sqrt{5})^3+27)Q_{35}^3-(60+28\sqrt{5})^3R_{35}^2=&0,
\end{aligned}$$

and

$$((4(4+\sqrt{17})^{2/3}(5+\sqrt{17}))^3+1)Q_{51}^3-(4(4+\sqrt{17})^{2/3}(5+\sqrt{17}))^3R_{51}^2=0.$$

Proof. In [4], [3, pp. 310, 311], we showed that

$$\begin{cases}
J_{11}=1, & J_{19}=3,\\
J_{27}=5\cdot 3^{1/3}, & J_{43}=30,\\
J_{67}=165, & J_{163}=20,010,\\
J_{35}=\sqrt{5}\left(\dfrac{1+\sqrt{5}}{2}\right)^4, & J_{51}=3(4+\sqrt{17})^{2/3}\left(\dfrac{5+\sqrt{17}}{2}\right).
\end{cases} \tag{2.7}$$

Using (2.7) in (2.6), we readily deduce all eight equations in Q_n and R_n. □

3. Eisenstein series and equations in π − first method

Define, after Ramanujan,

$$P(q):=1-24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}, \qquad |q|<1, \tag{3.1}$$

and put

$$P_n:=P(-e^{-\pi\sqrt{n}}). \tag{3.2}$$

Next, set

$$b_n=\{n(1728-j_n)\}^{1/2} \tag{3.3}$$

and

$$a_n = \frac{1}{6} b_n \left\{ 1 - \frac{Q_n}{R_n} \left(P_n - \frac{6}{\pi\sqrt{n}} \right) \right\}. \tag{3.4}$$

The numbers a_n and b_n arise in series representations for $1/\pi$ proved by the Chudnovskys [13] and the Borweins [8], namely,

$$\frac{1}{\pi} = \frac{1}{\sqrt{-j_n}} \sum_{k=0}^{\infty} \frac{(6k)!}{(3k)!(k!)^3} \frac{a_n + k b_n}{j_n^k}, \tag{3.5}$$

where $(c)_0 = 1, (c)_k = c(c+1)\cdots(c+k-1),$ for $k \geq 1$, and

$$j_n = j\left(\frac{3+\sqrt{-n}}{2}\right).$$

These authors have calculated a_n and b_n for several values of n, but we are uncertain if these calculations are theoretically grounded. We show how (3.3) and (3.4) lead to a formula from which Ramanujan's equalities in the third column on page 211 of [19] follow.

From (2.6), we easily see that

$$\frac{Q_n}{R_n} = \frac{1}{\sqrt{Q_n}} \left(\frac{\left(\frac{8}{3}J_n\right)^3 + 1}{\left(\frac{8}{3}J_n\right)^3} \right)^{-1/2}, \tag{3.6}$$

and from (3.4), we find that

$$\frac{Q_n}{R_n}\left(\frac{6}{\pi} - \sqrt{n} P_n\right) = 6\sqrt{n}\frac{a_n}{b_n} - \sqrt{n}. \tag{3.7}$$

The substitution of (3.6) into (3.7) leads to the following theorem.

Theorem 3.1. *If $P_n, b_n, a_n,$ and J_n are defined by* (3.2)–(3.4) *and* (2.4)*, respectively, then*

$$\frac{1}{\sqrt{Q_n}}\left(\sqrt{n} P_n - \frac{6}{\pi}\right) = \sqrt{n}\left(1 - 6\frac{a_n}{b_n}\right) \left(\frac{\left(\frac{8}{3}J_n\right)^3 + 1}{\left(\frac{8}{3}J_n\right)^3} \right)^{1/2}. \tag{3.8}$$

Examples 3.2 [19, p. 211]. *We have*

$$\frac{1}{\sqrt{Q_{11}}}\left(\sqrt{11} P_{11} - \frac{6}{\pi}\right) = \sqrt{2},$$
$$\frac{1}{\sqrt{Q_{19}}}\left(\sqrt{19} P_{19} - \frac{6}{\pi}\right) = \sqrt{6},$$
$$\frac{1}{\sqrt{Q_{27}}}\left(\sqrt{27} P_{27} - \frac{6}{\pi}\right) = 3\sqrt{\frac{6}{5}},$$
$$\frac{1}{\sqrt{Q_{43}}}\left(\sqrt{43} P_{43} - \frac{6}{\pi}\right) = 6\sqrt{\frac{3}{5}},$$

