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SCREW MOTION SURFACES IN $\mathbb{H}^2 \times \mathbb{R}$ AND $\mathbb{S}^2 \times \mathbb{R}$

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Abstract. In this paper we study the geometry of constant mean curvature H–screw motion surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$. We compute the Abresch-Rosenberg holomorphic quadratic differential Q. For instance, if the ambient space is $\mathbb{H}^2 \times \mathbb{R}$, we derive that, given ℓ, if $0 < 4H^2 < 1$, then there exists a complete H–screw motion surface with pitch ℓ immersed in $\mathbb{H}^2 \times \mathbb{R}$, such that $Q \neq 0$. An analogous result holds if the ambient space is $\mathbb{S}^2 \times \mathbb{R}$. We deduce a general *non-parametric* formula for the mean curvature $H(\rho)$. When H is constant, we find a first integral and use it to get an explicit two parameter family of complete, embedded, simply connected, minimal screw motion surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with pitch ℓ. If $\ell = 1$, each such surface has Gaussian curvature $K \equiv -1$. We deduce that any two isometric screw motion minimal immersions in $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{S}^2 \times \mathbb{R}$ are associate, i.e., the absolute values of their Hopf functions are the same.

1. Introduction

In this paper we study minimal and constant mean curvature surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$. The study of such surfaces has been initiated and influenced by Harold Rosenberg [11]. Recently, several mathematicians worked on the subject as we will describe in the sequel. See, for instance, the articles of Abresch-Rosenberg [1], Hauswirth [5], Meeks-Rosenberg [7], Nelli-Rosenberg [9] and Daniel [2]. For H–surfaces immersed in $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{S}^2 \times \mathbb{R}$, Uwe Abresch and Harold Rosenberg discovered a holomorphic quadratic differential Q, generalizing Hopf's holomorphic quadratic differential. The definition of Q is the following:

Let us assume now that the ambient space is $\mathbb{H}^2 \times \mathbb{R}$. Let $\boldsymbol{\nabla}$ be the Riemannian connection in $\mathbb{H}^2 \times \mathbb{R}$. If X, Y are tangent vectors and if N is the unit normal, we define

$$q(X, Y) := 2H\langle \boldsymbol{\nabla}_X Y, N\rangle + dt(X)dt(Y),$$

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where t is the height. Then

$$Q(X,Y) := \frac{1}{2} \cdot \Big[q(X,Y) - q(JX,JY)\Big] - \frac{1}{2} \cdot i\Big[q(JX,Y) + q(X,JY)\Big],$$

where J is the induced almost complex structure.

When the ambient space is $\mathbb{S}^2 \times \mathbb{R}$, the definition of q is slightly different, namely

$$q(X,Y) := 2H\langle \boldsymbol{\nabla}_X Y, N\rangle - dt(X)dt(Y).$$

We will investigate the geometry of surfaces that are invariant by screw motions in $\mathbb{H}^2 \times \mathbb{R} := \{(x,y,t), x^2+y^2 < 1, t \in \mathbb{R}\}$. We will consider a graph $\lambda = \lambda(\rho)$ lying in the vertical plane xt, where ρ is the hyperbolic distance from the origin along the x axis. We will consider a surface obtained by applying successive screw motions on $\lambda(\rho)$ with pitch ℓ, around the vertical t–axis, called standard screw motion surface, or simply screw motion surface. An analogous definition holds in $\mathbb{S}^2 \times \mathbb{R}$. For H–screw motion surfaces in $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{S}^2 \times \mathbb{R}$, we will find an explicit non-parametric integral formula for $\lambda(\rho)$, and we will develop techniques to compute explicitly Q and give a geometric description of some special cases. For instance, if $4H^2 > 1$, we will show that there exists a complete H–screw motion surface immersed in $\mathbb{H}^2 \times \mathbb{R}$, with pitch ℓ, such that $Q = 0$ if and only if $\ell = 0$. The generating curve of this immersion is periodic with period $2\mathcal{P}$. Moreover, this immersion is proper if and only if $2\mathcal{P}$ and $2\pi\ell$ are linearly dependent over the field of the rational numbers. We will also show that, given ℓ, if $0 < 4H^2 < 1$, then there exists a complete H–screw motion surface with pitch ℓ immersed in $\mathbb{H}^2 \times \mathbb{R}$, such that $Q \neq 0$. An analogous result holds if the ambient space is $\mathbb{S}^2 \times \mathbb{R}$.

We now summarize our results. If the ambient space is $\mathbb{H}^2 \times \mathbb{R}$, we will obtain embedded minimal surfaces, and we will determine the isometric screw motion minimal surfaces. In fact, we will find an explicit formula for the metric of a screw motion H–surface. Then we will deduce that two isometric screw motion minimal immersions in $\mathbb{H}^2 \times \mathbb{R}$ are associate, i.e., the absolute values of their Hopf functions are the same. We remark that Benoît Daniel gave an equivalent definition of associate and conjugate surfaces and proved some related results [2]. For H–screw motion surfaces, we will give an explicit formula for the Abresch-Rosenberg holomorphic quadratic differential Q.

Minimal surfaces of revolution were studied by Barbara Nelli and Harold Rosenberg [9]. Nelli and Rosenberg established the formula for the helicoids. This was already obtained in [11]: The helicoid is a minimal surface that is invariant by screw motions generated by a straight line. We will prove that the catenoid is conjugate to a helicoid of pitch $\ell < 1$, i.e., their Hopf functions have opposite signs. We will prove that an helicoid is conjugate to a catenoid if and only if $\ell < 1$.

If the ambient space is $\mathbb{S}^2 \times \mathbb{R}$, we will deduce similar results. Any two isometric screw motion minimal immersions in $\mathbb{S}^2 \times \mathbb{R}$ are associate. William

Meeks, III and Harold Rosenberg [7] constructed a two-parameter family of proper minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$ foliated by circles on each $\mathbb{S}^2 \times \{t\}$. Again on account of an explicit non-parametric integral formula for $\lambda(\rho)$, we will construct complete minimal and H–screw motion surfaces immersed in $\mathbb{S}^2 \times \mathbb{R}$. For instance, we will find complete minimal and H–screw motion surfaces immersed in $\mathbb{S}^2 \times \mathbb{R}$ whose generating curve is periodic with period $2\mathcal{P}$. If $2\mathcal{P}$ and $2\pi\ell$ are linearly dependent over $\mathbb{Q}$, then the immersion is proper.

We remark that H–surfaces of revolution in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$ were studied in the Abresch-Rosenberg paper [1], where the authors proved some general theorems when $Q \equiv 0$. For H–screw motion surfaces in $\mathbb{S}^2 \times \mathbb{R}$, we will compute an explicit formula for the Abresch-Rosenberg holomorphic quadratic differential Q.

We now give a more detailed outline of this paper. We will first prove a general *non-parametric* formula for the mean curvature $H(\rho)$. When H is constant, we will find a first integral and use it to get an explicit two parameter family of complete, embedded, simply connected, minimal screw motion surfaces. More precisely, we will obtain for $\ell > 1/\sqrt{2}$ a complete, embedded, simply connected, minimal screw motion surface in $\mathbb{H}^2 \times \mathbb{R}$ with pitch ℓ that is conformally equivalent to the hyperbolic plane $\mathbb{H}^2$. If $\ell = 1$, we will see that each of these surfaces has Gaussian curvature $K \equiv -1$.

We will then establish a Bour Lemma for screw motion surfaces in the product $\mathbb{H}^2 \times \mathbb{R}$ or in $\mathbb{S}^2 \times \mathbb{R}$, and use it to determine all isometric immersions with the same mean curvature H. To this end, we will follow ideas of Bour [3], Manfredo Do Carmo and Marcos Dajczer [4] in Euclidean space and ideas of Javier Ordóñes from his Doctoral Thesis at PUC-Rio [10] in space form. The geometric construction underlying this result can be sketched as follows. We will find natural coordinates s, τ such that s is the arc length of a geodesic curve in the surface orthogonal to the orbits (helices $s = cte$), and hence the metric is given by $d\mu^2 = ds^2 + U^2(s)d\tau^2$. By an obvious change of coordinates we get conformal coordinates $w = v + i\tau$. Incidentally, we will obtain a family of complete isometric surfaces in $\mathbb{H}^2 \times \mathbb{R}$, with constant (*intrinsic*) Gaussian curvature $K = -a^2, 0 < a^2 \leqslant 1$.

