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TOPOLOGICAL 0-1 LAWS FOR SUBSPACES OF A BANACH SPACE WITH A SCHAUDER BASIS

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ABSTRACT. For a Banach space X with an (unconditional) basis, topological 0-1 law type dichotomies are stated for block-subspaces of X as well as for subspaces of X with a successive finite-dimensional decomposition on its basis. A uniformity principle for properties of block-sequences, results about block-homogeneity, and a possible method to construct a Banach space with an unconditional basis which has a complemented subspace without an unconditional basis, are deduced.

1. Introduction and notation

The *second topological 0-1 law* (Theorem 8.47 in [13]) states that in an infinite product space of Polish spaces, a set with the Baire Property which is a tail set (i.e., invariant with respect to changing a finite number of coordinates) is either meager or comeager.

It is tempting to prove a principle of topological 0-1 law in Banach space theory, as many natural properties of Banach spaces (for example, all properties preserved by isomorphism) are invariant by a finite dimensional perturbation, and therefore correspond to tail sets in appropriate product spaces.

A natural setting is given by a Banach space with a Schauder basis and the set of its block-subspaces. This is motivated by [10], where W.T. Gowers considered different topologies on the set of block-sequences, with the aim of proving his famous dichotomy theorem. In this article, we obtain a principle of topological 0-1 law for the set of block-sequences of a Banach space with a Schauder basis (Theorem 2.1). With a little extra care in the proof, we also obtain a uniform version of this principle (Theorem 2.2). These results extend those of [9], where isomorphism classes of block-subspaces of a space with a Schauder basis were studied.

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The correct topological space in which to state this principle is the space $bb_d(X)$ of "rational normalized block-sequences" of a Banach space X with a Schauder basis, equipped with the product of the discrete topology on X. Indeed, this space is Polish, and the definition of its basic open sets do not involve small perturbations. Furthermore, by perturbation arguments, rational block-sequences capture enough of the structure of block-sequences on X.

A characterization of comeager sets in $bb_d(X)$ was obtained in [9]; we derive from it an explicit version of the principle of topological 0-1 law for block-sequences in $bb_d(X)$ as a dichotomy theorem (Theorem 2.4). From this we deduce two consequences regarding questions related to a conjecture by H. P. Rosenthal. Rosenthal asked if a Schauder basis $(e_n)_{n\in\mathbb{N}}$ of a Banach space, such that every normalized block-sequence has a subsequence equivalent to $(e_n)_{n\in\mathbb{N}}$, must be equivalent to the unit vector basis of c_0 or some $\ell_p, 1 \leq p < +\infty$. This question remains unsolved. However we show some ways of strengthening his definition to obtain a positive answer (Proposition 3.2 and Corollary 3.3), as a consequence of some general theorem about equivalence relations between block-sequences (Theorem 3.1).

We also state a particularly simple form of our topological 0-1 law theorem when the concerned property of block-sequences is stable by taking subsequences (Theorem 4.1).

Recall that, due to the work of W.T. Gowers [10] and of R. Komorowski and N. Tomczak-Jagermann [15], it is known that a homogeneous Banach space (i.e., one that is isomorphic to its closed infinite dimensional subspaces) must be isomorphic to ℓ_2. Their solution to this problem leaves many questions unanswered. If a Banach space has an unconditional basis and is isomorphic to its subspaces with an unconditional basis, must it be isomorphic to ℓ_2? If a Banach space has an unconditional basis and is isomorphic to its block-subspaces (call such a space *block-homogeneous*), must it be isomorphic to c_0 or $\ell_p, 1 \leq p < +\infty$? We give a positive answer in some special case (Proposition 5.3). Is there a direct proof that a homogeneous Banach space must be uniformly homogeneous (this was asked in [10])? Theorem 4.1 also has a uniform version, Theorem 4.2, and we deduce from this principle a general uniformity theorem. This gives a partial answer to the uniform homogeneity question, Proposition 5.5.

In the last section, we prove a principle of 0-1 topological law in a Banach space X with a Schauder basis, for subspaces with a successive finite dimensional decomposition on the basis (Proposition 6.4), again continuing some work from [9]. We derive a possible application to a long-standing open question in Banach space theory: does a complemented subspace of a Banach space with an unconditional basis necessarily have an unconditional basis (Corollary 6.5)?

The following notation will be used in this paper. Let X be a Banach space with a normalized Schauder basis $(e_n)_{n\in\mathbb{N}}$. We shall use some standard notation about finitely supported vectors on $(e_n)_{n\in\mathbb{N}}$; for example, we shall write $x < y$, and shall say that x and y are successive when $\max(\mathrm{supp}(x)) < \min(\mathrm{supp}(y))$. A block-sequence (or block-basis) is a sequence of successive vectors in X; see, e.g., [17] for basic facts about block-sequences.

It will be necessary to restrict our attention to normalized block-bases in X to use compactness properties. We denote by $bb(X)$ the set of normalized block-bases on X. Let $Q(X)$ be the set of normalized blocks of the basis that are a multiple of some block with rational coordinates (or coordinates in $\mathbb{Q} + i\mathbb{Q}$ in the complex case). We denote by $bb_d(X)$ the set of block-bases of vectors in $Q(X)$. (Here "d" stands for "discrete", this notation was introduced in [8].) We consider $bb_d(X)$ as a topological space, equipped with the product topology of the discrete topology on $Q(X)$, which turns it into a Polish space.

The notation $bb_d^{<\omega}(X)$ will denote the set of finite block-sequences with blocks in $Q(X)$. For a finite block sequence $\tilde{x} = (x_1, \ldots, x_n) \in bb_d^{<\omega}(X)$, we denote by $N(\tilde{x})$ the set of elements of $bb_d(X)$ whose first n vectors are $(x_1, \ldots, x_n)$; this is the basic open set associated to $\tilde{x}$.

If $(x_n)_{n\in J}$ is a finite or infinite block-sequence of X, then $[x_n]_{n\in\mathbb{N}}$ will stand for its closed linear span. If s is a finite block-basis and y is a finite or infinite block-basis supported after s, denote by $s^\frown y$ the concatenation of s and y. The notation $x = (x_n)_{n\in\mathbb{N}}$ will be reserved to denote an infinite block-sequence, and $[x]$ will denote its closed linear span; $\tilde{x}$ will denote a finite block-sequence, and $|\tilde{x}|$ its length as a sequence, $\mathrm{supp}(\tilde{x})$ the union of the supports of the terms of $\tilde{x}$. For two finite block-sequences $\tilde{x} = (x_1, \ldots, x_n)$ and $\tilde{y} = (y_1, \ldots, y_m)$, write $\tilde{x} < \tilde{y}$ to mean that they are successive, i.e., $x_n < y_1$. For a sequence of successive finite block-sequences $(\tilde{x}_i)_{i\in I}$, we denote the concatenation of the block-sequences by $\tilde{x}_1^\frown \ldots ^\frown \tilde{x}_n$ if the sequence is finite with $I = \{1, \ldots, n\}$, or $\tilde{x}_1^\frown \tilde{x}_2^\frown \ldots$ if it is infinite, and we denote by $\mathrm{supp}(\tilde{x}_i, i \in I)$ the support of the concatenation, by $[\tilde{x}_i]_{i\in I}$ the closed linear span of the concatenation.

