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DOUBLE DECKER SETS OF GENERIC SURFACES IN 3-SPACE AS HOMOLOGY CLASSES

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ABSTRACT. The double decker set Γ of a generic map $g : F_0^2 \to M^3$ is the preimage of the singularity of the generic surface $g(F_0)$. If both F_0 and M are oriented, then Γ is regarded as an oriented 1-cycle in F_0, which is shown to be null-homologous if $g(F_0) = 0 \in H_2(M; \mathbf{Z})$. We also investigate a double decker set of a surface diagram which is a generic surface in $\mathbf{R}^3$ with crossing information.

1. Introduction

For a connected closed surface F_0 and a 3-manifold M, a map $g : F_0 \to M$ is *generic* if the singularity set of the image $g(F_0)$ consists of double points and isolated triple/branch points. Such a set is called the *double point set* of g and denoted by Γ^*. The preimage $\Gamma = g^{-1}(\Gamma^*)$ in F_0 is called the *double decker set* of g (cf. [5]).

Every double decker set Γ is regarded as a union of immersed circles in F_0, which we call *decker curves*. If both of F_0 and M are oriented, then each decker curve is also oriented naturally, and hence Γ determines an oriented 1-cycle in F_0. Figure 1 shows an example of a generic torus $g(F_0)$ in $M = \mathbf{R}^3$ with the double decker set Γ consisting of a union of three decker curves on the torus F_0 (cf. [3]). We observe that

$$\Gamma = (1,0) + (0,1) + (-1,-1) \in H_1(F_0; \mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z}$$

for a suitable basis of $H_1(F_0; \mathbf{Z})$; hence Γ is null-homologous in F_0. Generalizing this example, we obtain the following theorem.

THEOREM 1. *Let $g : F_0 \to M$ be a generic map with F_0 and M oriented, and let Γ be the double decker set of g. If $g(F_0) = 0 \in H_2(M; \mathbf{Z})$, then $\Gamma = 0 \in H_1(F_0; \mathbf{Z})$.*

Therefore, if M is an oriented 3-manifold with trivial second homology, then every double decker set $\Gamma \subset F_0$ of a generic surface $g(F_0) \subset M$ is always

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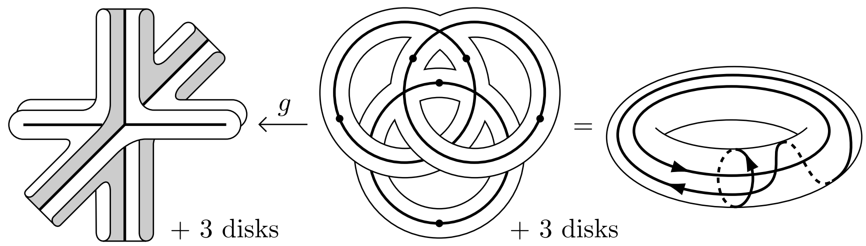


Figure 1

null-homologous in F_0. We notice that there exists a generic torus $g(F_0)$ in $M = S^1 \times S^2$ such that $g(F_0) \neq 0 \in H_2(M; \mathbf{Z})$ and $\Gamma \neq 0 \in H_1(F_0; \mathbf{Z})$.

On the other hand, if F_0 is non-orientable, the double decker set Γ defines an unoriented 1-cycle in F_0. In this case, we have the following theorem.

Theorem 2. *Let $g : F_0 \to M$ be a generic map with F_0 non-orientable, and let Γ be the double decker set of g. If $g(F_0) = 0 \in H_2(M; \mathbf{Z}_2)$, then $\Gamma \neq 0 \in H_1(F_0; \mathbf{Z}_2)$.*

Generic surfaces in $\mathbf{R}^3$ play an important role in 2-knot theory, which is to study embedded surfaces in $\mathbf{R}^4$ locally flatly (up to ambient isotopies of $\mathbf{R}^4$). To illustrate such an embedded surface $F \subset \mathbf{R}^4$, we often use a projection image $\pi(F)$ under a standard projection $\pi : \mathbf{R}^4 \to \mathbf{R}^3$ and we may assume that $\pi(F)$ is a generic surface in $\mathbf{R}^3$. The *surface diagram* of F, denoted by $D(F)$, is such a generic surface $\pi(F)$ with crossing information (according to the projection direction of π) along the double point set of $\pi(F)$. In particular, a surface diagram is *regular* if it does not contain branch points.