For H–screw motion surfaces in $\mathbb{H}^2 \times \mathbb{R}$, computations show that the Abresch-Rosenberg holomorphic quadratic differential Q is given by

$$\frac{1}{2}\left(Q(X_v, X_v) - Q(X_\tau, X_\tau)\right) - iQ(X_v, X_\tau) = \frac{1}{m^2}\left(d^2 - 4H^2 + \ell^2(4H^2 - 1)\right) - i\frac{2\ell}{m^2}d,$$

where d is a parameter given by the first integral formula (see Lemma 11; the geometric meaning will be pointed out in the study of the generating curve) and m is a parameter given by Bour's Lemma. We observe that if $d = 0, H > 0$

and $\ell = 0$ or $1 - 4H^2 \geqslant 0$, then $Q \neq 0$. Notice that if the surface is rotational ($\ell = 0$), we may suppose $m = 1$. Then $Q = 0$ if and only if $d = \pm 2H, H \geqslant 0$.

Now notice that if $1 < 4H^2$, and if $d^2 = 4H^2, H > 0$, then $Q = 0$ if and only if $\ell = 0$. In the case $1 - 4H^2 < 0, \ell = 0, H > 0$ and $d = -2H$ we infer formula 29 S in [1]. In the case $1 - 4H^2 > 0, H > 0, \ell = 0$ and $d = -2H$ we obtain the formula 31 D in [1]. If $d = 2H, \ell = 0$ we get formula 31 C in [1]. In the case $H = 1/2, \ell = 0$ and $d = -1$ we deduce formula 30 D in [1]. On the other hand, if $1 > 4H^2$, $H > 0$, and if $d = 0$, then $Q \neq 0$. If $d = 0, H > 0, \ell = 0$, and if $1 - 4H^2 > 0$, we infer the formula

$$\cosh\left(\frac{t\sqrt{1-4H^2}}{2H}\right) = \sinh\rho \cdot \frac{\sqrt{1-4H^2}}{2H}.$$

For the values of H and d given above, we will describe the geometric proprieties of the H–screw motion surface. Using quite different techniques, namely reduction methods, Stefano Montaldo and Irene I. Onnis gave a classification of screw motion surfaces in $\mathbb{H}^2 \times \mathbb{R}$, based on the profile curve in orbit space [8].

Finally, if the ambient space is $\mathbb{S}^2 \times \mathbb{R}$, then the Abresch-Rosenberg holomorphic quadratic differential Q for H–screw motion surfaces is given by

$$\begin{aligned} (1) \qquad & \frac{1}{2} \cdot \Big[Q(X_v, X_v) - Q(X_\tau, X_\tau)\Big] - iQ(X_v, X_\tau) \\ & \qquad = \frac{1}{m^2}\left(4H^2 - d^2 + \ell^2(1 + 4H^2)\right) + i\frac{2\ell}{m^2}d. \end{aligned}$$

2. Some formulas for immersions of $\Omega \subset \mathbb{C}$ in $M^2 \times \mathbb{R}$

Let M^2 be a two dimensional Riemannian manifold. Let (x, y, t) be local coordinates in $M^2 \times \mathbb{R}$, where $z = x + iy$ are conformal coordinates on M^2 and $t \in \mathbb{R}$. Let $\sigma^2|dz|^2$ be the conformal metric in M^2. Thus $ds^2 = \sigma^2|dz|^2 + dt^2$ is the metric in the product space $M^2 \times \mathbb{R}$. Let $\Omega \subset \mathbb{C}$ be a planar domain and let

$$X : \Omega \looparrowright M^2 \times \mathbb{R}, w \mapsto (h(w), f(w)), w = u + iv \in \Omega,$$

be an immersion of Ω in $M^2 \times \mathbb{R}$. We say that $h(w)$ is the horizontal component and that $f(w)$ is the vertical component. Let $\partial_x, \partial_y, \partial_t$ be a local frame field adapted to X. Let $\langle , \rangle$ be the inner product in the product space $M^2 \times \mathbb{R}$; that is, if $V = a\partial_x + b\partial_y + c\partial_t$ is a tangent vector, then $\langle V, V\rangle = (a^2 + b^2)\sigma^2 + c^2$. We will need the complex operators ∂_w and $\partial_{\overline{w}}$, defined as follows. If g is a differentiable function in a domain Ω, we set $g_w := (1/2)(g_u - ig_v)$ and $g_{\overline{w}} := (1/2)(g_u + ig_v)$. Thus, using the complex terminology, we can rewrite $g_u = g_w + g_{\overline{w}}$ and $g_v = ig_w - ig_{\overline{w}}$.

Proposition 1. *$X : \Omega \looparrowright M^2 \times \mathbb{R}, w = u + iv \mapsto X(w)$, is a conformal immersion if and only if*

$$(f_w)^2 = -\left(\sigma \circ h\right)^2 h_w \overline{h}_w.$$

Furthermore, the induced metric $ds^2 = \mu^2 |dw|^2$ is given by

$$\mu^2 = \left(\sigma \circ h\right)^2 \left(|h_w| + |h_{\overline{w}}|\right)^2.$$

Proof. Note that $\langle X_u, X_u \rangle = \langle X_v, X_v \rangle$ if and only if $\sigma^2(|h_u|^2 - |h_v|^2) = f_v^2 - f_u^2$, and $\langle X_u, X_v \rangle = 0$ if and only if $\sigma^2\, \Re \overline{h}_u h_v = -f_u f_v$. Now multiplying the second equation by $2i$ and subtracting from the first equation, we deduce that X is conformal if and only if

$$\sigma^2(|h_u|^2 - |h_v|^2 - 2i\Re\, h_v \overline{h}_u) = -(f_u^2 - f_v^2 - 2i f_u f_v)$$

Using the operators ∂_w and $\partial_{\overline{w}}$, we deduce that

$$\left(\sigma \circ h\right)^2 h_w \overline{h}_w = -\left(f_w\right)^2,$$

as desired. This completes the proof of the first part of the statement.

We now turn to the computation of the induced metric $ds^2 = \mu^2 |dw|^2$. We have

$$\begin{aligned}
\mu^2 &= \frac{1}{2}(\langle X_u, X_u \rangle + \langle X_v, X_v \rangle) \\
&= \frac{1}{2}\left((\sigma \circ h)^2(|h_u|^2 + |h_v|^2) + f_u^2 + f_v^2\right) \\
&= \frac{1}{2}\left(2(\sigma \circ h)^2(|h_w|^2 + |h_{\overline{w}}|^2) + 4(\sigma \circ h)^2 |h_w||\overline{h}_w|\right).
\end{aligned}$$

The last equation follows from complex terminology, using the first part of the statement. This completes the proof of the second part of the statement. $\square$

Let $\boldsymbol{\nabla}$ be the Riemannian connection in $M^2 \times \mathbb{R}$.

Proposition 2.

$$\begin{aligned}
\boldsymbol{\nabla}_{\partial_x} \partial_x &= \frac{\sigma_x}{\sigma} \partial_x - \frac{\sigma_y}{\sigma} \partial_y, \\
\boldsymbol{\nabla}_{\partial_y} \partial_y &= -\frac{\sigma_x}{\sigma} \partial_x + \frac{\sigma_y}{\sigma} \partial_y, \\
\boldsymbol{\nabla}_{\partial_y} \partial_x &= \boldsymbol{\nabla}_{\partial_x} \partial_y = \frac{\sigma_y}{\sigma} \partial_x + \frac{\sigma_x}{\sigma} \partial_y, \\
\boldsymbol{\nabla}_{\partial_x} \partial_t &= \boldsymbol{\nabla}_{\partial_y} \partial_t = \boldsymbol{\nabla}_{\partial_t} \partial_t = \boldsymbol{\nabla}_{\partial_t} \partial_x = \boldsymbol{\nabla}_{\partial_t} \partial_y = 0.
\end{aligned}$$

Proof. The proof is a standard computation, in view of the product metric $ds^2 = \sigma^2 |dz|^2 + dt^2$. $\square$