We shall identify a block-sequence $(x_k)_{k\in K}$ indexed on some infinite subset $K = \{k_1, k_2, \ldots\}$ of $\mathbb{N}$ (where $(k_n)_{n\in\mathbb{N}}$ is increasing) with the block-sequence $(x_{k_n})_{n\in\mathbb{N}}$ indexed on $\mathbb{N}$. Thus, given an infinite block-sequence, we may always choose the most convenient way to index it. We make a similar identification for finite block-sequences. The range $\mathrm{ran}(x_0)$ of $x_0 \in X$ is the smallest interval of integers containing the support of x_0. If $x = (x_n)_{n\in I}$ is a finite or infinite block-sequence, $\mathrm{ran}(x)$ will denote the union $\bigcup_{n\in I} \mathrm{ran}(x_n)$.

Finally, we shall sometimes use the classical fact that any normalized block-basic sequence in X is an arbitrarily small perturbation of a basic sequence in $bb_d(X)$, and, in particular, is $1+\epsilon$-equivalent to it, for arbitrarily small $\epsilon > 0$.

2. Topological 0-1 laws for block-sequences

Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be block-sequences and let $n_0 \in \mathbb{N}$. When for some $N \in \mathbb{N}$, $x_n = y_n$ for all $n > N$ and the set $\{n \leq N : \forall m \leq N, x_n \neq y_m\}$ is of cardinality at most n_0, we shall say that $(y_n)_{n\in\mathbb{N}}$ is an n_0-*modification of* $(x_n)_{n\in\mathbb{N}}$. Note that in this case, there exist permutations σ and σ' of the integers such that $x_{\sigma(n)} = y_{\sigma'(n)}$ for all $n > n_0$, and so (x_n) and (y_n) span subspaces which are isomorphic with a constant $c(n_0)$ which depends only on n_0.

A *finite modification* of $(x_n)_{n\in\mathbb{N}}$ is a block-sequence $(y_n)_{n\in\mathbb{N}}$ which is an n_0-modification of $(x_n)_{n\in\mathbb{N}}$ for some n_0; this is the same as saying that $x_n = y_n$ for all $n \geq m_0$, for some m_0 (which could be much larger than n_0).

We are now ready to state our principle of topological 0-1 law for block-sequences.

Theorem 2.1 (Topological 0-1 law for block-sequences). *Let X be a Banach space with a Schauder basis. Assume $A \subset bb_d(X)$ has the Baire Property and is invariant by finite modifications. Then A is either meager or comeager in $bb_d(X)$.*

This is a corollary of the following uniform version:

Theorem 2.2 (Uniform topological 0-1 law for block-sequences). *Let X be a Banach space with a Schauder basis. Let $(A_N)_{N\in\mathbb{N}}$ be an increasing sequence of subsets of $bb_d(X)$ with the Baire Property, and let $A = \bigcup_{N\in\mathbb{N}} A_N$. Assume that for any $N \in \mathbb{N}$ and $n_0 \in \mathbb{N}$, there exists $K(N, n_0) \in \mathbb{N}$ such that whenever $(x_n)_{n\in\mathbb{N}}$ belongs to A_N, then any n_0-modification of $(x_n)_{n\in\mathbb{N}}$ belongs to $A_{K(N,n_0)}$. Then either A is meager in $bb_d(X)$, or there exists $K \in \mathbb{N}$ such that A_K is comeager in $bb_d(X)$.*

Proof. We assume that A is non-meager. Then for some $N \in \mathbb{N}$, A_N is non-meager. Our proof is similar to a proof from [9] for classes of isomorphism. As A_N has the Baire property, it is comeager in some basic open set U, of the form $N(\tilde{x})$, for some finite block-sequence $\tilde{x} \in bb_d^{<\omega}(X)$.

We now prove that A_K is comeager in $bb_d(X)$ if we choose $K = K(N, 2\max(\mathrm{supp}(\tilde{x})))$. So let us assume $V = N(\tilde{y})$ is some basic open set in $bb_d(X)$ such that A_K is meager in V. We may assume that $|\tilde{y}| > |\tilde{x}|$ and write $\tilde{y} = \tilde{x}'^\frown\tilde{z}$ with $\tilde{x} < \tilde{z}$ and $|\tilde{x}'| \leq \max(\mathrm{supp}(\tilde{x}))$. Choose $\tilde{u}$ and $\tilde{v}$ in $bb_d^{<\omega}(X)$ such that $\tilde{u}, \tilde{v} > \tilde{z}$, $|\tilde{u}| = |\tilde{x}'|$ and $|\tilde{v}| = |\tilde{x}|$, and such that $\max(\mathrm{supp}(\tilde{u})) = \max(\mathrm{supp}(\tilde{v}))$. Let U' be the basic open set $N(\tilde{x}^\frown\tilde{z}^\frown\tilde{u})$ and let V' be the basic open set $N(\tilde{x}'^\frown\tilde{z}^\frown\tilde{v})$. Again A_N is comeager in U', while A_K is meager in V'.

Now let T be the canonical map from U' to V'. For all u in U', $T(u)$ is an $(|\tilde{x}| + |\tilde{x}'|)$-modification of u. As $|\tilde{x}| + |\tilde{x}'| \leq 2\max(\mathrm{supp}(\tilde{x}))$, it follows that for any $u \in A_N \cap U'$, $T(u)$ belongs to $A_{K(N,2\max(\mathrm{supp}(\tilde{x})))} = A_K$. So A_K is comeager in $V' \subset V$. By the choice of V this gives a contradiction. $\square$

If A is a subset of $bb_d(X)$ and $\Delta = (\delta_n)_{n\in\mathbb{N}}$ is a sequence of positive real numbers, we denote by A_Δ the Δ-*expansion of* A in $bb_d(X)$, that is, $x = (x_n) \in A_\Delta$ iff there exists $y = (y_n) \in A$ such that $\|y_n - x_n\| \leq \delta_n, \forall n \in \mathbb{N}$. Such an element y will be called a Δ-*perturbation of* x.

For a block-sequence $(x_n)_{n\in\mathbb{N}}$ of X, a pair $((x_n)_{n\in I}, (x_n)_{n\in J})$ of block-sequences associated to a partition of $\mathbb{N}$ in two infinite sets I and J will be called a *partition* of $(x_n)_{n\in\mathbb{N}}$. Conversely, when $x = (x_n)_{n\in I}, y = (y_n)_{n\in J}$ are infinite block-sequences whose ranges are disjoint, we call *concatenation of* x *and* y the unique (up to the choice of K) block-sequence $z = (z_n)_{n\in K}$ such that $\{z_n, n \in K\} = \{x_n, n \in I\} \cup \{y_n, n \in J\}$.

Finally, if $\tilde{a} \in bb^{<\omega}(X)$ and $x \in bb(X)$, we say that x *passes through* $\tilde{a}$ if it can be written

$$x = \tilde{y}^\frown \tilde{a}^\frown z,$$

for some $\tilde{y} \in bb^{<\omega}(X)$ and some $z \in bb(X)$.

We recall a characterization of comeager subsets of $bb_d(X)$ which was proved in [9].

Proposition 2.3 (V. Ferenczi and C. Rosendal [9]). *Let X be a Banach space with a Schauder basis. Let A be a comeager subset in $bb_d(X)$. Then for all $\Delta > 0$, there exists a successive sequence $(\tilde{a}_n)_{n\in\mathbb{N}} \in (bb_d^{<\omega}(X))^\omega$ such that any block-sequence of $bb_d(X)$ passing through infinitely many of the $\tilde{a}_n$'s is in A_Δ.*

As was noted in [9], the property in the conclusion of this proposition is essentially (i.e., up to perturbation) a characterization of comeager sets in $bb_d(X)$. Indeed, it easily implies that A_Δ is comeager. Combining this observation with the principle of 0-1 topological law, we obtain the following dichotomy theorem.