We have two equivalence relations for (regular) surface diagrams as follows:

(1) Two surface diagrams $D(F)$ and $D(F')$ are *equivalent* if there exists a finite sequence of surface diagrams $D(F) = D_1 \to D_2 \to \cdots \to D_n = D(F')$ such that each $D_i \to D_{i+1}$ is one of the seven local deformations shown in Figures 2(a) and 2(b).

(2) Two regular surface diagrams $D(F)$ and $D(F')$ are *regular-equivalent* if there exists a finite sequence of surface diagrams $D(F) = D_1 \to D_2 \to \cdots \to D_n = D(F')$ such that each D_i is regular and $D_i \to D_{i+1}$ is one of the four local deformations shown in Figure 2(a).

In Figure 2, we omit the crossing information of surface diagrams. We call the local moves in the figure *Roseman moves* [8]. It is known that $D(F)$ and $D(F')$ is equivalent if and only if F and F' are ambient isotopic in $\mathbf{R}^4$.

The double decker set Γ of a surface diagram $D(F)$ is that of the generic projection $\pi(F)$. If F is oriented and $D(F)$ is regular, then Γ is oriented again by using the crossing information of $D(F)$. We notice that this orientation of

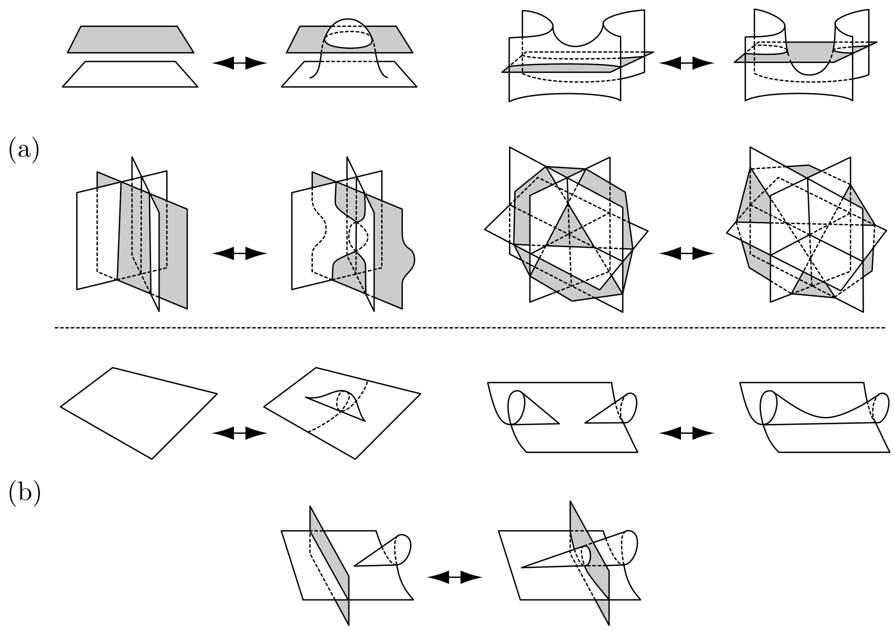


FIGURE 2

Γ is different from that defined only from $\pi(F)$. By regarding Γ as a 1-cycle in this sense, we have the following result:

THEOREM 3. *There exist two regular surface diagrams which are equivalent but not regular-equivalent.*

This paper is organized as follows. In Section 2, we review the notion of generic surfaces and introduce an orientation of double decker sets. In Section 3, we study a relationship between homology classes of double decker sets and Alexander numberings. In Section 4, we introduce another orientation of a double decker set of a surface diagram induced from the crossing information.

2. Double decker sets as homology classes

We first review the notion of generic surfaces in a 3-manifold; we refer to [5] for more details. Let F_0 denote a connected closed surface and M a 3-manifold. We say that a map $g : F_0 \to M$ is *generic* if for each point $x \in g(F_0)$ there is a 3-ball neighborhood $N(x)$ of x in M such that the pair $(N(x), g(F_0) \cap N(x))$ is homeomorphic to one of Figure 3(a)–(d). Such a surface $g(F_0)$ is a *generic surface* in M and denoted by F_0^*. In the cases (b), (c) and (d), the point $x \in F_0^*$ is called a *double point*, a *triple point* and a *branch point*, respectively.