Proposition 3. *Let*

$$X : \Omega \looparrowright M^2 \times \mathbb{R}, w \mapsto (h(w), f(w)), w = u + iv \in \Omega$$

be an immersion of a planar domain Ω *in* $M^2 \times \mathbb{R}$. *Then*

$$\begin{aligned}
\boldsymbol{\nabla}_{X_u} X_u &= \left(\Re h_{uu} + \left((\Re h_u)^2 - (\Im h_u)^2\right) \frac{\sigma_x}{\sigma} + 2\Re h_u \Im h_u \frac{\sigma_y}{\sigma}\right) \partial_x \\
&\quad + \left(\Im h_{uu} + \left((\Im h_u)^2 - (\Re h_u)^2\right) \frac{\sigma_y}{\sigma} + 2\Re h_u \Im h_u \frac{\sigma_x}{\sigma}\right) \partial_y + f_{uu}\partial_t \\
&= \left(\Re h_{uu} + (h_w + h_{\overline{w}})^2 \frac{\sigma_z}{\sigma} + \left(\overline{h}_w + \overline{h}_{\overline{w}}\right)^2 \frac{\sigma_{\overline{z}}}{\sigma}\right) \partial_x \\
&\quad + \left(\Im h_{uu} - i\,(h_w + h_{\overline{w}})^2 \frac{\sigma_z}{\sigma} + i\left(\overline{h}_w + \overline{h}_{\overline{w}}\right)^2 \frac{\sigma_{\overline{z}}}{\sigma}\right) \partial_y + f_{uu}\,\partial_t, \\
\boldsymbol{\nabla}_{X_v} X_v &= \left(\Re h_{vv} + \left((\Re h_v)^2 - (\Im h_v)^2\right) \frac{\sigma_x}{\sigma} + 2\Re h_v \Im h_v \frac{\sigma_y}{\sigma}\right) \partial_x \\
&\quad + \left(\Im h_{vv} + \left((\Im h_v)^2 - (\Re h_v)^2\right) \frac{\sigma_y}{\sigma} + 2\Re h_v \Im h_v \frac{\sigma_x}{\sigma}\right) \partial_y + f_{vv}\partial_t \\
&= \left(\Re h_{vv} - (h_w - h_{\overline{w}})^2 \frac{\sigma_z}{\sigma} - \left(\overline{h}_w - \overline{h}_{\overline{w}}\right)^2 \frac{\sigma_{\overline{z}}}{\sigma}\right) \partial_x \\
&\quad + \left(\Im h_{vv} + i\,(h_w - h_{\overline{w}})^2 \frac{\sigma_z}{\sigma} - i\left(\overline{h}_w - \overline{h}_{\overline{w}}\right)^2 \frac{\sigma_{\overline{z}}}{\sigma}\right) \partial_y + f_{vv}\,\partial_t, \\
\boldsymbol{\nabla}_{X_u} X_v &= \boldsymbol{\nabla}_{X_v} X_u \\
&= \left(\Re h_{uv} + (\Re h_u \Re h_v - \Im h_u \Im h_v) \frac{\sigma_x}{\sigma} + (\Re h_u \Im h_v + \Im h_u \Re h_v) \frac{\sigma_y}{\sigma}\right) \partial_x \\
&+ \left(\Im h_{uv} + (\Re h_u \Im h_v + \Im h_u \Re h_v) \frac{\sigma_x}{\sigma} + (\Im h_u \Im h_v - \Re h_u \Re h_v) \frac{\sigma_y}{\sigma}\right) \partial_y \\
&+ f_{uv}\partial_t \\
&= \left(\Re h_{uv} + i\left(h_w^2 - h_{\overline{w}}^2\right) \frac{\sigma_z}{\sigma} + i\left(\overline{h}_w^2 - \overline{h}_{\overline{w}}^2\right) \frac{\sigma_{\overline{z}}}{\sigma}\right) \partial_x \\
&\quad + \left(\Im h_{uv} + \left(h_w^2 - h_{\overline{w}}^2\right) \frac{\sigma_z}{\sigma} - \left(\overline{h}_w^2 - \overline{h}_{\overline{w}}^2\right) \frac{\sigma_{\overline{z}}}{\sigma}\right) \partial_y + f_{uv}\,\partial_t.
\end{aligned}$$

Proof. Applying Proposition 2 to any immersion $X(u,v)$, we can easily compute $\boldsymbol{\nabla}_{X_u} X_u, \boldsymbol{\nabla}_{X_v} X_v$ and $\boldsymbol{\nabla}_{X_v} X_u$ in terms of the derivatives of the horizontal component h and the vertical component f. Making use of complex terminology, we obtain the identities in the statement. □

Proposition 4. *Assume that*

$$X : \Omega \looparrowright M^2 \times \mathbb{R}, w \mapsto (h(w), f(w)), w \in \Omega$$

is a conformal immersion. Then a unit normal N *is given by*

$$N = \frac{\left(\frac{2}{\sigma}\,\Re F, \frac{2}{\sigma}\,\Im F, |F|^2 - 1\right)}{|F|^2 + 1},$$

where

$$F = \frac{f_w h_{\overline{w}} - f_{\overline{w}} h_w}{\sigma |h_{\overline{w}}| \left(|h_w| + |h_{\overline{w}}|\right)}$$

and

$$F^2 = -\frac{h_w}{\overline{h}_w}.$$

Proof. By a simple computation, we see that $\langle X_u, N\rangle = \langle X_v, N\rangle = 0$ and $\langle N, N\rangle = 1$, which proves the proposition. □

REMARK. Note that when the ambient space is the Euclidean space, i.e., $\sigma = 1$, the complex function F appearing in Proposition 4 is called the oriented Euclidean Gauss map, and denoted by g in the minimal surfaces literature.

Let X_s, X_τ, be adapted tangent vectors, where s, τ are natural coordinates, given by Theorem 19 and Theorem 20. We will compute the covariant derivatives, as follows.

PROPOSITION 5. *Let S be a screw motion surface immersed in $\mathbb{H}^2 \times \mathbb{R}$ given by*

$$X(\rho(s), \varphi(s,\tau)) = (\tanh\rho/2\,\cos\varphi, \tanh\rho/2\,\sin\varphi, \lambda(\rho) + \ell\varphi)\,, \tag{2}$$

where $\ell \geqslant 0$ is the pitch. If $\ell = 0$, the surface is rotational.

Then (using complex notation) we have

$$\begin{aligned}\boldsymbol{\nabla}_{X_s} X_s = \Bigg(& e^{i\varphi}\Big[\frac{\rho''}{1+\cosh\rho} - \varphi_s^2\,\tanh(\rho/2)\cosh\rho \\ & + i\left(\frac{2\varphi_s\rho'}{1+\cosh\rho} + \tanh(\rho/2)\varphi_{ss} + \frac{2\sinh^2\rho\,\varphi_s\rho'}{(1+\cosh\rho)^2}\right)\Big], \\ & \lambda''\rho'^2 + \lambda'\rho'' + \ell\varphi_{ss}\Bigg),\end{aligned} \tag{3}$$

$$\boldsymbol{\nabla}_{X_\tau} X_\tau = -e^{i\varphi}\varphi_\tau^2\cosh\rho\,\tanh(\rho/2), \tag{4}$$

$$\boldsymbol{\nabla}_{X_\tau} X_s = e^{i\varphi}\frac{\varphi_\tau\cosh\rho}{1+\cosh\rho}\left(-\varphi_s\sinh\rho + i\rho'\right). \tag{5}$$

In addition, a unit normal is given by

$$N = \frac{\rho'}{mU}\left(\frac{-e^{i\varphi}\,(\lambda'\sinh\rho + i\ell)}{1+\cosh\rho}, \sinh\rho\right). \tag{6}$$

Proof. The proof is a long calculation, setting

$$\Re h := \tanh(\rho(s)/2)\,\cos(\varphi(s,\tau)), \quad \Im h := \tanh(\rho(s)/2)\,\sin(\varphi(s,\tau)),$$

and $f := \lambda\circ\rho(s) + \ell\varphi(s,\tau)$, taking into account Proposition 3 and Proposition 4. □

Similarly, we state the following formulas for future reference.

Proposition 6. *Let S be a screw motion surface immersed in $\mathbb{S}^2 \times \mathbb{R}$ given by*

$$X(\rho(s),\varphi(s,\tau)) = (\tan\rho/2\,\cos\varphi, \tan\rho/2\,\sin\varphi, \lambda(\rho)+\ell\varphi)\,, \tag{7}$$

where $\ell \geqslant 0$ is the pitch. If $\ell = 0$, the surface is rotational.