Theorem 2.4. *Assume $A \subset bb_d(X)$ has the Baire Property, is invariant by finite modifications and by Δ-perturbations for some $\Delta > 0$. If A is not meager in $bb_d(X)$ (i.e., for any successive sequence $(\tilde{a}_n) \in (bb_d^{<\omega}(X))^\omega$, there exists $x \in bb_d(X)$ passing through $\tilde{a}_n$ for infinitely many n's, such that x is in A), then A is comeager in $bb_d(X)$ (i.e., there exists a successive sequence $(\tilde{a}_n) \in (bb_d^{<\omega}(X))^\omega$, such that any $x \in bb_d(X)$, passing through $\tilde{a}_n$ for infinitely many n's must belong to A).*

3. Applications to isomorphism and permutative equivalence

In this section, we deduce from Theorem 2.4 a general result about equivalence relations on $bb_d(X)$. Its proof is inspired in part by the classical decomposition method of Pełczyński for isomorphisms between Banach spaces.

If R is an equivalence relation on $bb_d(X)$, we say that R is *compatible with concatenation* if for all $(x,y) \in bb_d(X)^2$, for all $z \in bb_d(X)$ whose range is disjoint from the ranges of x and y, we have

$$xRy \Rightarrow x^\frown zRy^\frown z.$$

THEOREM 3.1. *Let R be an equivalence relation on $bb_d(X)$ which is invariant by finite modifications and Δ-perturbations for some $\Delta > 0$. Assume also that R is compatible with concatenation. If there is some R-class A with the Baire Property such that every successive sequence $(\tilde{x}_n)_{n\in\mathbb{N}} \in (bb_d^{<\omega}(X))^\omega$ has a subsequence $(\tilde{x}_{n_k})_{k\in\mathbb{N}}$ such that $\tilde{x}_{n_1}^\frown \tilde{x}_{n_2}^\frown \ldots$ is in A, then $bb_d(X) = A$.*

Proof. We first note that by our assumption and Theorem 2.4, we have that A is comeager in $bb_d(X)$, which means that there exists a sequence of successive finite normalized block-sequences $(\tilde{a}_n)_{n\in\mathbb{N}} \in (bb_d^{<\omega}(X))^\omega$ such that every block-sequence in $bb_d(X)$ passing through infinitely many $(\tilde{a}_n)_{n\in\mathbb{N}}$'s must belong to A. In particular, $\tilde{a}_{n_1}^\frown \tilde{a}_{n_2}^\frown \ldots$ is in A for any subsequence $(n_k)_{k\in\mathbb{N}}$ of the integers.

Let $(y_n)_{n\in\mathbb{N}}$ be arbitrary in $bb_d(X)$. We may find two block-sequences $(y_n^1)_{n\in\mathbb{N}}$ and $(y_n^2)_{n\in\mathbb{N}}$ which partition $(y_n)_{n\in\mathbb{N}}$ and a subsequence (n_k) of the integers such that (y_n^1) (resp. (y_n^2)) and $\tilde{a}_{n_2}^\frown \tilde{a}_{n_4}^\frown \ldots$ (resp. $\tilde{a}_{n_1}^\frown \tilde{a}_{n_3}^\frown \ldots$) have disjoint ranges. We let $(a_n^1)_{n\in\mathbb{N}}$ be the sequence $\tilde{a}_{n_1}^\frown \tilde{a}_{n_3}^\frown \ldots$, and $(a_n^2)_{n\in\mathbb{N}}$ be the sequence $\tilde{a}_{n_2}^\frown \tilde{a}_{n_4}^\frown \ldots$.

Applying the hypothesis about A, we see that (y_n^1) has a subsequence (z_n) which is in A. We may assume $(y_n^1) = (z_n)^\frown(w_n)$ for some infinite block-sequence (w_n). We deduce by R-compatibility

$$(y_n^1)R(a_n^2)^\frown(w_n).$$

As the sequence $(a_n^2)^\frown(w_n)$ passes through all the $\tilde{a}_{n_{2k}}$ for $k \in \mathbb{N}$, it must belong to A. We deduce that (y_n^1) is in A. The same reasoning gives that (y_n^2) is in A.

Finally, using R-compatibility again, we obtain

$$(y_n)_{n\in\mathbb{N}} = (y_n^1)^\frown(y_n^2)R(a_n^1)^\frown(a_n^2) = a_{n_1}^\frown a_{n_2}^\frown \ldots.$$

So $(y_n)_{n\in\mathbb{N}}$ is in the R-class A. □

Recall that two basic sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are said to be *permutatively equivalent* if there is a permutation σ on $\mathbb{N}$ such that $(x_n)_{n\in\mathbb{N}}$ is equivalent to $(y_{\sigma(n)})_{n\in\mathbb{N}}$, in which case we shall write $(x_n)_{n\in\mathbb{N}} \sim^{perm} (y_n)_{n\in\mathbb{N}}$.

A Schauder basis $(e_n)_{n\in\mathbb{N}}$ of a Banach space is said to be a *Rosenthal basis* if every normalized block-sequence of $(e_n)_{n\in\mathbb{N}}$ has a subsequence equivalent to $(e_n)_{n\in\mathbb{N}}$. As mentioned in the introduction, it is an open conjecture whether any Rosenthal basis is equivalent to the canonical basis of c_0 or ℓ_p. This question is motivated by Zippin's theorem ([17], Theorem 2.a.9), according to which a Schauder basis $(e_n)_{n\in\mathbb{N}}$ of a Banach space which is equivalent to all its normalized block-sequences must be equivalent to the canonical basis of c_0 or ℓ_p.

The question can also be asked for permutative equivalence, or isomorphism of the closed linear span, instead of equivalence. The question for permutative equivalence, however, turns out to be the same as the original question (by Proposition 6.2 from [1]). It is also open for isomorphism. (In this case one has to replace, in the conclusion, equivalence with the unit vector basis of c_0 or ℓ_p, by isomorphism with c_0 or ℓ_p.)

The form of Theorem 3.1 suggests strengthening the Rosenthal hypothesis by considering successive finite sequences of block-sequences instead of block-sequences. In this case, we shall obtain a positive answer for permutative equivalence as a non-trivial application of Theorem 2.4.

Recall that a *constant coefficient block* on $(e_i)_{i\in\mathbb{N}}$ is a finitely supported vector of the form $\lambda(\sum_{i\in I} e_i)$. The set of normalized sequences of successive constant coefficient blocks is a subset of $bb_d(X)$ that we shall denote $ccb(X)$.

A Schauder basis $(e_n)_{n\in\mathbb{N}}$ of a Banach space X is said to be *unconditional* if there is some $C \geq 1$ such that for any $I \subset \mathbb{N}$, any norm 1 vector $x = \sum_{n\in\mathbb{N}} a_n e_n \in X$, $\left\|\sum_{n\in I} a_n e_n\right\| \leq C$. In particular, any subspace generated by a subsequence of $(e_n)_{n\in\mathbb{N}}$ is complemented in X.

Proposition 3.2. *Let X be a Banach space with an unconditional basis $(e_n)_{n\in\mathbb{N}}$ such that every successive sequence $(\tilde{x}_n)_{n\in\mathbb{N}} \in (bb^{<\omega}(X))^\omega$ has a subsequence $(\tilde{x}_{n_k})_{k\in\mathbb{N}}$ with $\tilde{x}_{n_1}^\frown \tilde{x}_{n_2}^\frown \ldots$ permutatively equivalent to $(e_n)_{n\in\mathbb{N}}$. Then $(e_n)_{n\in\mathbb{N}}$ is equivalent to the canonical basis of c_0 or ℓ_p.*

Proof. We give two different proofs of the result.