$$\begin{aligned}
\frac{1}{\sqrt{Q_{67}}}\left(\sqrt{67}P_{67}-\frac{6}{\pi}\right) =&19\sqrt{\frac{6}{55}},\\
\frac{1}{\sqrt{Q_{163}}}\left(\sqrt{163}P_{163}-\frac{6}{\pi}\right) =&362\sqrt{\frac{3}{3335}},\\
\frac{1}{\sqrt{Q_{35}}}\left(\sqrt{35}P_{35}-\frac{6}{\pi}\right) =&(2+\sqrt{5})\sqrt{\frac{2}{\sqrt{5}}},\\
\frac{1}{\sqrt{Q_{51}}}\left(\sqrt{51}P_{51}-\frac{6}{\pi}\right) =&\dots
\end{aligned}$$

Ramanujan's formulation of the first of these examples is apparently given by

$$\frac{\sqrt{11}-\dfrac{6}{\pi}+\cdots}{\sqrt{1-240\left(\dfrac{1^3}{e^{\pi\sqrt{11}}}\cdots\right)}}=\sqrt{2}.$$

(The denominator with 1^3 in the numerator is unreadable.) Further equalities are even briefer, with $\sqrt{Q_n}$ replaced by $\sqrt{\cdot}$. Note that P_n is replaced by "$1+\cdots$" in Ramanujan's examples. Also observe that Ramanujan did not record the right side when $n=51$. Because it is unwieldy, we also have not recorded it. However, readers can readily complete the equality, since J_{51} is given in (2.7) and a_{51} and b_{51} are given in the next table.

Proof. The first six values of a_n and b_n were calculated by the Borweins [8, pp. 371, 372]. The values for $n=35$ and 51 were calculated by the present authors. We record all 8 pairs of values for a_n and b_n in the following table.

n	a_n	b_n
11	60	616
19	300	4104
27	1116	18216
43	9468	195048
67	122124	3140424
163	163096908	6541681608
35	$1740+768\sqrt{5}$	$32200+14336\sqrt{5}$
51	$11820+2880\sqrt{17}$	$265608+64512\sqrt{17}$

If we substitute these values of a_n and b_n in Theorem 3.1, we obtain, after some calculation and simplification, Ramanujan's equalities. □

Theorem 3.1 and the last set of examples yield approximations to π. Let r_n denote the algebraic expression on the right side in (3.8). If we use the expansions

$$P_n = 1 + 24e^{-\pi\sqrt{n}} - \cdots \qquad \text{and} \qquad \sqrt{Q_n} = 1 - 120e^{-\pi\sqrt{n}} + \cdots,$$

we easily find that

$$\pi = \frac{6}{\sqrt{n} - r_n}\left(1 - \frac{24\sqrt{n} + 120r_n}{\sqrt{n} - r_n}e^{-\pi\sqrt{n}} + \cdots\right).$$

We thus have proved the following theorem.

THEOREM 3.3. *We have*

$$\pi \approx \frac{6}{\sqrt{n} - r_n} =: A_n,$$

with the error approximately equal to

$$144\frac{\sqrt{n} + 5r_n}{(\sqrt{n} - r_n)^2}e^{-\pi\sqrt{n}},$$

where r_n is the algebraic expression on the right side in (3.8).

See Ramanujan's paper [15], [18, p. 33] for other approximations to π of this sort.

In the table below, we record the decimal expansion of each approximation A_n and the number N_n of digits of π agreeing with the approximation.

n	A_n	N_n
11	3.1538 ...	1
19	3.1423 ...	2
27	3.1416621 ...	3
43	3.141593 ...	5
67	3.14159266 ...	7
163	3.14159265358980 ...	12
35	3.141601 ...	3
51	3.14159289 ...	6

4. Eisenstein series and equations in π – second method

Set $\mathbf{P} := \mathbf{P}(q) := P(-q)$, $\mathbf{Q} := \mathbf{Q}(q) := Q(-q)$, $\mathbf{R} := \mathbf{R}(q) := R(-q)$, $\boldsymbol{\Delta} := \boldsymbol{\Delta}(q) := \mathbf{Q}^3(q) - \mathbf{R}^2(q)$, and $\mathbf{J} := \mathbf{J}(q) := 1728/j\left(\frac{3+\tau}{2}\right)$, where $q = e^{2\pi i\tau}$. Set

$$z^4 := \mathbf{Q} = \left(\frac{\boldsymbol{\Delta}}{\mathbf{J}}\right)^{1/3}, \tag{4.1}$$

by (2.1). Then, by (4.1) and the definition of $\mathbf{\Delta}$,

$$\mathbf{R} = \sqrt{\mathbf{Q}^3 - \mathbf{\Delta}} = \sqrt{\frac{\mathbf{\Delta}}{\mathbf{J}}}\sqrt{1-\mathbf{J}} = z^6\sqrt{1-\mathbf{J}}. \tag{4.2}$$