Then the set $\mathrm{cl}\{x \in M \mid \#g^{-1}(x) > 1\}$ consists of (possibly empty) double points and isolated triple/branch points. This singular set is called the *double point set* of g and denoted by Γ^*. The preimage $g^{-1}(\Gamma^*) \subset F_0$ of the double point set Γ^* is called a *double decker set* of g and denoted by Γ.

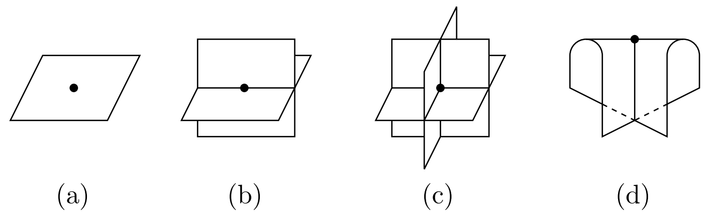


FIGURE 3

A double point set Γ^* is regarded as a union of immersed curves in M, which are called *double curves*. Each double curve is homeomorphic to a circle or an arc whose endpoints are branch points. We see that the preimage of a double curve consists of (one or two) immersed circles in F_0. Thus, the double decker set Γ is regarded as a union of immersed circles in F_0. Such an immersed circle in Γ is called a *decker curve*. Figure 4 shows an example of a generic projective plane in $\mathbf{R}^3$ which is called the *Boy's surface* [1]. In this example, the double point set consists of one double curve and the double decker set consists of one decker curve.

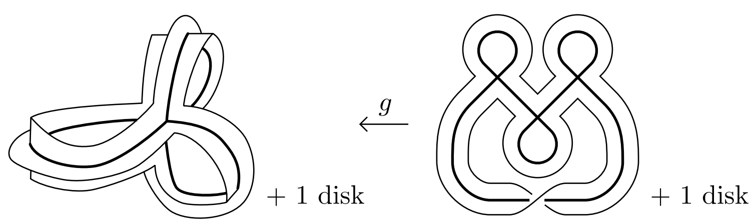


FIGURE 4

Assume that F_0 and M are oriented. We give the generic surface F_0^* the orientation which comes from that of F_0. Then each decker curve is oriented as follows. Let H^* and H'^* be two sheets in F_0^* which intersect along a double curve C^*, and let $C \subset H$ and $C' \subset H'$ be the decker curves which are the preimages of C^*; see Figure 5(a). Let $\vec{n}$ and $\vec{n}'$ be the orientation normals to H^* and H'^*, respectively. Then the orientation $\vec{v}$ of $C \subset H$ is determined by the condition that the ordered triple $(\vec{n}, \vec{n}', g(\vec{v}))$ matches the orientation of M. The orientation $\vec{v}'$ of C' is also defined similarly; hence $g(\vec{v}) = -g(\vec{v}')$. We see that the orientations of decker curves near a preimage of a branch point are coincident; see Figure 5(b). Hence, the double decker set Γ is regarded as a union of *oriented* immersed circles in F_0 and determines a homology class in $H_1(F_0; \mathbf{Z})$; see again Figure 1.

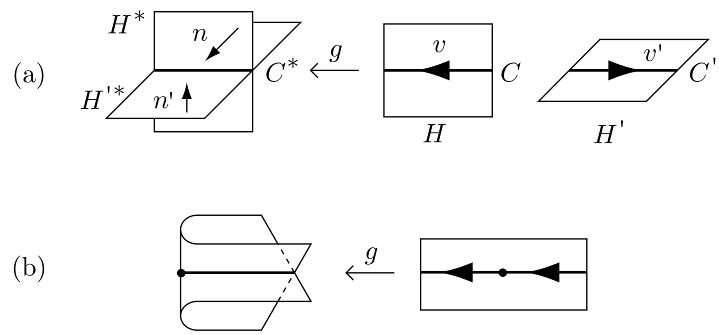


FIGURE 5

THEOREM 1. *Let $g : F_0 \to M$ be a generic map with F_0 and M oriented and Γ be the double decker set of g. If $g(F_0) = 0 \in H_2(M; \mathbf{Z})$, then $\Gamma = 0 \in H_1(F_0; \mathbf{Z})$.*