For the first proof, we note that every subsequence of $(e_n)_{n\in\mathbb{N}}$ has a further subsequence which is permutatively equivalent to $(e_n)_{n\in\mathbb{N}}$. By [1], Proposition 6.2, it follows that some permutation of $(e_n)_{n\in\mathbb{N}}$ is subsymmetric. Passing to a subsymmetric subsequence and up to renorming, we may assume that $(e_n)_{n\in\mathbb{N}}$ is normalized, 1-unconditional, 1-subsymmetric.

Now by Krivine's Theorem [16], we may find some $p \in [1, +\infty]$ and a successive sequence $(\tilde{x}_n)_{n\in\mathbb{N}}$ in $(bb^{<\omega}(X))^\omega$ such that each $\tilde{x}_n$ is of length n and 2-equivalent to the unit basis of ℓ_p^n. For some subsequence $(\tilde{x}_{n_k})_{k\in\mathbb{N}}$, $\tilde{x}_{n_1}^\frown \tilde{x}_{n_2}^\frown \ldots$ is C-permutatively equivalent to $(e_n)_{n\in\mathbb{N}}$ for some constant C. In particular, each $\tilde{x}_{n_k}$ is C-permutatively equivalent to some subsequence of $(e_n)_{n\in\mathbb{N}}$ of length n_k, or by subsymmetry, to $(e_i)_{i\leq n_k}$. We deduce that the unit basis of $\ell_p^{n_k}$ is $2C$-equivalent to $(e_i)_{i\leq n_k}$, for any $k \in \mathbb{N}$, which proves the result.

The second proof uses Theorem 3.1. It implies a stronger version of this proposition as we will only consider sequences of constant coefficient blocks.

We recall that $ccb(X)$ is the set of constant coefficient block-sequences in X. We let $\widetilde{ccb}(X)$ be the set of sequences in $bb_d(X)$ which are equivalent to some sequence in $ccb(X)$.

For (x_n) and (y_n) in $bb_d(X)$, we define an equivalence relation R by letting $(x_n)R(y_n) \Leftrightarrow (x_n) \sim^{perm} (y_n)$ if (x_n) and (y_n) are in $\widetilde{ccb}(X)$, and $(x_n)R(e_n)$ whenever $(x_n) \notin \widetilde{ccb}(X)$. The R-class of (e_n) is analytic, and so has the Baire Property, and R is stable by Δ-perturbations for small enough $\Delta > 0$, and by finite modifications. It is also compatible with concatenation, by the unconditionality of the basis. By Theorem 3.1, we deduce that there is only one R-class. In particular, all constant coefficient block sequences are permutatively equivalent. So $(e_n)_{n\in\mathbb{N}}$ is equivalent to the unit basis of c_0 or ℓ_p, by [1] Proposition 6.2. □

In the case of isomorphism, we recall that a Banach space with a Schauder basis, which is isomorphic to its block-subspaces, is said to be block-homogeneous. We have the following immediate corollary of Theorem 3.1:

Corollary 3.3. *Let X be a Banach space with an unconditional basis $(e_n)_{n\in\mathbb{N}}$. Assume that there exists some Banach space Y such that every sequence of successive finite normalized block-sequences $(\tilde{x}_n)_{n\in\mathbb{N}}$ has a subsequence $(\tilde{x}_{n_k})_{k\in\mathbb{N}}$ such that $[\tilde{x}_{n_1}^\frown \tilde{x}_{n_2}^\frown \ldots]$ is isomorphic to Y. Then X is block-homogeneous.*

Proof. We apply Theorem 3.1 to the relation of isomorphism, which is compatible with concatenation by the unconditionality of the basis. □

It is an open question whether a block-homogeneous Banach space must be isomorphic to c_0 or $\ell_p, 1 \leq p < +\infty$. In Section 5, we shall obtain more results about this property.

4. Topological 0-1 law for properties which are stable by taking subsequences

When a set A of block-sequences is stable by taking subsequences, we obtain a simpler version of the principle of the topological 0-1 law for A:

Theorem 4.1. *Let X be a Banach space with a Schauder basis. Let A be a subset of $bb_d(X)$ with the Baire Property, which is stable by Δ-perturbations for some $\Delta > 0$, by finite modifications, and by taking subsequences. Assume that any sequence $(\tilde{x}_n)_{n\in\mathbb{N}} \in (bb_d^{<\omega}(X))^\omega$ of successive finite block-sequences admits a subsequence $(\tilde{x}_{n_k})_{k\in\mathbb{N}}$ such that the block-sequence $\tilde{x}_{n_1}^\frown \tilde{x}_{n_2}^\frown \ldots$ belongs to A. Then every block-sequence in $bb_d(X)$ admits a partition in a pair of elements of A. If, furthermore, the set A is stable by concatenation of pairs of block-sequences, then $bb_d(X) = A$.*

Once again this is a corollary of a uniform version:

Theorem 4.2. *Let X be a Banach space with a Schauder basis. Let $(A_N)_{N\in\mathbb{N}}$ be an increasing sequence of subsets of $bb_d(X)$ with the Baire Property, satisfying:*

(a) *There exists $\Delta > 0$ such that for any $N \in \mathbb{N}$, there exists $K_1(N) \in \mathbb{N}$ such that $(A_N)_\Delta \subset A_{K_1(N)}$.*
(b) *For any $N \in \mathbb{N}$ and $n_0 \in \mathbb{N}$, there exists $K_2(N, n_0) \in \mathbb{N}$ such that whenever $(x_n)_{n\in\mathbb{N}}$ belongs to A_N, then any n_0-modification of $(x_n)_{n\in\mathbb{N}}$ belongs to $A_{K_2(N,n_0)}$.*
(c) *For any $N \in \mathbb{N}$, there exists $K_3(N) \in \mathbb{N}$ such that whenever $(x_n)_{n\in\mathbb{N}}$ belongs to A_N then any subsequence of $(x_n)_{n\in\mathbb{N}}$ belongs to $A_{K_3(N)}$.*

Let $A = \bigcup_{N\in\mathbb{N}} A_N$. Assume that for any sequence $(\tilde{x}_n)_{n\in\mathbb{N}} \in (bb_d^{<\omega}(X))^\omega$ of successive finite block-sequences, there is a subsequence $(\tilde{x}_{n_k})_{k\in\mathbb{N}}$ such that the block-sequence $\tilde{x}_{n_1}^\frown\tilde{x}_{n_2}^\frown\dots$ belongs to A. Then there exists $N \in \mathbb{N}$ such that every block-sequence in $bb_d(X)$ has a partition in two elements of A_N. If furthermore,

(d) *for any $N \in \mathbb{N}$, there exists $K_4(N) \in \mathbb{N}$ such that any concatenation of a pair of block-sequences in A_N^2 belongs to $A_{K_4(N)}$,*

then $bb_d(X) = A_N$ for some $N \in \mathbb{N}$.

Proof. The part which is a consequence of (d) is obvious once we prove the first part of the proposition. We note that by Proposition 2.2, either A is meager, or A_N is comeager for some $N \in \mathbb{N}$. By (a), there is some $\Delta > 0$ such that $A = A_\Delta$. It follows that $A_\Delta \cap A^C = \emptyset$, that is, $(A^C)_\Delta \cap A = \emptyset$.

If A is meager, Proposition 2.3 gives us a sequence of successive finite block-sequences $(\tilde{x}_n)_{n\in\mathbb{N}}$ such that, in particular, $\tilde{x}_{n_1}^\frown\tilde{x}_{n_2}^\frown\dots$ is in $(A^C)_\Delta$ for every subsequence $(\tilde{x}_{n_k})_{k\in\mathbb{N}}$. So for no subsequence $(\tilde{x}_{n_k})_{k\in\mathbb{N}}$, $\tilde{x}_{n_1}^\frown\tilde{x}_{n_2}^\frown\dots$ is in A.