Then (using complex notation)

$$\begin{aligned}\nabla_{X_s}X_s = \Bigg(& e^{i\varphi}\Big[\frac{\rho''}{1+\cos\rho} - \varphi_s^2\,\tan(\rho/2)\cos\rho \\ &+ i\left(\frac{2\varphi_s\rho'}{1+\cos\rho} + \tan(\rho/2)\varphi_{ss} - \frac{2\sin^2\rho\,\varphi_s\rho'}{(1+\cos\rho)^2}\right)\Big], \\ &\lambda''\rho'^2 + \lambda'\rho'' + \ell\varphi_{ss}\Bigg),\end{aligned} \tag{8}$$

$$\nabla_{X_\tau}X_\tau = -e^{i\varphi}\varphi_\tau^2\cos\rho\,\tan(\rho/2), \tag{9}$$

$$\nabla_{X_\tau}X_s = e^{i\varphi}\frac{\varphi_\tau\cos\rho}{1+\cos\rho}\,(-\varphi_s\sin\rho + i\rho')\,. \tag{10}$$

In addition, a unit normal is given by

$$N = \frac{\rho'}{mU}\left(\frac{-e^{i\varphi}\,(\lambda'\sin\rho + i\ell)}{1+\cos\rho}, \sin\rho\right). \tag{11}$$

Proof. The proof is a calculation similar to that of Proposition 5. □

Proposition 7. *Assume that*

$$X : \Omega \looparrowright M^2\times\mathbb{R}, w \mapsto (h(w), f(w)), w \in \Omega,$$

is a conformal immersion with induced metric $ds^2 = \mu^2|dw|^2$. Then the mean curvature vector $\overrightarrow{\mathbf{H}}$ is given by

$$2\mu^2\,\overrightarrow{\mathbf{H}} = 4\Re\left(h_{w\overline{w}} + 2\frac{\sigma_z}{\sigma}h_w h_{\overline{w}}\right)\partial_x + 4\Im\left(h_{w\overline{w}} + 2\frac{\sigma_z}{\sigma}h_w h_{\overline{w}}\right)\partial_y + \triangle f\,\partial_t.$$

Proof. Using Proposition 3 we see that

$$\begin{aligned}&\nabla_{X_u}X_u + \nabla_{X_v}X_v \\ &\quad= \left(\Re h_{uu} + \Re h_{vv} + 4h_w h_{\overline{w}}\frac{\sigma_z}{\sigma} + 4\overline{h}_w\overline{h}_{\overline{w}}\frac{\sigma_{\overline{z}}}{\sigma}\right)\partial_x \\ &\qquad+ \left(\Im h_{uu} + \Im h_{vv} - 4ih_w h_{\overline{w}}\frac{\sigma_z}{\sigma} + 4i\overline{h}_w\overline{h}_{\overline{w}}\frac{\sigma_{\overline{z}}}{\sigma}\right)\partial_y + \triangle f\,\partial_t \\ &\quad= 4\Re\left(h_{w\overline{w}} + 2\frac{\sigma_z}{\sigma}h_w h_{\overline{w}}\right)\partial_x + 4\Im\left(h_{w\overline{w}} + 2\frac{\sigma_z}{\sigma}h_w h_{\overline{w}}\right)\partial_y + \triangle f\,\partial_t,\end{aligned}$$

as desired. Since X is conformal, this completes the proof of the proposition. □

REMARK. We make some fairly general comments concerning harmonic maps. We recall that a function

$$h : \Omega \subset \mathbb{C} \to (M^2, \sigma^2 |dz|^2), w \mapsto h(w),$$

is a harmonic map if it satisfies

$$h_{w\overline{w}} + 2\frac{\sigma_z}{\sigma} h_w h_{\overline{w}} = 0. \tag{12}$$

We also recall that for any harmonic map $h : \Omega \subset \mathbb{C} \mapsto M^2$ there exists a related Hopf holomorphic function given by

$$\phi = (\sigma \circ h)^2 \, h_w \overline{h}_w; \tag{13}$$

see [13], [14]. For the reader's benefit we show now that ϕ is indeed holomorphic. A simple computation, using complex differentiation and the chain rule, ensures that

$$\phi_{\overline{w}} = (\sigma \circ h)^2 \left[\overline{h}_w \left(h_{w\overline{w}} + \frac{2\sigma_z}{\sigma} h_w h_{\overline{w}} \right) + h_w \overline{\left(h_{w\overline{w}} + \frac{2\sigma_z}{\sigma} h_w h_{\overline{w}} \right)} \right].$$

Hence ϕ is holomorphic if h is an harmonic map. It is worth noting that if h is a harmonic map, the zeros of h_w and $h_{\overline{w}}$ are isolated unless $\phi \equiv 0$. In view of Proposition 1 and Proposition 7, we infer that

$$X : \Omega \looparrowright M^2 \times \mathbb{R}, w \mapsto (h(w), f(w)), w \in \Omega$$

is a conformal *minimal* (possibly ramified) immersion if and only if the following three conditions hold: $(f_w)^2 = -\phi$, the horizontal component h is a harmonic map, and the vertical component f is a harmonic function.

Let $\Omega \subset \mathbb{C}$ be a planar domain, and let M^2 be a two dimensional space of constant Gaussian curvature. Let $X, \widetilde{X} : \Omega \looparrowright M^2 \times \mathbb{R}$ be two conformal immersions with same constant mean curvature H. Assume that X and $\widetilde{X}$ are isometric, that is, $d\widetilde{s}^2 = ds^2 = \mu^2 |dw|^2$, where $w \in \Omega$ is a complex parameter. Assume also for the moment that $M^2 = \mathbb{R}^2$, and that X and $\widetilde{X}$ are isometric immersions in $\mathbb{R}^3$. We now describe the standard notion of associate immersions in $\mathbb{R}^3$.

Let ϕ_{R^3} and $\widetilde{\phi_{R^3}}$ be the *classical Hopf functions* of X, and $\widetilde{X}$, respectively. A standard computation show that

$$|\phi_{R^3}|^2 = \left|\widetilde{\phi_{R^3}}\right|^2 = \left(H^2 - K\right)\mu^4,$$

where K is the Gaussian curvature. It follows that

$$\widetilde{\phi_{R^3}} = e^{i\theta}\,\phi_{R^3}, \theta \in \mathbb{R}.$$

Now turning to the product space $\mathbb{H}^2 \times \mathbb{R}$, let $X(w) = (h(w), f(w))$ be a minimal conformal immersion with Hopf function ϕ given by (13). Let $\widetilde{\phi}$ be the Hopf function associate to $\widetilde{X}$, given by (13). We then have the following theorem.

THEOREM 8 (Hauswirth, Sa Earp, Toubiana [6]). *Let $X, \widetilde{X} : U \looparrowright \mathbb{H}^2 \times \mathbb{R}$ be two conformal and minimal isometric immersions. If $\phi = \widetilde{\phi}$, then, up to rigid motions of $\mathbb{H}^2 \times \mathbb{R}$, the two immersions X and $\widetilde{X}$ are the same.*

This result suggests the following definition.

DEFINITION. We say that two conformal isometric immersions $X, \widetilde{X} : \Omega \looparrowright \mathbb{H}^2 \times \mathbb{R}$ are *associate*, if the Hopf functions satisfy the relation $\widetilde{\phi} = e^{i\theta}\, \phi$. If $\widetilde{\phi} = -\phi$, we say that the two surfaces are *conjugate*.

Of course, these definitions can be extended to $\mathbb{S}^2 \times \mathbb{R}$, and more generally to $M^2 \times \mathbb{R}$. We consider the following question:

Is it true that two isometric minimal immersions $X, \widetilde{X}$ of Ω in $\mathbb{H}^2 \times \mathbb{R}$ are associate?

As we will see, this question has a positive answer in the context of minimal standard screw motion surfaces immersed in $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{S}^2 \times \mathbb{R}$.

3. Some formulas for H–surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$ that are invariant by screw motions

Let $\mathbb{H}^2 = \{(x, y), x^2 + y^2 < 1\}$ be the hyperbolic plane equipped with the hyperbolic metric, i.e., $\sigma = 2/(1 - |z|^2)$. The metric in $\mathbb{H}^2 \times \mathbb{R}$ in cylindrical coordinates (ρ, θ, t), where ρ is the hyperbolic distance measure from the origin of $\mathbb{H}^2$, i.e., $R = \tanh \rho/2, R = \sqrt{x^2 + y^2}$, and t is the height, is given by $ds^2 = d\rho^2 + \sinh^2 \rho\, d\theta^2 + dt^2$. We will study surfaces that are invariant by a 1-parameter group of screw motions. As we indicated in the introduction, the idea is to take a graph $t = \lambda(\rho)$ in the vertical plane xt and turn it around the t axis by performing screw motions. More precisely, we consider smooth immersions of the form (2):

PROPOSITION 9. *A unit normal vector N is (using complex notation)*

$$N = \frac{1}{\sqrt{\ell^2 + \sinh^2 \rho + \lambda'^2 \sinh^2 \rho}} \left(\frac{-\, e^{i\varphi}(\lambda' \sinh \rho + i\ell)}{1 + \cosh \rho}, \sinh \rho \right).$$

Proof. For the sake of clarity, this formula for the unit normal could be derived with the aid of the formula obtained in Proposition 4, assuming the existence of natural parameters s, τ. However, the formula also follows readily, by a simple computation, from the equations $\langle N, N \rangle = 1$ and $\langle N, X_\varphi \rangle = \langle N, X_\rho \rangle = 0$, where

$$X_\varphi = (-\tanh \rho/2\, \sin \varphi, \tanh \rho/2\, \cos \varphi, \ell)$$

and

$$X\rho = \left(\frac{\cos\varphi}{1+\cosh\rho}, \frac{\sin\varphi}{1+\cosh\rho}, \lambda'(\rho)\right)$$

are the tangent vectors adapted to X. □

We will need the mean curvature equation for screw motion surfaces.