Recall the differential equations [16], [17, p. 142]

$$q\frac{dP}{dq} = \frac{P^2(q) - Q(q)}{12},\ q\frac{dQ}{dq} = \frac{P(q)Q(q) - R(q)}{3},\ q\frac{dR}{dq} = \frac{P(q)R(q) - Q^2(q)}{2}, \tag{4.3}$$

which yield the associated differential equations

$$q\frac{d\mathbf{P}}{dq} = \frac{\mathbf{P}^2(q) - \mathbf{Q}(q)}{12},\ q\frac{d\mathbf{Q}}{dq} = \frac{\mathbf{P}(q)\mathbf{Q}(q) - \mathbf{R}(q)}{3},\ q\frac{d\mathbf{R}}{dq} = \frac{\mathbf{P}(q)\mathbf{R}(q) - \mathbf{Q}^2(q)}{2}. \tag{4.4}$$

Now, by rearranging the second equation in (4.4), with the help of (4.1) and (4.2), we find that

$$\mathbf{P}(q) = \frac{\mathbf{R}(q)}{\mathbf{Q}(q)} + \frac{12q}{z}\frac{dz}{dq}. \tag{4.5}$$

From the chain rule and (4.5), it follows that, for any positive integer n,

$$\mathbf{P}(q^n) = \frac{\mathbf{R}(q^n)}{\mathbf{Q}(q^n)} + \frac{12q}{nz(q^n)}\frac{dz(q^n)}{dq}.$$

Subtracting (4.5) from the last equality and setting

$$m := \frac{z(q)}{z(q^n)},$$

we find that

$$\begin{aligned} n\mathbf{P}(q^n) - \mathbf{P}(q) =& n\frac{\mathbf{R}(q^n)}{\mathbf{Q}(q^n)} - \frac{\mathbf{R}(q)}{\mathbf{Q}(q)} + 12\frac{q}{z(q^n)}\frac{dz(q^n)}{dq} - 12\frac{q}{z(q)}\frac{dz(q)}{dq} \\ =& n\frac{\mathbf{R}(q^n)}{\mathbf{Q}(q^n)} - \frac{\mathbf{R}(q)}{\mathbf{Q}(q)} - 12\frac{q}{m}\frac{dm}{dq}. \end{aligned} \tag{4.6}$$

Our next aim is to replace $\frac{dm}{dq}$ in (4.6) by $\frac{dm}{d\mathbf{J}}(\mathbf{J}(q), \mathbf{J}(q^n))$. From (2.1), the definition of $\mathbf{J}$, (4.4), (4.1), and (4.2), upon differentiation, we find that

$$\begin{aligned} q\frac{d\mathbf{J}}{dq} =& \frac{(3\mathbf{Q}^2\mathbf{Q}' - 2\mathbf{R}\mathbf{R}')\mathbf{Q}^3 - 3\mathbf{Q}^2\mathbf{Q}'(\mathbf{Q}^3 - \mathbf{R}^2)}{\mathbf{Q}^6} \\ =& \frac{\{\mathbf{Q}^2(\mathbf{P}\mathbf{Q} - \mathbf{R}) - \mathbf{R}(\mathbf{P}\mathbf{R} - \mathbf{Q}^2)\}\mathbf{Q}^3 - \mathbf{Q}^2(\mathbf{P}\mathbf{Q} - \mathbf{R})(\mathbf{Q}^3 - \mathbf{R}^2)}{\mathbf{Q}^6} \\ =& \frac{\mathbf{R}\mathbf{Q}^3 - \mathbf{R}^3}{\mathbf{Q}^4} = \frac{\mathbf{R}\mathbf{\Delta}}{\mathbf{Q}^4} = z^6\sqrt{1-\mathbf{J}}\frac{\mathbf{J}}{\mathbf{Q}} = z^2\mathbf{J}\sqrt{1-\mathbf{J}}, \end{aligned} \tag{4.7}$$

which implies that

$$z^2(q) = \frac{1}{\mathbf{J}(q)\sqrt{1-\mathbf{J}(q)}} q \frac{d\mathbf{J}(q)}{dq}. \tag{4.8}$$