So A_N is comeager for some $N \in \mathbb{N}$. Applying Proposition 2.3, and up to modifying N, let $(\tilde{a}_n)_{n\in\mathbb{N}}$ be a sequence of successive block-sequences such that every block-sequence passing through infinitely many of the $\tilde{a}_n$'s is in A_N.

Let now $(x_n)_{n\in\mathbb{N}}$ be an arbitrary block-sequence in $bb_d(X)$. We note that we may find a partition of $(x_n)_{n\in\mathbb{N}}$ in two subsequences $(x_n)_{n\in I}$ and $(x_n)_{n\in J}$, and a subsequence $(\tilde{a}_{n_k})_{k\in\mathbb{N}}$ of $(\tilde{a}_n)_{n\in\mathbb{N}}$ such that $(x_n)_{n\in I}$ and $(\tilde{a}_{n_{2k}})_{k\in\mathbb{N}}$ have disjoint ranges (let $(i_n)_{n\in\mathbb{N}}$ denote their concatenation) and such that $(x_n)_{n\in J}$ and $(\tilde{a}_{n_{2k-1}})_{k\in\mathbb{N}}$ have disjoint ranges (let $(j_n)_{n\in\mathbb{N}}$ denote their concatenation).

Now $(i_n)_{n\in\mathbb{N}}$ belongs to A_N, so by (c), for some $N' \in \mathbb{N}$, $(x_n)_{n\in I}$ belongs to $A_{N'}$, and likewise $(x_n)_{n\in J}$ belongs to $A_{N'}$. □

In particular, we deduce a uniformity principle from Proposition 4.2. Under its hypotheses, and if every block-sequence of $bb_d(X)$ is in A, there exists $N \in \mathbb{N}$ such that every block-sequence of $bb_d(X)$ is in A_N. This method was first used in [9] to study the property of complementable embeddability ([9], Proposition 17).

To conclude this section, it is worth noting the form that our topological 0-1 law takes when A is really an isomorphic property of the span of a block-sequence in $bb_d(X)$.

Theorem 4.3. *Let X be a Banach space with an unconditional basis $(e_n)_{n\in\mathbb{N}}$. Let P be an isomorphic property of Banach spaces such that $A = \{(x_n)_{n\in\mathbb{N}} \in bb_d(X) : [x_n]_{n\in N}$ has $P\}$ has the Baire Property, and which is stable by taking complemented subspaces.*

Assume that for any sequence of successive finite block-sequences $(\tilde{x}_n)_{n\in\mathbb{N}}$, there is a subsequence $(\tilde{x}_{n_k})_{k\in\mathbb{N}}$ such that the block-subspace $[\tilde{x}_{n_1}^\frown \tilde{x}_{n_2}^\frown \ldots]$ satisfies P. Then every block-subspace of X is the sum of two disjointly supported block-subspaces satisfying P.

Assume furthermore that any direct sum of two spaces with P satisfies P. Then every block-subspace of X satisfies P.

5. Applications to homogeneity and uniformity problems

The following homogeneity question remains unsolved:

Question 5.1. *Let X be a Banach space with an unconditional basis which is isomorphic to all its subspaces with an unconditional basis. Does it follow that X is isomorphic to ℓ_2?*

We recall that a Banach space X with a basis is *block-homogeneous* if X is isomorphic to all its block-subspaces. The following question is also still open:

Question 5.2. *Let X be a Banach space with an unconditional basis $(e_n)_{n\in\mathbb{N}}$, which is block-homogeneous. Does it follow that X is isomorphic to c_0 or ℓ_p?*

Note that an unconditional basis $(e_n)_{n\in\mathbb{N}}$ of a block-homogeneous Banach space X need not be equivalent to the canonical basis of c_0 or ℓ_p: for $1 < p < +\infty$, $X = (\oplus_{n\in\mathbb{N}}\ell_2^n)_p$ is isomorphic to ℓ_p, and every block-subspace of X (with the associated canonical basis) is complemented in X ([17], Proposition 2.a.12) and thus isomorphic to ℓ_p as well.

Note also that a positive answer to Question 5.2 would imply a positive answer to Question 5.1. Indeed it is known that each space c_0 or $\ell_p, p \neq 2$, contains a subspace with an unconditional basis which is not isomorphic to the whole space. This is easy for c_0 and $\ell_p, p < 2$, and in this case a continuum of non-isomorphic subspaces with this property can be obtained [7]; for $\ell_p, p > 2$, this requires results of Szankowski about the Approximation Property; see [17], p. 91.

Question 5.2 is motivated by the fact that $(e_n)_{n\in\mathbb{N}}$ must be equivalent to the basis of c_0 or ℓ_p when all block-sequences are equivalent (Zippin), or even permutatively equivalent (Bourgain, Casazza, Lindenstrauss, Tzafriri). The problem seems to be much harder for isomorphism.

We first show how, in a special case, some results of uniqueness of an unconditional basis will allow us to pass from isomorphism to permutative equivalence and deduce a positive answer. We recall some definitions and results from [5]. A sequence space X is said to be *left (resp. right) dominant* if there exists a constant $C \geq 1$ such that whenever $(u_i)_{i\leq n}$ and $(v_i)_{i\leq n}$ are finite block-sequences, with $\|u_i\| \geq \|v_i\|$ (resp. $\|u_i\| \leq \|v_i\|$) and $v_i > u_i$ for all $i \leq n$, then $\|\sum_{i=1}^n v_i\| \leq C\,\|\sum_{i=1}^n u_i\|$ (resp. $\|\sum_{i=1}^n u_i\| \leq C\,\|\sum_{i=1}^n v_i\|$). When X is left or right dominant, then there exists exactly one $r = r(X)$ such that l_r (or c_0 if $r = +\infty$) is finitely disjointly representable in X, and we call r the index of X.

We refer to [17], [12] for definitions and background on Banach lattices. If X and Y are Banach lattices, a bounded linear operator $V : X \to Y$ is called a *lattice homomorphism* if $V(x_1 \vee x_2) = Vx_1 \vee Vx_2$ for all $x_1, x_2 \in X$. Following [5], we define a Banach lattice X to be *sufficiently lattice-euclidean* if there exists $C \geq 1$ such that for all $n \in \mathbb{N}$, there exist operators $S : X \to \ell_2^n$ and $T : \ell_2^n \to X$ such that $ST = I_{\ell_2^n}$, $\|S\|\,\|T\| \leq C$ and such that S is a lattice homomorphism. This is equivalent to saying that ℓ_2 is finitely representable as a complemented sublattice of X. A Banach lattice which is not sufficiently lattice-euclidean is said to be *anti-lattice euclidean.*

For an unconditional basis $(x_n)_{n\in\mathbb{N}}$ of a Banach space (viewed as a Banach lattice), being sufficiently lattice-euclidean is the same as having, for some $C \geq 1$ and every $n \in \mathbb{N}$, a C-complemented, C-isomorphic copy of ℓ_2^n whose basis is disjointly supported on $(x_n)_{n\in\mathbb{N}}$.