PROPOSITION 10. *The mean curvature equation is*

$$\begin{aligned} 2H(\rho)\left(\ell^2+\sinh^2\rho+\lambda'^2\sinh^2\rho\right)^{3/2} \\ = \lambda''\sinh\rho(\ell^2+\sinh^2\rho) \\ + \lambda'\cosh\rho(2\ell^2+\sinh^2\rho+\lambda'^2\sinh^2\rho). \end{aligned} \tag{14}$$

Proof. We shall take natural coordinates s, τ so that the induced metric is given by $d\mu^2 = ds^2 + U^2 d\tau^2$. In fact, it is easy to see that this can be done locally (see Theorem 19). We have therefore natural conformal coordinates $\upsilon + i\tau$, where $\upsilon = \int(1/U)ds$. Of course, the induced metric becomes

$$d\mu^2 = U^2(d\upsilon^2 + d\tau^2).$$

Notice that the mean curvature H is given by

$$2H(\rho)U^2 = U^2\langle \boldsymbol{\nabla}_{X_s}X_s, N\rangle + \langle \boldsymbol{\nabla}_{X_\tau}X_\tau, N\rangle.$$

Now invoking Proposition 5, Proposition 9, and (2), we obtain, after a long calculation,

$$\begin{aligned} U^2\langle \boldsymbol{\nabla}_{X_s}X_s, N\rangle = U^2\Bigg(& \frac{\lambda''(\ell^2+\sinh^2\rho)\sinh\rho}{(\ell^2+\sinh^2\rho+\lambda'^2\sinh^2\rho)^{3/2}} \\ & + \frac{\cosh\rho\lambda'\ell^2}{(\ell^2+\sinh^2\rho+\lambda'^2\sinh^2\rho)^{3/2}} \\ & + \frac{\cosh\rho\lambda'\ell^2}{(\ell^2+\sinh^2\rho)(\ell^2+\sinh^2\rho+\lambda'^2\sinh^2\rho)^{1/2}}\Bigg), \\ \langle \boldsymbol{\nabla}_{X_\tau}X_\tau, N\rangle = U^2 & \frac{\cosh\rho\lambda'\sinh^2\rho}{(\ell^2+\sinh^2\rho)(\ell^2+\sinh^2\rho+\lambda'^2\sinh^2\rho)^{1/2}}. \end{aligned}$$

Adding these equalities we obtain the desired formula. □

The following formula is crucial for our study.

LEMMA 11.

$$\left(\frac{\lambda'\sinh^2\rho}{\sqrt{\ell^2+\sinh^2\rho+\lambda'^2\sinh^2\rho}}\right)' = 2\sinh\rho\, H(\rho).$$

In particular, if H is constant, we obtain

$$(*) \qquad \frac{\lambda'^2 \sinh^4 \rho}{\ell^2 + \sinh^2 \rho + \lambda'^2 \sinh^2 \rho} = (d + 2\cosh \rho\, H)^2 \quad \textit{(first integral).}$$

Proof. Using (14), we find that the derivative of the quantity

$$\frac{\lambda' \sinh^2 \rho}{\sqrt{\ell^2 + \sinh^2 \rho + \lambda'^2 \sinh^2 \rho}}$$

is exactly $2 \sinh \rho\, H(\rho)$. This completes the proof of the Proposition. □

Now let $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$ be the sphere equipped with the spherical metric, i.e., $\sigma = 2/(1+|z|^2)$. The metric in $\mathbb{S}^2 \times \mathbb{R}$ in cylindrical coordinates (ρ, θ, t), where ρ is the sphere distance measure from the origin of $\mathbb{S}^2$, i.e., $R = \tan \rho/2, R = \sqrt{x^2+y^2}$, and t is the height, is given by $ds^2 = d\rho^2 + \sin^2 \rho\, d\theta^2 + dt^2$. We investigate surfaces that are invariant by a 1-parameter group of screw motions. We consider smooth immersions of the form (7). Using the formula for the unit normal obtained above, we infer the following result.

Proposition 12. *A unit normal vector N is (using complex notation)*

$$N = \frac{1}{\sqrt{\ell^2 + \sin^2 \rho + \lambda'^2 \sin^2 \rho}} \left(\frac{-e^{i\varphi}(\lambda' \sin \rho + i\ell)}{1 + \cos \rho}, \sin \rho \right).$$

The mean curvature equation is given in the following result.

Proposition 13. *We have*

$$(15) \qquad \begin{aligned} &2H(\rho) \left(\ell^2 + \sin^2 \rho + \lambda'^2 \sin^2 \rho\right)^{3/2} \\ &\quad = \lambda'' \sin \rho (\ell^2 + \sin^2 \rho) \\ &\qquad + \lambda' \cos \rho (2\ell^2 + \sin^2 \rho + \lambda'^2 \sin^2 \rho). \end{aligned}$$

The companion result to Lemma 11 is the following.

Lemma 14.

$$\left(\frac{\lambda' \sin^2 \rho}{\sqrt{\ell^2 + \sin^2 \rho + \lambda'^2 \sin^2 \rho}} \right)' = 2 \sin \rho\, H(\rho).$$

In particular, if H is constant, we obtain

$$(**) \qquad \frac{\lambda'^2 \sin^4 \rho}{\ell^2 + \sin^2 \rho + \lambda'^2 \sin^2 \rho} = (d - 2\cos \rho\, H)^2 \quad \textit{(first integral).}$$

We observe that the proofs of Proposition 12, Proposition 13 and Lemma 14 are exactly the same as those of the analogous results in $\mathbb{H}^2 \times \mathbb{R}$ with the substitutions $\sinh \rho \to \sin \rho$ and $\cosh \rho \to \sin \rho$. We omit the details.

Of course, the reader may expect that there exists some discovery process leading to Lemma 11 and Lemma 14. This is related to the knowledge of the Hopf function and of the Abresch-Rosenberg quadratic differential.

In the remainder of the paper, we will employ the formulas established in this section to study minimal and constant (mean) curvature surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$.

4. Complete minimal screw motion immersions in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$

We wish to find complete embedded minimal screw motion surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and complete minimal screw motion surfaces in $\mathbb{S}^2 \times \mathbb{R}$ and describe these geometrically.

We first investigate minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$. Notice that the helicoids are obtained from the first integral $(*)$ by setting $d = 0$. Assume $d \neq 0$. Up to vertical translations or symmetry about the xy plane, we infer from $(*)$ that the generating graph $t = \lambda(\rho)$ of a minimal screw motion immersion in $\mathbb{H}^2 \times \mathbb{R}$ is given by (assuming $d > 0$; if $d = 0$ we obtain the helicoids)

$$\lambda(\rho) = d \int_a^\rho \frac{\sqrt{\ell^2 + \sinh^2 r}}{\sinh r \sqrt{\sinh^2 r - d^2}} dr, \quad d = \sinh a. \tag{16}$$

Before we proceed to construct the minimal (and constant mean curvature) surfaces, we fix some notation and prove some basic formulas for future reference.

We shall always denote by $t = \lambda\circ\rho(R) := \lambda(\rho(R))$ the height, where R is the Euclidean distance in $\{x^2 + y^2 < 1\}$ or $\mathbb{C} \cup \{\infty\}$ measured from the origin, when the ambient space is $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{S}^2 \times \mathbb{R}$, respectively. By the chain rule we easily obtain the following relations:

$$\begin{aligned} \frac{d\lambda\circ\rho}{dR} &= \begin{cases} \lambda'(\rho)(1 + \cosh\rho) & \text{if the ambient space is } \mathbb{H}^2 \times \mathbb{R}, \\ \lambda'(\rho)(1 + \cos\rho) & \text{if the ambient space is } \mathbb{S}^2 \times \mathbb{R}, \end{cases} \\ \frac{d^2\lambda\circ\rho}{dR^2} &= \begin{cases} \lambda''(\rho)(1 + \cosh\rho)^2 + \lambda'(\rho)(\sinh\rho)(1 + \cosh\rho) \\ \qquad \text{if the ambient space is } \mathbb{H}^2 \times \mathbb{R}. \end{cases} \end{aligned} \tag{17}$$

In the following result, we obtain a family of embedded surfaces with Gaussian curvature $K \equiv -1$.

THEOREM 15. *The generating function $t = \lambda\circ\rho(R)$ of a minimal immersion invariant by screw motions is an increasing strictly concave function for $R > (\sqrt{1 + d^2} - 1)/d$ cutting orthogonally the x–axis at $(\sqrt{1 + d^2} - 1)/d$. When $\rho \to \infty$, i.e., $R \to 1$, then $t = \lambda\circ\rho$ has a finite limit and the tangent line to the graph has a limit angle α with the x–axis such that $\tan\alpha = d$. Thus, extending the graph by symmetry about the x–axis, we obtain a complete embedded curve with bounded height generating a complete minimal properly immersed surface* *in* $\mathbb{H}^2 \times \mathbb{R}$. *Each such surface is conformally equivalent to the hyperbolic plane* $\mathbb{H}^2$.