Replacing q by q^n in (4.8) and simplifying, we deduce that

$$z^2(q^n) = \frac{1}{n\mathbf{J}(q^n)\sqrt{1-\mathbf{J}(q^n)}} q \frac{d\mathbf{J}(q^n)}{dq}. \tag{4.9}$$

Using (4.8) and (4.9), we conclude that

$$m^2 = n \frac{\mathbf{J}(q^n)\sqrt{1-\mathbf{J}(q^n)}}{\mathbf{J}(q)\sqrt{1-\mathbf{J}(q)}} \frac{d\mathbf{J}(q)}{d\mathbf{J}(q^n)}. \tag{4.10}$$

It is well known that there is a relation (known as the class equation) between $j(\tau)$ and $j(n\tau)$ for any integer n [14, p. 231, Theorem 11.18(i)]. With the definition of $\mathbf{J}$ given at the beginning of this section, the class equation translates to a relation between $\mathbf{J}(q)$ and $\mathbf{J}(q^n)$. It follows that,

$$\frac{d\mathbf{J}(q)}{d\mathbf{J}(q^n)} = F(\mathbf{J}(q), \mathbf{J}(q^n)), \tag{4.11}$$

for some rational function $F(x,y)$. Thus, by (4.10) and (4.11), we may differentiate m with respect to $\mathbf{J}$, and so, by (4.7) and the definition of $m(q)$,

$$\begin{aligned} 2\frac{q}{m(q)}\frac{dm}{dq} &= 2z(q)z(q^n)\frac{q}{z^2(q)}\frac{dm}{dq} \\ &= 2z^2(q^n)m(q)\frac{dm}{d\mathbf{J}} q \frac{d\mathbf{J}}{dq} = z^2(q^n)\mathbf{J}\sqrt{1-\mathbf{J}}\frac{dm^2(q)}{d\mathbf{J}}. \end{aligned}$$

Using this in (4.6), we deduce that

$$\frac{n\mathbf{P}(q^n) - \mathbf{P}(q)}{z(q)z(q^n)} = n\frac{\mathbf{R}(q^n)}{\mathbf{Q}(q^n)} - \frac{\mathbf{R}(q)}{\mathbf{Q}(q)} - 6z^2(q^n)\mathbf{J}(q)\sqrt{1-\mathbf{J}(q)}\frac{dm^2}{d\mathbf{J}}. \tag{4.12}$$

If we put $q = e^{-\pi/\sqrt{n}}, n > 0$, (4.12) takes the shape

$$\begin{aligned} n\mathbf{P}(e^{-\pi\sqrt{n}}) - \mathbf{P}(e^{-\pi/\sqrt{n}}) =& n\frac{\mathbf{R}(e^{-\pi\sqrt{n}})}{\mathbf{Q}(e^{-\pi\sqrt{n}})} - \frac{\mathbf{R}(e^{-\pi/\sqrt{n}})}{\mathbf{Q}(e^{-\pi/\sqrt{n}})} \\ & - 6z^2(e^{-\pi\sqrt{n}})\mathbf{J}(e^{-\pi/\sqrt{n}})\sqrt{1-\mathbf{J}(e^{-\pi/\sqrt{n}})} \\ & \times \frac{dm^2}{d\mathbf{J}}\left(\mathbf{J}(e^{-\pi\sqrt{n}}), \mathbf{J}(e^{-\pi/\sqrt{n}})\right). \end{aligned} \tag{4.13}$$

It is well known that [12, p. 84]

$$\mathbf{J}(e^{-\pi/\sqrt{n}}) = \mathbf{J}(e^{-\pi\sqrt{n}}). \tag{4.14}$$

Furthermore, if

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} \quad \text{and} \quad \psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}, \qquad |q| < 1, \tag{4.15}$$

then [2, p. 127, Entries 13(iii), (iv)],

$$Q(q) = z_2^4(1 + 14x_2 + x_2^2), \tag{4.16}$$

and

$$R(q) = z_2^6(1 + x_2)(1 - 34x_2 + x_2^2), \tag{4.17}$$

where [2, p. 122–123, Entries 10(i), 11(iii)]

$$z_2 := \varphi^2(q) \quad \text{and} \quad x_2 := 16q\frac{\psi^4(q^2)}{\varphi^4(q)}. \tag{4.18}$$