Proposition 5.3. *Let X be a Banach space with a normalized unconditional basis $(e_n)_{n\in\mathbb{N}}$ such that all subsequences span isomorphic subspaces of X. Assume $(e_n)_{n\in\mathbb{N}}$ is right or left dominant with $r(X) \neq 2$ and that $(e_n)_{n\in\mathbb{N}}$ is equivalent to $(e_{2n})_{n\in\mathbb{N}}$. Then $(e_n)_{n\in\mathbb{N}}$ is equivalent to the canonical basis of $l_{r(X)}$ (or c_0 if $r(X) = +\infty$).*

Proof. Let $(y_n)_{n\in\mathbb{N}}$ be any subsequence of $(e_n)_{n\in\mathbb{N}}$, and $Y = [y_n]_{n\in\mathbb{N}}$. The sequence $(y_n)_{n\in\mathbb{N}}$ is equivalent to an unconditional basis $(u_n)_{n\in\mathbb{N}}$ of X. It is enough to note now that the proof of [5], Theorem 5.7, is still valid as long as we prove that $(u_n)_{n\in\mathbb{N}}$ is anti-lattice euclidean. But this is clear because $r(Y) = r(X) \neq 2$. So $(y_n)_{n\in\mathbb{N}}$ must be permutatively equivalent to $(e_n)_{n\in\mathbb{N}}$.

It follows from [1], Proposition 6.2, that some subsequence $(v_n)_{n\in\mathbb{N}}$ of $(e_n)_{n\in\mathbb{N}}$ is subsymmetric. By [5], X is asymptotically c_0 or ℓ_p for some $p \neq 2$, so $(v_n)_{n\in\mathbb{N}}$ is equivalent to the canonical basis of c_0 or ℓ_p, $p \neq 2$, and $(e_n)_{n\in\mathbb{N}}$ as well. $\square$

The right or left dominant hypothesis in Proposition 5.3 cannot be removed: the canonical basis $(e_n)_{n\in\mathbb{N}}$ of Schlumprecht's space S [19] is unconditional, subsymmetric, but S does not even contain a copy of c_0 or ℓ_p.

It is of interest to note that S is however quite homogeneous in some sense: any constant coefficient block-subspace of S is isomorphic to S (see [14], Remark before Proposition 9). So S is an example of a non c_0 or ℓ_p, yet "constant coefficient block-homogeneous" sequence space. This is in contrast to the theorem of Zippin (resp. the theorem of Bourgain, Casazza, Lindenstrauss, Tzafriri) for equivalence (resp. permutative equivalence), which can be proved using only constant coefficient block-sequences in X; see [1].

The question of uniformity in the homogeneous Banach space problem was raised by Gowers [10]. Since a homogeneous Banach space must be isomorphic to ℓ_2, it is trivial that if X is homogeneous, then there exists a constant $C \geq 1$ such that X is C-isomorphic to any of its subspaces. However, there does not seem to be a direct proof of this fact. Note also that uniformity is the first step in the proof of the theorem of Zippin. So the following question is natural:

Question 5.4. *Let X be a Banach space with an unconditional basis $(e_n)_{n\in\mathbb{N}}$. Assume X is block homogeneous. Does there exist $C \geq 1$ such that X is C-block homogeneous?*

A Banach space with a Schauder basis is C-block homogeneous when it is C-isomorphic to its block-subspaces.

As a partial result, we may use the primeness of the spaces c_0 and ℓ_p and Theorem 4.2 to get a positive answer to Question 5.4 when X is isomorphic to ℓ_p or c_0:

Proposition 5.5. *Let $p \geq 1$. Let X be a Banach space with an unconditional basis. Assume X is block-homogeneous and isomorphic to ℓ_p. Then there exists $C \geq 1$, such that all block-subspaces of X are C-isomorphic to ℓ_p. A similar result holds for c_0.*

Proof. We may assume that the unconditional basis of X is 1-unconditional (so that all canonical projections on subspaces spanned by subsequences are of norm 1), and we give the proof for the case of ℓ_p. The set $A_N = \{(x_n)_{n\in\mathbb{N}} \in bb_d(X) : [x_n]_{n\in\mathbb{N}} \simeq^N \ell_p\}$ is analytic and so has the Baire Property. (This is true of any isomorphism class in $bb_d(X)$; see [9] about this.) We check the hypotheses of Theorem 4.2. Given $\epsilon > 0$, there exists $\Delta > 0$ such that the Δ-perturbation of a block-sequence $(x_n)_{n\in\mathbb{N}}$ in $bb_d(X)$ spans a space which is $1+\epsilon$ isomorphic to $[x_n]_{n\in\mathbb{N}}$, so (a) follows. (b) is true with $K(N, n_0) = Nc(n_0)$. (Here $c(n)$ is a constant such that in any Banach space, any two subspaces of codimension n are $c(n)$-isomorphic; see the observation following the definition of an n_0-modification.) If $[x_n]_{n\in\mathbb{N}}$ is C-isomorphic to ℓ_p, and if $(x_{n_k})_{k\in\mathbb{N}}$ is a subsequence of $(x_n)_{n\in\mathbb{N}}$, then as $[x_{n_k}]$ is 1-complemented in $[x_n]_{n\in\mathbb{N}}$, it is

C-isomorphic to a C-complemented subspace of ℓ_p, so is $K(C)$-isomorphic to ℓ_p, for some constant $K(C)$. (This a consequence of the use of Pełczyński's Decomposition Method to prove the isomorphism of $[x_{n_k}]$ with l_p.) Finally, it is easy to check that if x, y in $bb_d(X)$ are disjointly supported, and $[x]$ and $[y]$ are C-isomorphic to ℓ_p, then the concatenation of x and y will span a subspace which is $k(C)$ isomorphic to ℓ_p, for some constant $k(C)$. $\square$

6. Topological 0-1 law for subspaces with a finite dimensional decomposition

We now turn to subspaces of a space X with a Schauder basis, which have a successive finite dimensional decomposition on the basis. There is a natural discretization of the set of such spaces, introduced in [9], where their classes of isomorphism were studied.

Let $\mathrm{Fin}(X)$ be the set of finite-dimensional subspaces of X of dimension at least 1. We say that F and G in $\mathrm{Fin}(X)$ are *successive*, and write $F < G$, if for any $0 \neq x \in F$, $0 \neq y \in G$, x and y are successive. A space with a *successive finite dimensional decomposition (or successive FDD) in* X is a subspace of X of the form $\oplus_{k\in\mathbb{N}} F_k$, with successive subspaces F_k in $\mathrm{Fin}(X)$. The associated sequence $(F_k)_{k\in\mathbb{N}}$ will be called a *sequence of successive finite dimensional subspaces*. The set of such infinite sequences is denoted $fdd(X)$.

Let $\mathrm{Fin}_{\mathbb{Q}}(X)$ be the set of finite-dimensional subspaces in $\mathrm{Fin}(X)$ which have a basis of vectors of $\mathbb{Q}(X)$. We let $fdd_d(X)$ be the Polish space of infinite sequences of successive finite-dimensional subspaces in $\mathrm{Fin}_{\mathbb{Q}}(X)$, equipped with the product of the discrete topology on $\mathrm{Fin}_{\mathbb{Q}}(X)$. The set of finite sequences of successive finite-dimensional subspaces in $\mathrm{Fin}_{\mathbb{Q}}(X)$ will be denoted by $fdd_d^{<\omega}(X)$. $\tilde{F}$ will denote a finite sequence of successive finite-dimensional spaces, and $(\tilde{F}_n)_{n\in\mathbb{N}}$ an infinite sequence of such finite sequences. The usual notation about concatenation of finite sequences will be used. For $S \in fdd_d(X)$, $[S]$ will denote the linear span of S.

For E, F in $\mathrm{Fin}(X)$, we let $d_H(E, F)$ be the Hausdorff distance between the unit spheres S_E of E and S_F of F (i.e., $d_H(E, F) = \max_{x\in S_E} d(x, S_F) \vee \max_{y\in S_F} d(y, S_E)$).