In addition, if $\ell > 1/\sqrt{2}$, *then for any parameter* d *we obtain a minimal simply connected embedded screw motion surface. If* $\ell = 1$, *then each such surface has Gaussian curvature* $K \equiv -1$.

Proof. Of course, in view of (16) we see that $\lambda'(\rho) > 0$ if $\rho > a$ and that $\lambda(\rho)$ is vertical at $\rho = a$. According to (17) we deduce $t = \lambda\circ\rho$ is a strictly increasing function for $R > (\sqrt{1+d^2}-1)/d$ cutting orthogonally the x–axis at $(\sqrt{1+d^2}-1)/d$. Differentiating twice (16), taking into account (17), after a straightforward computation we infer

$$\begin{aligned}\frac{d^2\lambda\circ\rho}{dR^2}\cdot\frac{1}{(1+\cos\rho)}&\\ &= d\cdot\frac{\sinh^2\rho\,(\ell^2+\sinh^2\rho)\left[(\sinh^2\rho-d^2)-(\cosh^2\rho+\cosh\rho)\right]}{\sinh^2\rho\,(\ell^2+\sinh^2\rho)^{1/2}\,(\sinh^2\rho-d^2)^{3/2}}\\ &\quad - d\cdot\frac{\ell^2(1+\cosh\rho)\cosh\rho(\sinh^2\rho-d^2)}{\sinh^2\rho\,(\ell^2+\sinh^2\rho)^{1/2}\,(\sinh^2\rho-d^2)^{3/2}}.\end{aligned}$$

Thus $d^2\lambda\circ\rho/dR^2 < 0$ if $R > (\sqrt{1+d^2}-1)/d$, for any pitch ℓ. Clearly, using (16) once again, we deduce $\lambda\circ\rho$ has a finite limit when $R \to 1$ (hence the height $t = \lambda\circ\rho$ is bounded) and the tangent line to the graph has a limit angle α with the x–axis such that $\tan\alpha = d$. Notice that the curvature

$$k(R) := \frac{d^2\lambda\circ\rho/dR^2}{\left(1+(d\lambda\circ\rho/dR)^2\right)^{3/2}}$$

has a finite limit at $R = (\sqrt{1+d^2}-1)/d$. Clearly, the minimal surface equation (see (14)) is invariant by vertical reflection, i.e., $\lambda(-\rho) = -\lambda(\rho)$. Thus, we can extend the graph smoothly across the x–axis by vertical reflection. We have therefore constructed a complete embedded curve satisfying the required geometric properties which generates a complete proper minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ that is invariant by screw motions with pitch ℓ for any $\ell \geqslant 0$. This completes the proof of the first part of the statement.

Notice that by Corollary 21 we deduce that for $\ell = 1$ each surface has Gaussian curvature $K \equiv -1$ (since $K = -U''/U$). To investigate embedded surfaces, we shall construct under our assumptions a family of surfaces (containing the helicoids) such that the height $t = \lambda(\rho)$ of the generating curve for each element of this family satisfies $\lim_{\rho\to\infty} t < \pi\ell$. In view of (16), this can be achieved by elementary one variable calculus estimates, as follows.

By the change of variables $\xi = \sinh\rho$ we get

$$\lambda(\rho) = d\int_d^{\sinh\rho}\sqrt{\frac{\ell^2+\xi^2}{1+\xi^2}}\cdot\frac{1}{\sqrt{\xi+d}}\cdot\frac{d\xi}{\xi\sqrt{\xi-d}}.$$

We define

$$c := \max \left\{ 1, \sqrt{\frac{\ell^2 + d^2}{1 + d^2}} \right\}.$$

By an elementary argument we deduce the inequality

$$\lambda(\rho) \leqslant \frac{c\sqrt{d}}{\sqrt{2}} \int_d^{\sinh \rho} \frac{d\xi}{\xi\sqrt{\xi - d}}.$$

Setting

$$I(\rho) := \int_d^{\sinh \rho} \frac{d\xi}{\xi\sqrt{\xi - d}},$$

we have

$$\begin{aligned} I(\rho) &= \int_0^{\sqrt{\sinh \rho - d}} \frac{2d\varsigma}{\varsigma^2 + d} \quad (\varsigma = \sqrt{\xi - d}) \\ &= \frac{2}{\sqrt{d}} \int_0^{\sqrt{\frac{\sinh \rho}{d} - 1}} \frac{d\zeta}{1 + \zeta^2} \quad (\zeta = \frac{\varsigma}{\sqrt{d}}) \\ &= \frac{2}{\sqrt{d}} \arctan \sqrt{\frac{\sinh \rho}{d} - 1}. \end{aligned}$$

Hence

$$I(\rho) \longrightarrow \frac{\pi}{\sqrt{d}} \quad (\text{as}\, \rho \to \infty).$$

But then

$$\lim_{\rho \to \infty} \lambda(\rho) \leqslant \frac{c\pi}{\sqrt{2}}.$$

Now if $\ell > 1/\sqrt{2}$, we deduce that $c\pi/\sqrt{2} < \pi\ell$. Hence we obtain

$$\lim_{\rho \to \infty} \lambda(\rho) < \pi\ell,$$

as desired.

Finally, we observe that by invoking Theorem 19 we readily see that we can parametrize our screw motion surfaces with the aid of natural coordinates s, τ for all real values. In terms of the natural conformal coordinates $\upsilon + i\tau$, where $\upsilon = \int (1/U) ds$, it suffices to deduce that υ is bounded to conclude that the conformal type is $\mathbb{H}^2$. To see this, we first combine Theorem 19 with (16) to infer the formula

$$\rho'^2(s) = \frac{\sinh^2 \rho - d^2}{\sinh^2 \rho}.$$

Then, making the change of variables $\nu = \rho(\xi)$, we obtain

$$\begin{aligned} \upsilon(s) &= m \int_0^s \frac{d\xi}{\sqrt{\ell^2 + \sinh^2 \rho(\xi)}} \\ &= m \int_a^{\rho(s)} \frac{\sinh \nu \, d\nu}{\sqrt{\sinh^2 \nu - d^2}\sqrt{\ell^2 + \sinh^2 \nu}}. \end{aligned}$$

Clearly, the above expression for υ shows that υ is bounded. This completes the proof of the Theorem. □

Next we will display our first figure generated with the aid of Maple software.

Generating curve of minimal screw motion surface in $\mathbb{H}^2 \times \mathbb{R}$: the ρt–plane

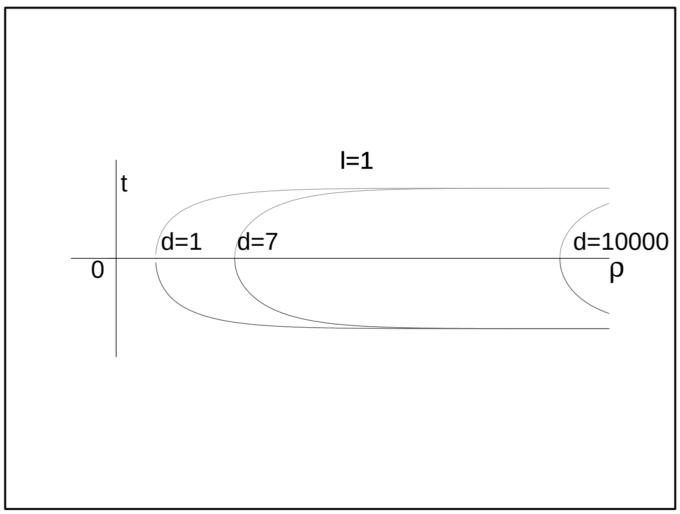

Figure 1

We observe that the generating curve plotted above leaves the x–axis vertically and *very fast* becomes horizontal, as expected in view of (16). Indeed, fixing $\ell = 1$, we have: if $d = 1$, then $t = \lambda(20) = 1.570796323$; if $d = 7$, then $t = \lambda(20) = 1.570796298$; if $d = 10000$, then $t = \lambda(20) = 1.570755104$. Moreover, for either $d = 1$ and $\rho = 40, 80, 200, 500$, or $d = 7$ and $\rho = 40, 80, 200, 500$, or $d = 10000$ and $\rho = 40, 80, 200, 500$, Maple gives the same output (with 11 digit accuracy), namely $t = 1.5707963268$.