Replacing q by $-q$ in (4.16) and (4.17), and using (4.18), we find that

$$\mathbf{Q}(q) = \varphi^8(-q) - 224q\varphi^4(-q)\psi^4(q^2) + 16^2q^2\psi^8(q^2) \tag{4.19}$$

and

$$\begin{aligned} \mathbf{R}(q) &= (\varphi^4(-q) - 16q\psi^4(q^2)) \\ &\quad \times (\varphi^8(-q) + 544q\varphi^4(-q)\psi^4(q^2) + 16^2q^2\psi^8(q^2)). \end{aligned} \tag{4.20}$$

Using the transformation formula [2, p. 43, Entry 27(ii)],

$$\varphi(e^{-\pi/t}) = 2e^{-\pi t/4}\sqrt{t}\psi(e^{-2\pi t})$$

in (4.19) and (4.20), we deduce that

$$\mathbf{R}(e^{-\pi/\sqrt{n}}) = -n^3\mathbf{R}(e^{-\pi\sqrt{n}}) \tag{4.21}$$

and

$$\mathbf{Q}(e^{-\pi/\sqrt{n}}) = n^2\mathbf{Q}(e^{-\pi\sqrt{n}}). \tag{4.22}$$

Using (4.14), (4.21), and (4.22), we may rewrite (4.13) as

$$\begin{aligned} n\mathbf{P}(e^{-\pi\sqrt{n}}) &- \mathbf{P}(e^{-\pi/\sqrt{n}}) \\ &= 2n\frac{\mathbf{R}(e^{-\pi\sqrt{n}})}{\mathbf{Q}(e^{-\pi\sqrt{n}})} - 6z^2(e^{-\pi\sqrt{n}})\mathbf{J}_n\sqrt{1 - \mathbf{J}_{1/n}}\frac{dm^2}{d\mathbf{J}}\left(\mathbf{J}_n, \mathbf{J}_n\right), \\ &= \left(2n\sqrt{1 - \mathbf{J}_n} - 6\mathbf{J}_n\sqrt{1 - \mathbf{J}_{1/n}}\frac{dm^2}{d\mathbf{J}}\left(\mathbf{J}_n, \mathbf{J}_n\right)\right)z^2(e^{-\pi\sqrt{n}}), \end{aligned} \tag{4.23}$$

where

$$\mathbf{J}_k = \mathbf{J}(e^{-\pi\sqrt{k}}), \qquad k > 0. \tag{4.24}$$

This gives the first relation between $\mathbf{P}(e^{-\pi\sqrt{n}})$ and $\mathbf{P}(e^{-\pi/\sqrt{n}})$.

Recall the definitions of Ramanujan's function $f(-q)$ and the Dedekind eta-function $\eta(\tau)$, namely,

$$f(-q) := \prod_{k=1}^{\infty}(1-q^k) =: e^{-2\pi i\tau/24}\eta(\tau), \qquad q = e^{2\pi i\tau}, \qquad \operatorname{Im}\tau > 0.$$

The function f satisfies the well-known transformation formula [2, p. 43]

$$n^{1/4}e^{-\pi\sqrt{n}/24}f(e^{-\pi\sqrt{n}}) = e^{-\pi/(24\sqrt{n})}f(e^{-\pi/\sqrt{n}}), \qquad n > 0. \tag{4.25}$$

Logarithmically differentiating (4.25) with respect to n, multiplying both sides by $48n^{3/2}/\pi$, rearranging terms, and employing the definition of $P(q)$ given in (3.1), we find that

$$\frac{12\sqrt{n}}{\pi} = n\mathbf{P}(e^{-\pi\sqrt{n}}) + \mathbf{P}(e^{-\pi/\sqrt{n}}). \tag{4.26}$$

This gives a second relation between $\mathbf{P}(e^{-\pi\sqrt{n}})$ and $\mathbf{P}(e^{-\pi/\sqrt{n}})$.