Proof. Since the $\mathbf{Z}$-intersection form $\mathrm{Int}_{F_0} : H_1(F_0; \mathbf{Z}) \times H_1(F_0; \mathbf{Z}) \to \mathbf{Z}$ is non-singular, it is sufficient to prove that $\mathrm{Int}_{F_0}(\ell, \Gamma) = 0$ for any oriented simple closed curve ℓ in F_0. We may assume that $\ell^* = g(\ell)$ is embedded in $F_0^* = g(F_0)$ and misses the triple/branch points of F_0^*. We take a loop $\bar{\ell}^*$ embedded in M which goes parallel to ℓ^* and intersects F_0^* transversely. By using the orientation of F_0^*, we may assume that each point of $\bar{\ell}^* \cap F_0^*$ appears near $g(\ell \cap \Gamma)$ only; see Figure 6. Then we see that $\mathrm{Int}_{F_0}(\ell, \Gamma) = \mathrm{Int}_M(\bar{\ell}^*, F_0^*)$, where $\mathrm{Int}_M : H_1(M; \mathbf{Z}) \times H_2(M; \mathbf{Z}) \to \mathbf{Z}$ denotes the intersection form in M. Since $F_0^* = 0 \in H_2(M; \mathbf{Z})$, we have $\mathrm{Int}_{F_0}(\ell, \Gamma) = \mathrm{Int}_M(\bar{\ell}^*, F_0^*) = 0$. □

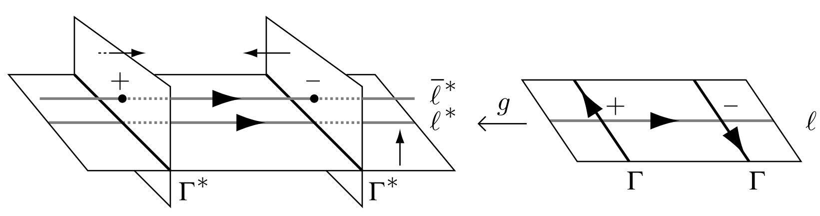


FIGURE 6

We remark that Theorem 1 is trivial in the case when F_0 is a sphere.

In the case when $F_0^* \neq 0 \in H_2(M; \mathbf{Z})$, not every double decker set is null-homologous in F_0. To see this, we take a 2-sphere $S = \{*\} \times S^2$ and a torus $T = S^1 \times C$ in $M = S^1 \times S^2$, where C is a circle embedded in S^2. We notice that S and T intersect along the circle $\{*\} \times C$. By attaching a 1-handle between S and T without producing new singularities, we obtain a generic torus F_0^* in $S^1 \times S^2$. Then it is easy to see that F_0^* presents a generator of $H_2(S^1 \times S^2; \mathbf{Z}) \cong \mathbf{Z}$ and the double decker set Γ is not null-homologous in the torus F_0.

Assume that F_0 is non-orientable. Then the double decker set Γ of a generic map $g : F_0 \to M$ defines a homology class in $H_1(F_0; \mathbf{Z}_2)$. For example, the double decker set of Boy's surface is the generator of $H_2(F_0; \mathbf{Z}_2) \cong \mathbf{Z}_2$ (recall that F_0 is homeomorphic to a projective plane).

Theorem 2. *Let $g : F_0 \to M$ be a generic map with F_0 non-orientable and Γ be the double decker set of g. If $g(F_0) = 0 \in H_2(M; \mathbf{Z}_2)$, then $\Gamma \neq 0 \in H_1(F_0; \mathbf{Z}_2)$.*

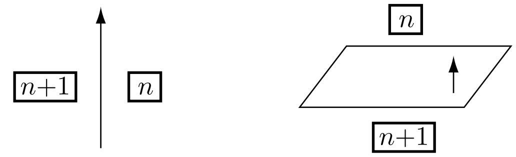
Proof. It is sufficient to prove that there exists a closed curve ℓ in F_0 such that the $\mathbf{Z}_2$-intersection number $\mathrm{Int}_{F_0}(\ell, \Gamma)$ is equal to $1 \in \mathbf{Z}_2$. We take ℓ such that its regular neighborhood in F_0 is homeomorphic to an immersed Möbius band. Then we can show, as in the proof of Theorem 1, that the number of the crossings $\ell \cap \Gamma$ is odd, and hence we have $\mathrm{Int}_{F_0}(\ell, \Gamma) = 1$. □

Remark. The proof of Theorem 2 shows that $\mathrm{Int}_{F_0}(\,\cdot\,, \Gamma) : H_1(F_0; \mathbf{Z}_2) \to \mathbf{Z}_2$ corresponds to the first Stiefel-Whitney class $w_1 \in H^1(F_0; \mathbf{Z}_2)$, since $\mathrm{Int}_{F_0}(\ell, \Gamma) = 1$ (resp. 0) if and only if the regular neighborhood of ℓ in F_0 is homeomorphic to an immersed Möbius band (resp. annulus).