Generating curve of minimal screw motion surface in $\mathbb{H}^2 \times \mathbb{R}$

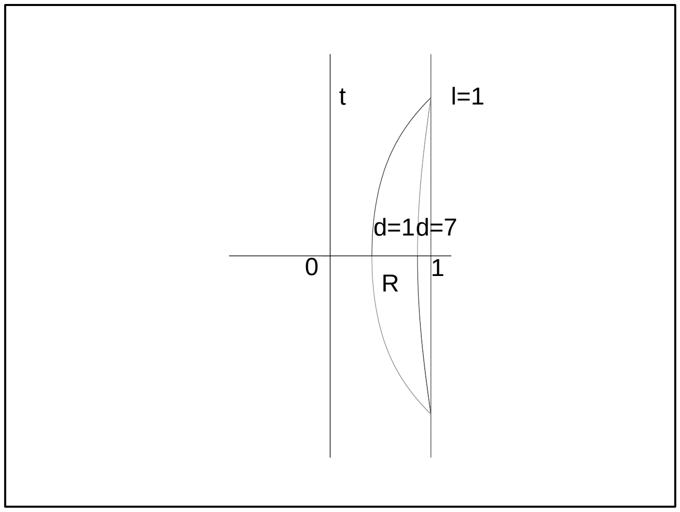

FIGURE 2

Minimal screw motion surface in $\mathbb{H}^2 \times \mathbb{R}$: $\ell = 1, d = 1$

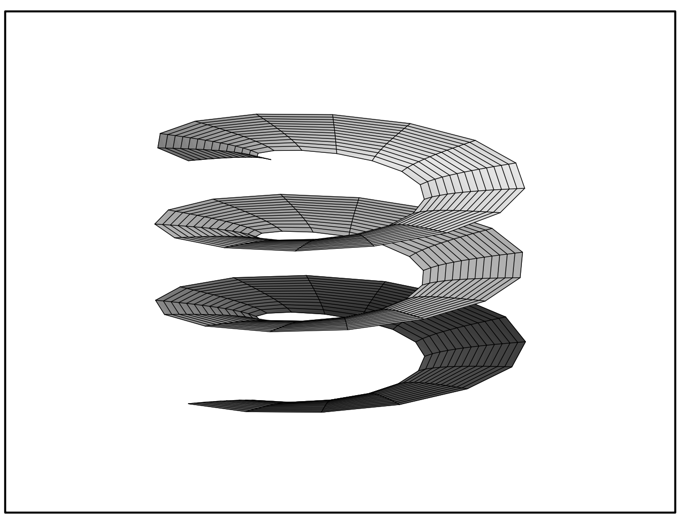

FIGURE 3

Next let us turn our attention to minimal surfaces that are invariant by screw motions immersed in $\mathbb{S}^2 \times \mathbb{R}$, given by (7). We shall simplify the discussion.

Up to vertical translations or symmetry about the xy plane, we infer from $(**)$ that the generating graph $t = \lambda(\rho)$ of a minimal screw motion immersion in $\mathbb{S}^2 \times \mathbb{R}$, is given by (we may assume $d > 0$, since if $d = 0$ we obtain the helicoids)

(18)
$$\lambda(\rho) = d \int_a^\rho \frac{\sqrt{\ell^2 + \sin^2 r}}{\sin r \sqrt{\sin^2 r - d^2}} dr, \quad a < \rho < \pi - a \quad (d = \sin a, 0 < a < \pi/2).$$

THEOREM 16. *We assume $d > 0$. The generating function $t = \lambda \circ \rho(R)$ of a minimal immersion that is invariant by screw motions is an increasing function for $d/(1+\sqrt{1-d^2}) < R < d/(1-\sqrt{1-d^2})$ and is vertical at $d/(1+\sqrt{1-d^2})$ and $d/(1-\sqrt{1-d^2})$. By extending the graph by vertical symmetry we obtain a complete embedded curve (periodic in the t direction), generating a complete minimal immersed screw motion surface in $\mathbb{S}^2 \times \mathbb{R}$. Any such rotational minimal surface is an embedded annulus.*

In addition, the period $2\mathcal{P}$ of any such minimal screw motion surface satisfies the estimates

$$2\pi\ell < 2\pi\sqrt{\ell^2 + d^2} \leqslant 2\mathcal{P} \leqslant 2\pi\sqrt{\ell^2 + 1}.$$

If the period $2\mathcal{P}$ and $2\pi\ell$ are linearly dependent over $\mathbb{Q}$, then the immersion is proper. If the period $2\mathcal{P}$ and $2\pi\ell$ are linearly independent over $\mathbb{Q}$, then the immersion is not proper and the surface is dense in the region Ω of $\mathbb{S}^2 \times \mathbb{R}$ (bounded by two cylinders) given by $d/(1+\sqrt{1-d^2}) < R < d/(1-\sqrt{1-d^2})$, $t \in \mathbb{R}$.

Proof. Of course, according to (18) and (17), $t = \lambda \circ \rho$ is a strictly increasing function for $d/(1+\sqrt{1-d^2}) < R < d/(1-\sqrt{1-d^2})$, departing vertically from $d/(1+\sqrt{1-d^2})$. Notice that $\lambda(\pi - \rho) = 2\lambda(\pi/2) - \lambda(\rho)$ for $a < \rho < \pi - a$. Hence the graph of $\lambda(\rho)$ is symmetric in the ρt plane about the point $(\rho = \pi/2, t = \lambda(\pi/2))$. But then the graph of $\lambda(\rho)$ is vertical at $\rho = \pi - a$, and therefore $t = \lambda \circ \rho$ arrives vertically at $d/(1-\sqrt{1-d^2})$. Owing to the minimal equation (15) we can extend the graph by vertical reflection at the points $d/(1+\sqrt{1-d^2})$ and $d/(1-\sqrt{1-d^2})$, as in the proof of Theorem 15, to construct a complete embedded curve periodic in the t direction. We obtain a complete minimal immersion in $\mathbb{S}^2 \times \mathbb{R}$. Of course, any such minimal rotational surface is embedded. This completes the proof of the first part of the statement.

In view of formula (18), we conclude that the period $2\mathcal{P}$ of the generating curve is given by

$$\mathcal{P} = d \int_a^{\pi-a} \frac{\sqrt{\ell^2 + \sin^2\rho}}{\sin\rho\sqrt{\sin^2\rho - d^2}} d\rho. \tag{19}$$

Hence

$$B\sqrt{\ell^2 + d^2} \leqslant \mathcal{P} \leqslant B\sqrt{\ell^2 + 1},$$

where

$$B := d \int_a^{\pi-a} \frac{1}{\sin\rho\sqrt{\sin^2\rho - d^2}} d\rho. \tag{20}$$

To complete the proof of the second part of the statement, it suffices to proof the following claim.

Claim. $B = \pi$.

To prove the Claim we will perform various changes of variables to show that (20) is elementary.

$$\begin{aligned}
B &= d \int_{-\cos a}^{\cos a} \frac{dr_1}{(1 - r_1^2)\sqrt{1 - r_1^2 - d^2}} \quad (\text{substituting } r_1 := \cos\rho) \\
&= d \int_{-1}^{1} \frac{dr_2}{(1 - (1 - d^2)r_2^2)\sqrt{1 - r_2^2}} \quad (\text{substituting } r_2 := \frac{r_1}{\sqrt{1 - d^2}}) \\
&= d \int_{-\pi/2}^{\pi/2} \frac{dr_3}{1 - (1 - d^2)\sin^2 r_3} \quad (\text{substituting } r_2 := \sin r_3) \\
&= d \int_{-\infty}^{\infty} \frac{dr_4}{1 + d^2 r_4^2} \quad (\text{substituting } r_4 := \tan r_3) \\
&= \pi.
\end{aligned}$$

In view of (7) and the preceding geometric construction (the periodicity of the generating curve), notice that the last part of the statement follows from the following elementary property of the real numbers: Consider the subgroup Λ of the real number $(\mathbb{R}, +)$ given by $\Lambda := \{2\mathcal{P}k + 2\pi\ell n; k, n \in \mathbb{Z}\}$. There are two possibilities (see [12]): Either the period $2\mathcal{P}$ and $2\pi\ell$ are linearly dependent over $\mathbb{Q}$ and then Λ is isomorphic to $\mathbb{Z}$, or else Λ is dense in $\mathbb{R}$. In the first case the immersion is proper, and in the second case it is dense in the region between the two cylinders, as we asserted in the statement. Note that if $2\pi\ell n = 2\mathcal{P}k$, $n, k \in \mathbb{Z}$, then the screw motion surface "closes" after turning n times around the vertical axis t with self-intersections. $\square$

Generating curve of a minimal screw motion surface in $\mathbb{S}^2 \times \mathbb{R}$:
$\ell = 1, d = \sin(\pi/10)$