We define a distance d on $\mathrm{Fin}(X)$ by $d(E, F) = 1$ if $\dim E \neq \dim F$, and $d(E, F) = \min(1, 4k\sqrt{k} d_H(E, F))$ if $\dim E = \dim F = k$. When $\delta < 1$ and $\dim E = \dim F = k$, the inequality $d_H(E, F) \leq \delta/(4k\sqrt{k})$ implies that we can find a map T from E onto F such that for all $x \in E$, $\|Tx - x\| \leq \delta \|x\|$. (Use [17], Prop. 1.a.9, together with the fact that any k-dimensional space has a Schauder basis with constant $\sqrt{k}$.) So we observe that for $\delta < 1$, $d(E, F) \leq \delta$ will imply the existence of $T : E \to F$ with for all $x \in E$, $\|Tx - x\| \leq \delta \|x\|$.

Let $\Delta = (\delta_n)_{n\in\mathbb{N}} > 0$. Let A be a subset of $fdd_d(X)$. The Δ-expansion A_Δ of A is the set of sequences of successive finite dimensional spaces $(F_k)_{k\in\mathbb{N}} \in fdd_d(X)$ such that there exists $(E_k)_{k\in\mathbb{N}}$ in A with $d(E_k, F_k) \leq \delta_k$ for all $k \in \mathbb{N}$.

Lemma 6.1. *Let K be the constant of the basis of X. If $\epsilon < 1$ and $\sum_{n\in\mathbb{N}} \delta_n \leq \epsilon/(8K)$, then any Δ-perturbation of a sequence $(F_k)_{k\in\mathbb{N}} \in fdd(X)$ spans a subspace which is $1+\epsilon$-isomorphic to $\oplus_{k\in\mathbb{N}} F_k$.*

Proof. Denote $(G_k)_{k\in\mathbb{N}} \in fdd(X)$ a Δ-perturbation of $(F_k)_{k\in\mathbb{N}}$. By the observation, we find for each k an onto map $T_k : E_k \to F_k$ such that, for all $x \in E_k$, $\|T_k x - x\| \leq \delta_k \|x\|$.

For $x = \sum_{k\in\mathbb{N}} x_k$, with $x_k \in E_k$, let $Tx = \sum_{k\in\mathbb{N}} T_k x_k$. Then

$$\|x - Tx\| \leq \sum_{k\in\mathbb{N}} \|x_k - T_k x_k\| \leq \max_{k\in\mathbb{N}} \|x_k\| \sum_{k\in\mathbb{N}} \delta_k,$$

so

$$\|x - Tx\| \leq 2K \left(\sum_{k\in\mathbb{N}} \delta_k \right) \|x\| \leq \epsilon/4 \|x\|,$$

and it follows that $\|T\| \left\|T^{-1}\right\| \leq 1+\epsilon$. □

This lemma shows that by choosing a Δ-net for small enough $\Delta > 0$, we shall always be able to capture the properties of sequences in $fdd(X)$ up to some arbitrary constant $\epsilon > 0$.

A sequence $(F_k)_{k\in\mathbb{N}} \in fdd(X)$ *passes through* a finite sequence of successive finite dimensional subspaces $(A_i)_{1\leq i\leq I}$ if there exists $k \in \mathbb{N}$ such that $F_{k+i} = A_i$ for all $1 \leq i \leq I$. If the sequence $(A_i)_i$ is a length 1 sequence (A), we shall just say that $(F_k)_{k\in\mathbb{N}}$ passes through A.

The following theorem was essentially proved in [9].

Theorem 6.2. *Let X be a Banach space with a Schauder basis. If A is comeager in $fdd_d(X)$, then for any $\Delta > 0$, there exists a successive sequence $(\tilde{F}_n)_{n\in\mathbb{N}} \in (fdd_d^{<\omega}(X))^\omega$, such that all elements of $fdd_d(X)$ passing through infinitely many $\tilde{F}_n$'s are in A_Δ.*

Proof. The proof is verbatim the same as that given for the case of block-sequences in [9] (which corresponds to Proposition 2.3 in this article), if we replace blocks in $Q(X)$ by finite-dimensional spaces in $\mathrm{Fin}_{\mathbb{Q}}(X)$, and block-sequences in $bb_d(X)$ by sequences of successive finite-dimensional subspaces in $fdd_d(X)$. □

We shall use this theorem when A is in fact a property of $[x_n]_{n\in\mathbb{N}}$. In this case, each sequence $\tilde{F}_n$ can be chosen to be of length 1, and the formulation becomes a bit more tractable. We have:

Theorem 6.3 (Topological 0-1 law for FDD subspaces). *Let X be a Banach space with an unconditional basis. Let P be a property of Banach spaces which is preserved by isomorphism. Assume that the set of $(F_n)_{n\in\mathbb{N}}$ in $fdd_d(X)$ such that $[F_n]_{n\in\mathbb{N}}$ satisfies P has the Baire Property. Assume that* *for every sequence $(F_n)_{n\in\mathbb{N}}$ in $fdd(X)$, there exists a sequence which passes through infinitely many F_n's and whose closed linear span satisfies P. Then there exists a sequence (F_n) such that any sequence of $fdd(X)$ passing through infinitely many F_n's has a closed linear span satisfying P.*

Proof. Let $A = \{(F_n)_{n\in\mathbb{N}} \in fdd_d(X) : [F_n]_{n\in\mathbb{N}}$ *has* $P\}$. Let $\Delta > 0$ be small enough so that $A_\Delta = A$ and $(A^C)_\Delta = A^C$. In Theorem 6.2 applied with Δ to A or A^C, the sequence $\tilde{F}_n$ may be chosen to be of length 1 for each $n \in \mathbb{N}$. It follows from our hypotheses about P that A cannot be meager. (Otherwise apply Theorem 6.2 with Δ to A^C.) For $\tilde{E} = (E_1, \ldots, E_p) \in fdd_d^{<\omega}(X)$, denote by $N(\tilde{E})$ the set of sequences $(F_n)_{n\in\mathbb{N}} \in fdd_d(X)$ such that $F_n = E_n$ for all $n \leq p$. As A has the Baire Property, it is comeager in some open set $N(\tilde{E})$, and without loss of generality $\tilde{E}$ is a length 1 sequence (E_1). We now prove that A is comeager in $fdd_d(X)$. Then our result follows from Theorem 6.2.

Otherwise, A is meager in some open set $N(\tilde{F})$, $\tilde{F} \in fdd_d^{<\omega}(X)$, and without loss of generality $\tilde{F}$ is a length 1 sequence (F_1). Now we may find E_2 and F_2 in $\mathrm{Fin}_{\mathbb{Q}}(X)$, with $E_1 < E_2$, $\dim E_2 = \dim F_1$, $F_1 < F_2$, $\dim F_2 = \dim E_1$, and $\max(\mathrm{supp}(E_2)) = \max(\mathrm{supp}(F_2))$. Let f be the canonical bijection between the sets $N((E_1, E_2))$ and $N((F_1, F_2))$, defined by $f((E_1, E_2)^\frown S) = (F_1, F_2)^\frown S$ for all $S \in fdd_d(X)$. It is routine to check that f is an homeomorphism, and that for all $S \in N((E_1, E_2))$, $[f(S)]$ is isomorphic to $[S]$; in particular, $S \in A$ if and only if $f(S) \in A$. This gives a contradiction to the fact that A is meager in $N((F_1, F_2))$ and comeager in $N((E_1, E_2))$. □

Corollary 6.4. *Let X be a Banach space with an unconditional basis. Let P be an isomorphic property of Banach spaces. Assume that the set $\{(F_n)_{n\in\mathbb{N}} \in fdd(X) : [F_n]_{n\in\mathbb{N}}$ has $P\}$ has the Baire Property, and that P is stable by passing to complemented subspaces and by squaring. If every sequence in $fdd(X)$ has a subsequence whose closed linear span satisfies P, then all subspaces with a successive FDD in X satisfy P.*

Proof. Let $A = \{(F_n)_{n\in\mathbb{N}} \in fdd(X) : [F_n]_{n\in\mathbb{N}}$ *has* $P\}$. By the previous theorem, there exists a sequence (F_n) such that any sequence of $fdd(X)$ passing through infinitely many F_n's has a closed linear span satisfying P. By the properties of P, and because the basis of X is assumed to be unconditional, A is stable by taking subsequences and by concatenation of disjoint sequences. Using the same method as at the end of the proof of Theorem 4.2, we obtain that $A = fdd_d(X)$. □

One important open question in Banach space theory is whether any complemented subspace of a Banach space with an unconditional basis must have an unconditional basis (or even an unconditional finite-dimensional decomposition, also noted UFDD). Note that it is known that a complemented subspace of a space with a UFDD does not necessarily have a Schauder basis (this is due to Szarek), or even a FDD (due to Read). We refer to [3] for more on this.