3. Alexander numberings

In this section, we give an alternative proof of Theorem 1 (and Theorem 2) by using Alexander numberings. We first recall an Alexander numbering in the case of a union of oriented curves γ immersed in a connected, oriented closed surface F_0. The curve γ divides F_0 into regions. Let $\mathcal{R}$ be the set of the closures of all the regions $F_0 - \gamma$. Then an *Alexander numbering* is a map $c : \mathcal{R} \to \mathbf{Z}$ such that $c(R_{\mathrm{right}}) + 1 = c(R_{\mathrm{left}})$ for any adjacent two regions, where R_{right} (resp. R_{left}) denotes the region to the right (resp. left) of the bounded oriented curve; see Figure 7(a). Of course, not every $\gamma \subset F_0$ admits an Alexander numbering; more precisely, an Alexander numbering exists if and only if γ is null-homologous in F_0 (cf. [2]). Roughly speaking, γ is presented by the boundary $\partial(\sum_{k=1}^{n} c_k R_k)$, where $c_k \in \mathbf{Z}$ and $\mathcal{R} = \{R_k\}_{k=1,\dots,n}$, if and only if $\{c_k\}$ gives an Alexander numbering for $\mathcal{R}$.

An analogous definition of Alexander numberings can be given for a generic surface in a 3-manifold (cf. [6]). Let $g : F_0 \to M$ be a generic map with F_0 and M oriented. Then the generic surface $F_0^* = g(F_0)$ divides M into some regions.



(a) (b)

FIGURE 7

We denote by $\mathcal{S}$ the set of the closures of these regions. Then an Alexander numbering for $\mathcal{S}$ is a map $d : \mathcal{S} \to \mathbf{Z}$ such that $d(S_{\text{over}})+1 = d(S_{\text{under}})$ for any adjacent two regions, where S_{over} (resp. S_{under}) denotes the region over (resp. under) the bounded sheet with respect to the orientation of the sheet; see Figure 7(b). We can show similarly that an Alexander numbering for $M - F_0^*$ exists if and only if F_0^* is null-homologous in M.

By using Alexander numberings, we have the following alternative proof of Theorem 1.

Proof of Theorem 1. We use the above notations. It is sufficient to show that the double decker set $\Gamma \subset F_0$ admits an Alexander numbering $c : \mathcal{R} \to \mathbf{Z}$. Since $F_0^* = 0 \in H_2(M; \mathbf{Z})$, the generic surface $F_0^* \subset M$ admits an Alexander numbering $d : \mathcal{S} \to \mathbf{Z}$. For each $R \in \mathcal{R}$, the divided region $S \in \mathcal{S}$ is uniquely determined such that S is the region over $g(R)$. Then we define the map $c : \mathcal{R} \to \mathbf{Z}$ such that $c(R) = d(S)$. It is easy to see that c gives an Alexander numbering for $F_0 - \Gamma$. □

In the case when F_0 is non-orientable, we can give an alternative proof of Theorem 2 by using a *checkerboard coloring* which is a map of the set of regions $\mathcal{R}$ to $\mathbf{Z}_2$ such that any pair of adjacent regions are assigned distinct elements. This proof is left to the reader.

REMARK. We can generalize Theorems 1 and 2 to the case when F_0 is disconnected.

4. Regular surface diagrams

Let $f : F_0 \to \mathbf{R}^4$ be a locally-flat embedding of a connected closed surface F_0 into $\mathbf{R}^4$ and $\pi : \mathbf{R}^4 \to \mathbf{R}^3$ the projection defined by $(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3)$. By a slight perturbation of f if necessary, we may assume that $\pi \circ f : F_0 \to \mathbf{R}^3$ is a generic map. The *surface diagram* of an embedded surface $F = f(F_0)$ in $\mathbf{R}^4$ is a generic projection $\pi(F)$ with the crossing information. Here, the *crossing information* describes which of the two sheets along a double curve is higher than the other with respect to the x_4-coordinate. To indicate this information, we remove the neighborhood of the double curve in the sheet (under-sheet) which lies lower than the other sheet (over-sheet); we refer to [5] for more details. We denote by $D(F)$ the surface diagram obtained from an embedded surface F in $\mathbf{R}^4$. In particular, we say that a surface diagram is *regular* if it does not contain branch points.