Now adding (4.23) and (4.26) and dividing by 2, we arrive at

$$n\mathbf{P}(e^{-\pi\sqrt{n}}) = \frac{6\sqrt{n}}{\pi} + \left(n\sqrt{1-\mathbf{J}_n} - 3\mathbf{J}_n\sqrt{1-\mathbf{J}_{1/n}}\frac{dm^2}{d\mathbf{J}}(\mathbf{J}_n,\mathbf{J}_n)\right)z^2(e^{-\pi\sqrt{n}}),$$

or, by (4.1),

$$\frac{1}{\sqrt{Q_n}}\left(P_n - \frac{6}{\sqrt{n}\pi}\right) = \sqrt{1-\mathbf{J}_n}\left(1 - 3\mathbf{J}_n\frac{\sqrt{1-\mathbf{J}_{1/n}}}{n\sqrt{1-\mathbf{J}_n}}\frac{dm^2}{d\mathbf{J}}(\mathbf{J}_n,\mathbf{J}_n)\right),$$

where Q_n is defined by (2.3), and P_n is defined by (3.2). (Be careful; $\mathbf{J}_n \neq J_n$, where J_n is defined by (2.4).)

We record the last result in the following theorem, which should be compared with Theorem 3.1.

THEOREM 4.1. *If P_n, Q_n, and $\mathbf{J}_n$ are defined by* (3.2), (2.3), *and* (4.24), *respectively, then*

$$\frac{1}{\sqrt{Q_n}}\left(P_n - \frac{6}{\sqrt{n}\pi}\right) = \sqrt{1-\mathbf{J}_n}\,t_n, \tag{4.27}$$

where

$$t_n := \left(1 - 3\mathbf{J}_n\frac{\sqrt{1-\mathbf{J}_{1/n}}}{n\sqrt{1-\mathbf{J}_n}}\frac{dm^2}{d\mathbf{J}}(\mathbf{J}_n,\mathbf{J}_n)\right). \tag{4.28}$$

Observe that, by (4.1), (4.2), and Theorem 2.1,

$$\sqrt{1-\mathbf{J}_n} = \left(\frac{\left(\frac{8}{3}J_n\right)^3 + 1}{\left(\frac{8}{3}J_n\right)^3}\right)^{1/2}. \tag{4.29}$$

Hence, the values of $\sqrt{1-\mathbf{J}_n}$ for those n given on page 211 of the Lost Notebook follow immediately from (2.7). In order to rederive Examples 3.2, it suffices to compute t_n.

THEOREM 4.2. *If $n > 1$ is an odd positive integer, then t_n lies in the ring class field of $\mathbb{Z}[\sqrt{-n}]$.*

Proof. By differentiating (4.10) with respect to $\mathbf{J}(q)$, we conclude that

$$\frac{dm^2}{d\mathbf{J}(q)} = \frac{\sqrt{1-\mathbf{J}(q^n)}}{\sqrt{1-\mathbf{J}(q)}} G(\mathbf{J}(q), \mathbf{J}(q^n)), \tag{4.30}$$

for some rational function $G(x, y)$. Using (4.28) and (4.30), we find that t_n can be expressed in terms of $\mathbf{J}_n$. Since $\mathbf{J}_n$ is in the ring class field of $\mathbb{Z}[\sqrt{-n}]$ when $n > 1$ is odd and squarefree [14, p. 220, Theorem 11.1], we complete our proof. □

An equivalent form of Theorem 4.2 is first mentioned without proof and conditions on n by the Chudnovskys on page 391 of [13].

Theorem 4.2 allows us to devise an empirical process for deriving t_n whenever the class group of $\mathbb{Q}(\sqrt{-n})$ is of the type $\mathbb{Z}_2^r$, $r \in \mathbb{N}$, where $r = 0$ refers to imaginary quadratic fields with class number 1. By (3.6), (4.27), and (4.29), we find that

$$t_n = \frac{Q_n}{R_n}\left(P_n - \frac{6}{\sqrt{n}\pi}\right). \tag{4.31}$$

When $r = 0$, we numerically compute the right hand side of (4.31) using the definitions of P_n, Q_n, and R_n, and then using the command "minpoly(t_n,1)" on the computer algebra system MAPLE V, we derive the values of t_n for $n = 3, 7, 11, 19, 27, 43, 67$ and 163. We summarize our findings in the following table.

n	t_n
3	0
7	$\frac{5}{21}$
11	$\frac{32}{77}$
19	$\frac{32}{57}$
27	$\frac{160}{253}$
43	$\frac{640}{903}$
67	$\frac{33440}{43617}$
163	$\frac{77265280}{90856689}$

When $r = 1$, we are in a situation where t_n satisfies a polynomial of degree 2. We use the command "minpoly(t_n,2)" and derive the resulting polynomial satisfied by t_n. Using this method, we could, in particular, determine t_n for $n = 35$ and 51, namely,

$$t_{35} = \frac{1504 + 576\sqrt{5}}{4123} \quad \text{and} \quad t_{51} = \frac{144}{329} + \frac{400\sqrt{17}}{5593}.$$

Using the values of t_n given in the tables above and Theorem 4.1, we rederive Examples 3.2.