The following corollary gives a possible approach towards answering the first question in the negative. (Here we use that "spanning a subspace with an unconditional basis" is analytic and thus has the Baire Property in $fdd_d(X)$.)

Corollary 6.5. *Let X be a Banach space with an unconditional basis. Assume:*

(1) *Every sequence in $fdd(X)$ has a subsequence which spans a subspace with an unconditional basis.*
(2) *There exists a sequence in $fdd(X)$ which spans a subspace without an unconditional basis.*

Then there exists a subspace $F = \oplus_{n\in\mathbb{N}} F_n$ of X with a successive FDD on the basis, which has an unconditional basis, and a subsequence $(G_k)_{k\in\mathbb{N}}$ of $(F_n)_{n\in\mathbb{N}}$, such that $G = \oplus_{k\in\mathbb{N}} G_k$, though complemented in F, does not have an unconditional basis.

Note that, by construction, a counterexample produced by Corollary 6.5 will be equipped with a UFDD. We conclude this paper by discussing some of the properties that a Banach space X with (1) and (2) must have, supposing it exists.

Recall that a Banach space X is said to have Gordon-Lewis *l.u.st.* if there is a constant $C \geq 1$ such that for every finite dimensional subspace E of X, there exists a finite dimensional space F with a 1-unconditional basis, and maps $T : E \to F$, $U : F \to X$, such that $UT(x) = x$ for all $x \in E$ and such that $\|T\|\,\|U\| \leq C$. We note that having l.u.st. is an analytic property of Banach spaces, which is stable by passing to complemented subspaces and squaring. As (1) implies that every sequence in $fdd_d(X)$ has a subsequence which spans a subspace with l.u.st., it follows from Theorem 6.4 that if X satisfies (1) and (2), then every subspace of X with a successive FDD must have l.u.st..

By the theorem of Komorowski and Tomczak-Jaegermann [15] mentioned in the introduction, it follows that X must be ℓ_2-saturated. Also by [4], Theorem 3.8, every subspace of X with a uniform FDD on the basis must have an unconditional basis.

Another interesting fact is that the unconditional basis for the space $(\oplus F_n)_{n\in\mathbb{N}}$ in the conclusion of Corollary 6.5 cannot be obtained in the obvious way, that is, by constructing in each F_n a C-unconditional basis, and proving that the sequence which is the reunion of each basis is a $K(C)$-unconditional basis for $(\oplus F_n)_{n\in\mathbb{N}}$, for some constant $K(C)$. In this case, any subspace $(\oplus G_k)_{k\in\mathbb{N}}$ associated to a subsequence $(G_k)_{k\in\mathbb{N}}$ of $(F_n)_{n\in\mathbb{N}}$ would inherit an unconditional basis (which is just a subsequence of the unconditional basis of $(\oplus F_n)_{n\in\mathbb{N}}$).

A natural candidate for X is the Orlicz sequence space l_F considered by P. Casazza and N.J. Kalton in [4]. It is reflexive, has cotype 2 and type $2-\epsilon$ for any $\epsilon > 0$, and is ℓ_2-saturated. Among other interesting properties, every subspace of l_F with a uniform UFDD has an unconditional basis. We do not know whether l_F satisfies the hypotheses of Corollary 6.5.

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References

[1] J. Bourgain, P. G. Casazza, J. Lindenstrauss, and L. Tzafriri, *Banach spaces with a unique unconditional basis, up to permutation*, Mem. Amer. Math. Soc. **54** (1985). MR 782647 (86i:46014)

[2] P. G. Casazza, *The Schroeder-Bernstein property for Banach spaces*, Banach space theory (Iowa City, IA, 1987), Contemp. Math., vol. 85, Amer. Math. Soc., Providence, RI, 1989, pp. 61–77. MR 983381 (90d:46019)

[3] ———, *Approximation properties*, Handbook of the geometry of Banach spaces, Vol. I, North-Holland, Amsterdam, 2001, pp. 271–316. MR 1863695 (2003f:46012)

[4] P. G. Casazza and N. J. Kalton, *Unconditional bases and unconditional finite-dimensional decompositions in Banach spaces*, Israel J. Math. **95** (1996), 349–373. MR 1418300 (97k:46010)

[5] ———, *Uniqueness of unconditional bases in Banach spaces*, Israel J. Math. **103** (1998), 141–175. MR 1613564 (99d:46007)

[6] P. G. Casazza and T. J. Shura, *Tsirelson's space*, Lecture Notes in Mathematics, vol. 1363, Springer-Verlag, Berlin, 1989. MR 981801 (90b:46030)

[7] V. Ferenczi and E. M. Galego, *Some equivalence relations which are Borel reducible to isomorphism between Banach spaces*, Israel J. Math. **152** (2006), 61–82.

[8] V. Ferenczi and C. Rosendal, *On the number of non-isomorphic subspaces of a Banach space*, Studia Math. **168** (2005), 203–216. MR 2146123

[9] ———, *Ergodic Banach spaces*, Adv. Math. **195** (2005), 259–282. MR 2145797 (2006b:46007)

[10] W. T. Gowers, *An infinite Ramsey theorem and some Banach-space dichotomies*, Ann. of Math. (2) **156** (2002), 797–833. MR 1954235 (2005a:46032)

[11] W. T. Gowers and B. Maurey, *The unconditional basic sequence problem*, J. Amer. Math. Soc. **6** (1993), 851–874. MR 1201238 (94k:46021)

[12] N. J. Kalton, *Lattice structures on Banach spaces*, Mem. Amer. Math. Soc. **103** (1993). MR 1145663 (93j:46024)

[13] A. S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995. MR 1321597 (96e:03057)

[14] D. Kutzarova and P.-K. Lin, *Remarks about Schlumprecht space*, Proc. Amer. Math. Soc. **128** (2000), 2059–2068. MR 1654081 (2000m:46031)

[15] R. A. Komorowski and N. Tomczak-Jaegermann, *Banach spaces without local unconditional structure*, Israel J. Math. **89** (1995), 205–226. MR 1324462 (96g:46007)

[16] H. Lemberg, *Nouvelle démonstration d'un théorème de J.-L. Krivine sur la finie représentation de l_p dans un espace de Banach*, Israel J. Math. **39** (1981), 341–348. MR 636901 (83b:46015)

[17] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces*, Springer-Verlag, Berlin, 1973. MR 0415253 (54 #3344)

[18] C. Rosendal, *Incomparable, non-isomorphic and minimal Banach spaces*, Fund. Math. **183** (2004), 253–274. MR 2128711 (2006b:46009)

[19] T. Schlumprecht, *An arbitrarily distortable Banach space*, Israel J. Math. **76** (1991), 81–95. MR 1177333 (93h:46023)

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