In this section we always assume that F_0 is oriented. Then each double curve of a surface diagram D is oriented as follows. Let $\vec{n}_O$, $\vec{n}_U$ be the normals to the over- and under-sheet, respectively. Then the orientation $\vec{v}$ of the double curve is determined by the condition that the ordered triple $(\vec{n}_O, \vec{n}_U, \vec{v})$ matches the right-handed orientation of $\mathbf{R}^3$. We define the orientation of each decker curve which inherits that of the associated double curve. This orientation of the decker curves is not coincident with that used in Sections 2 and 3. In particular, the orientations of decker curves near a preimage of a branch point of D are non-coherent. This is not the convention used in [5]. Hence, if D is a regular surface diagram, then the double decker set Γ is regarded as a union of oriented circles immersed in F_0, that is, we have $\Gamma \in H_1(F_0; \mathbf{Z})$. In this section, we use this orientation for every double decker set.

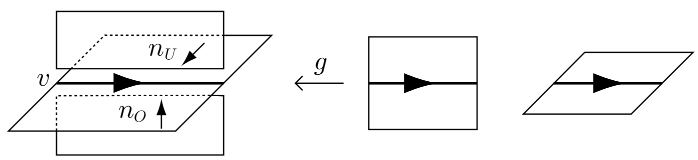


Figure 8

Let $F, F' \subset \mathbf{R}^4$ be two embeddings of F_0, and $D(F)$ and $D(F')$ the corresponding surface diagrams. We say that $D(F)$ and $D(F')$ are *equivalent* if F and F' are ambient isotopic in $\mathbf{R}^4$ with preserving their orientations. There is a set of moves, called *Roseman moves*, that are similar to the Reidemeister moves in classical knot theory. These are the finite set of local moves (which are depicted in Figure 2 without the crossing information) that are sufficient to connect two equivalent diagrams via a finite sequence.

Theorem 4 ([8]). *Two surface diagrams D and D' are equivalent if and only if there exists a finite sequence of surface diagrams $D = D_1 \to D_2 \to \cdots \to D_n = D'$ such that each deformation $D_k \to D_{k+1}$ is one of Roseman moves.* □

It is known that any surface diagram is equivalent to a regular surface diagram in the case when F_0 is orientable (cf. [4]). We introduce an equivalence relation among regular surface diagrams as follows: two regular surface diagrams D and D' are *regular-equivalent* if there exists a finite sequence of surface diagrams $D = D_1 \to D_2 \to \cdots \to D_n = D'$ such that each D_k is regular and each $D_k \to D_{k+1}$ is one of the four Roseman moves shown in Figure 2(a).

Theorem 3. *There exist two regular surface diagrams which are equivalent but not regular-equivalent.*

Proof. Let D and D' be two regular surface diagrams and Γ and Γ' their double decker sets in F_0, respectively. By checking each of the four Roseman moves in Figure 2(a), we see that if D and D' are regular-equivalent then Γ and Γ' are homologous in F_0.

We consider two regular surface diagrams $D(F)$ and $D(F')$ as follows: $D(F)$ is shown in Figure 9 and $D(F')$ is an *embedded* torus in $\mathbf{R}^3$. We see that $D(F)$ and $D(F')$ are equivalent diagrams, for the embeddings $F \subset \mathbf{R}^4$ and $F' \subset \mathbf{R}^4$ are both unknotted. On the other hand, the double decker set Γ of $D(F)$ is a union of two parallel longitudes with same orientation and Γ' of $D(F')$ is empty. Since Γ is not null-homologous in the original torus F_0, $D(F)$ and $D(F')$ are not regular-equivalent. $\square$

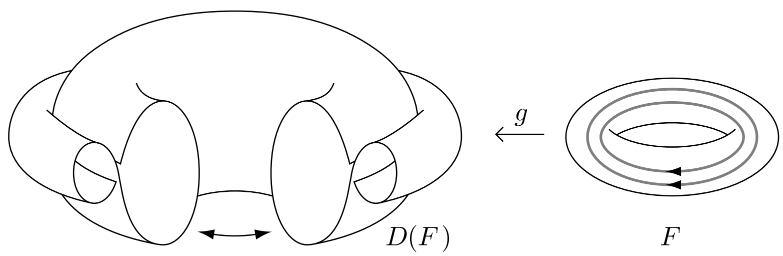


Figure 9

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