So far, our method is still similar to that of the Borweins and Chudnovskys. Of course, we could continue to use "minpoly" and derive minimal polynomials satisfied by t_n for $r = 2, 3$, and 4. However, this means that we have to solve polynomials of degress 4, 8, and 16, respectively, which is cumbersome when the degree of the polynomial exceeds 4. We now describe a new process for deriving t_n, for $r > 1$. We illustrate our method using $n = 21$.

When $n = 21$, we numerically compute t_{21} and $t_{7/3}$ and use "minpoly" to deduce that

$$t_{21} + t_{7/3} = \frac{30493}{34279} + \frac{3820}{34279}\sqrt{7}$$

and

$$\sqrt{3}(t_{21} - t_{7/3}) = \frac{28800}{34279} + \frac{1935}{34279}\sqrt{7}.$$

Solving for t_{21} yields

$$t_{21} = \frac{30493}{68558} + \frac{645}{68558}\sqrt{21} + \frac{4800}{34279}\sqrt{3} - \frac{1910}{34279}\sqrt{7}.$$

The largest n for which t_n can be computed using this new empirical method is $n = 3315$, namely,

$$\begin{aligned} t_{3315} :=& \frac{1095255033002752301233099478037584}{2050242335692983321671746996556833} \\ &+ \frac{1006588064225996719872149534306400}{34854119706780716468419698941466161}\sqrt{17}\sqrt{5} \\ &+ \frac{6927791681751285514532804270700000}{34854119706780716468419698941466161}\sqrt{17} \\ &- \frac{1364345361637794925035656184576960}{2050242335692983321671746996556833}\sqrt{5} \\ &+ \frac{4001793228797818605212992092480000}{26653150364008783181732710955238829}\sqrt{13} \end{aligned}$$

$$+ \frac{107756441301588202151920972676 2688}{4531035561881493140894560862390 60093} \sqrt{13}\,\sqrt{17}\,\sqrt{5}$$

$$+ \frac{120226784218523863048087030809600}{6472907945544990201277944089129 4299} \sqrt{17}\,\sqrt{13}$$

$$+ \frac{239369594240980944219359445009600}{2665315036400878318173271095523 8829} \sqrt{13}\,\sqrt{5}.$$

This value is deduced from determining the quadratic numbers

$$t_{3315} + t_{1105/3} + t_{663/5} + t_{221/15},$$

$$\sqrt{221}(t_{3315} - t_{1105/3} + t_{663/5} - t_{221/15}),$$

$$\sqrt{13}(t_{3315} + t_{1105/3} - t_{663/5} - t_{221/15}),$$

and

$$\sqrt{5}(t_{3315} - t_{1105/3} - t_{663/5} + t_{221/15})$$

in $\mathbb{Q}(\sqrt{85})$. We show in the next section that knowledge of t_{3315} and $\mathbf{J}_{3315}$ leads to a new series for $1/\pi$ which converges at a rate of 73/74 decimal places per term. This series appears to be the fastest known convergent series for $1/\pi$ which involves quadratic radicals. The previous fastest convergent series, giving 50 decimal places per term, was obtained by the Borweins [8]. Their series involves quartic radicals which arise from the field $\mathbb{Q}(\sqrt{-1555})$, which has class number 4.

5. t_n, $\mathbf{J}_n$ and the Ramanujan–Borweins–Chudnovskys series for $1/\pi$

Let $\varphi(q)$ and $\psi(q)$ be given as in (4.15), and recall the definition of the hypergeometric function ${}_2F_1$,

$${}_2F_1(a, b; c; u) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{u^k}{k!}, \qquad |u| < 1.$$

In [20], using Ramanujan's differential equations (4.3) and the relations

$$Q(q^2) = z_2^4(1 - x_2 + x_2^2) \quad \text{and} \quad R(q^2) = z_2^6(1 + x_2)(1 - 2x_2)(1 - x_2/2),$$

where z_2 and x_2 are given by (4.18), Venkatachaliengar proved the inversion formula

$$z_2 = {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; x_2).$$

His method can be applied whenever we have relations of the form

$$Q(q) = z^4 F(x) \quad \text{and} \quad R(q) = z^6 G(x),$$

for some function z and x. The relations (4.1) and (4.2) indicate that we are in such a situation with $z := \mathbf{Q}$ and $x := \mathbf{J}$. Invoking Venkatachaliengar's method (see [11], [10] for examples of such calculations), we readily deduce that

$$z = {}_2F_1(\tfrac{1}{12}, \tfrac{5}{12}; 1; \mathbf{J}). \tag{5.1}$$

We are now ready to give a proof of an equivalent form of the general series for $1/\pi$ given by (3.5). As far as we know, a proof of this series has never been written down in the literature. First, by Clausen's formula [1, p. 116], we find that

$$z^2 = \left({}_2F_1(\tfrac{1}{12}, \tfrac{5}{12}; 1; \mathbf{J})\right)^2 = {}_3F_2(\tfrac{1}{6}, \tfrac{5}{6}, \tfrac{1}{2}; 1, 1; \mathbf{J}), \tag{5.2}$$

where

$${}_3F_2(a, b, c; d, e; u) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k (c)_k}{(d)_k (e)_k} \frac{u^k}{k!}, \quad |u| < 1.$$

This implies, by (5.1), that

$$2z \frac{dz}{d\mathbf{J}} = \sum_{k=1}^{\infty} A_k k \mathbf{J}^{k-1}, \tag{5.3}$$

where

$$A_k := \frac{\left(\frac{1}{6}\right)_k \left(\frac{5}{6}\right)_k \left(\frac{1}{2}\right)_k}{(k!)^3}.$$

By (4.5) and (4.7), we deduce that

$$2z \frac{dz}{d\mathbf{J}} = \frac{1}{6\mathbf{J}} \left(\frac{\mathbf{P}}{\sqrt{1-\mathbf{J}}} - z^2 \right). \tag{5.4}$$

Substituting (5.4) into (5.3) and using (5.2), we deduce that

$$\frac{\mathbf{P}}{\sqrt{1-\mathbf{J}}} = \sum_{k=0}^{\infty} A_k (6k+1) \mathbf{J}^k. \tag{5.5}$$

Next, set $q = e^{-\pi\sqrt{n}}$ and deduce from (5.5) that

$$\frac{P_n}{\sqrt{1-\mathbf{J}_n}} = \sum_{k=0}^{\infty} A_k (6k+1) \mathbf{J}_n^k. \tag{5.6}$$

On the other hand, by Theorem 4.1 and (4.1),

$$P_n = \frac{6}{\pi\sqrt{n}} + z_n^2 \sqrt{1-\mathbf{J}_n}\, t_n. \tag{5.7}$$

Using (5.7), (5.2) (with z replaced by z_n and $\mathbf{J}$ replaced by $\mathbf{J}_n$) and (5.6), we conclude that

$$\frac{6}{\sqrt{n}\sqrt{1-\mathbf{J}_n}} \frac{1}{\pi} = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{6}\right)_k \left(\frac{5}{6}\right)_k \left(\frac{1}{2}\right)_k}{(k!)^3} \mathbf{J}_n^k (6k+1-t_n). \tag{5.8}$$

It can be shown that (5.8) is equivalent to the Ramanujan–Borweins–Chudnovskys series. The series (5.8) enables us to write down for each n a series for $1/\pi$ if we know the values of t_n and $\mathbf{J}_n$.

We have already determined the value of t_{3315} in Section 4. It suffices to compute $\mathbf{J}_{3315}$ in order to write down a series for $1/\pi$ associated with $n = 3315$. We first quote the identity [4] [5],

$$j\left(\frac{3+\sqrt{-3n}}{2}\right) = -27\frac{(\lambda_n^2-1)(9\lambda_n^2-1)^3}{\lambda_n^2}, \tag{5.9}$$

where

$$\lambda_n = \frac{e^{\pi\sqrt{n}/2}}{3\sqrt{3}}\frac{f^6(e^{-\pi\sqrt{n/3}})}{f^6(e^{-\pi\sqrt{3n}})},$$

where $f(-q)$ is defined prior to (4.25). Since [5]

$$\lambda_{1105}^2 = \left(\frac{\sqrt{5}+1}{2}\right)^{24}(4+\sqrt{17})^6\left(\frac{15+\sqrt{221}}{2}\right)^6(8+\sqrt{65})^6,$$

the value of $\mathbf{J}_{3315}$ follows immediately from (5.9). The values $\mathbf{J}_{3315}$ and t_{3315}, when substituted into (5.8), yield the series which we mentioned at the end of Section 4.

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