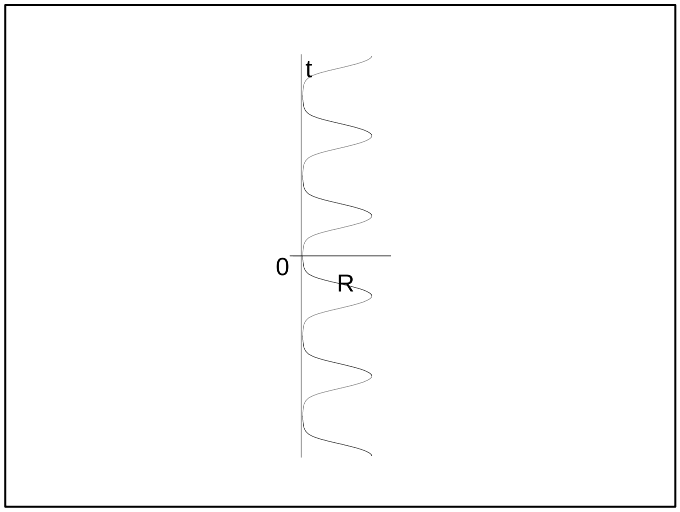

FIGURE 4

Minimal screw motion surface in $\mathbb{S}^2 \times \mathbb{R} : \ell = 1, d = \sin(\pi/10)$

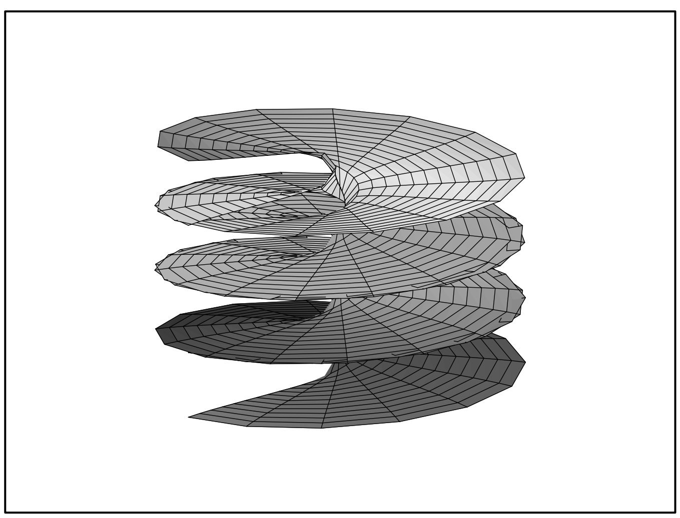

FIGURE 5

5. Complete H–screw motion surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$

In this section we give a geometric description of screw motion surfaces with constant (non-zero) mean curvature H in $\mathbb{H}^2 \times \mathbb{R}$.

Notice that, up to a vertical translation or a symmetry about the xy plane, the first integral $(*)$ yields

$$\lambda(\rho) = \int_*^\rho \frac{(d + 2\cosh r\, H)\sqrt{\ell^2 + \sinh^2 r}}{\sinh r\sqrt{\sinh^2 r - (d + 2\cosh r\, H)^2}} dr. \tag{21}$$

We recall again and we will deduce later (see Proposition 26) that the Abresch-Rosenberg quadratic holomorphic form for rotational H–surfaces vanishes if and only if $d = \pm 2H$.

We will give a full description in the case when $d = -2H$, and we will outline the description when $1 - 4H^2 > 0, H > 0$ and $d = 0$ or $d = 2H$. In the latter situation, we shall state the result, but leave the details to the reader. Notice, however, that these values of d are important. In fact, if $1 < 4H^2$, and if $d^2 = 4H^2, H > 0$, then the Abresch-Rosenberg holomorphic form Q vanishes if and only if $\ell = 0$ (the rotational surfaces). If $1 > 4H^2, H > 0$, and if $d = 0$, then the Abresch-Rosenberg holomorphic form Q does not vanish. We omit the discussion of other values of d.

Theorem 17.

A. *Let $d = -2H, H > 0$. Assume $1 - 4H^2 \geqslant 0$. Then for any pitch ℓ, up to a vertical translation or a vertical reflection, the graph of $t = \lambda \circ \rho$ is strictly increasing and strictly convex for $0 < R < 1$, and $t \to \infty$ (as $R \to 1$). The graph can be extended to a complete embedded curve by odd reflection. Hence we obtain a complete properly immersed screw motion H–surface in $\mathbb{H}^2 \times \mathbb{R}$ that contains the vertical axis t. The tangent to the curve at the origin forms an angle α with the x–axis such that $\tan\alpha = 2H\ell$. As $\ell \to 0$ the screw motion family of surfaces converges to two embedded rotational surfaces, symmetric about the xy plane and fitting together at the origin.*

B. *Let $d = -2H, H > 0$. Assume $1 - 4H^2 < 0$. Then, up to a vertical translation, the graph of $t = \lambda \circ \rho$ is strictly convex for $0 < R < 1/(2H)$, and is vertical at $1/(2H)$. The graph can be extended to a complete embedded periodic curve in the t direction, by vertical reflection at $1/(2H)$ and odd reflection at the origin. We obtain a complete immersed (periodic in the t direction) screw motion H–surface in $\mathbb{H}^2 \times \mathbb{R}$ that contains the vertical axis t. The tangent to the curve at the origin forms an angle α with the x–axis such that $\tan\alpha = 2H\ell$. As $\ell \to 0$ the screw motion family of surfaces converges to a family of closed rotational (genus 0) surfaces fitting together along the vertical axis. In addition, the period $2\mathcal{P}$ of any such screw motion surface* *satisfies the estimate*

$$2\ell\pi < 2\mathcal{P} \leqslant 2\pi\sqrt{\ell^2 + [4H/(4H^2-1)]^2}. \tag{22}$$

If the period $2\mathcal{P}$ and $2\pi\ell$ are linearly dependent over $\mathbb{Q}$, then the immersion is proper. If the period $2\mathcal{P}$ and $2\pi\ell$ are linearly independent over $\mathbb{Q}$, then the immersion is not proper and the surface is dense in the region Ω of $\mathbb{H}^2 \times \mathbb{R}$ (bounded by a cylinder) given by $-1/(2H) < R < 1/(2H)$, $t \in \mathbb{R}$.

C. *Assume $1 - 4H^2 > 0, H > 0$ and $d = 2H$ or $d = 0$. Then the generating curve, up to a vertical translation, is complete embedded in the xt plane, symmetric about the x–axis, and each symmetric part is a graph asymptotic to the vertical line $R = 1$. For $d = 2H$ the curve cuts orthogonally the x–axis at a point $R = 2H$, and for $d = 0$ the curve cuts orthogonally the x–axis at $\left(1 - \sqrt{1-4H^2}\right)/2H$. In any case the part of the curve near these points in the upper half plane $t > 0$ is strictly concave. We obtain therefore a complete properly immersed screw motion H–surface in $\mathbb{H}^2 \times \mathbb{R}$.*

In addition, any such rotational surface is embedded.

Proof. Of course, in view of (21), for $d = -2H, H > 0$, the height $t = \lambda(\rho)$, up to a vertical translation or vertical reflection, is given by

$$\lambda(\rho) = \int_0^\rho \frac{H\sqrt{\ell^2 + \sinh^2 r}}{\cosh(r/2)\sqrt{\cosh^2(r/2)\,(1-4H^2) + 4H^2}}dr. \tag{23}$$

After performing a long, but elementary computation, we obtain

$$\lambda''(\rho) = H\frac{\sinh(\rho/2)\ \left[(1+\cosh\rho)^2 + \cosh\rho\,\ell^2(4H^2-1) - \ell^2\right]}{2\sqrt{\ell^2+\sinh^2\rho}\ \sqrt{\left[4H^2 + \cosh^2(\rho/2)\,(1-4H^2)\right]^3}\ \cosh^2(\rho/2)}$$

and

(24)

$$\frac{d^2\lambda\circ\rho}{dR^2} = H\,\frac{(1+\cosh\rho)\sinh(\rho/2)}{\sqrt{\ell^2+\sinh^2\rho}\ \sqrt{\left[4H^2+\cosh^2(\rho/2)\,(1-4H^2)\right]^3}}\Big[4H^2\,\ell^2 + (1-4H^2)(1+\cosh\rho)\sinh^2\rho + 8H^2\sinh^2\rho + (1+\cosh\rho)^2\Big].$$

Let us first prove Assertion A. Using (23) and (24), in view of (17), if $1-4H^2 \geqslant 0$, we observe that $t = \lambda\circ\rho$ is strictly increasing and strictly convex for $0 < R < 1$, and satisfies $t \to \infty$ (as $R \to 1$). Clearly, in view of (23), we see that the tangent to the curve at the origin forms an angle α with the

x–axis such that $\tan\alpha = 2H\ell$. Now letting $R \to 0$, we infer by a computation that the curvature

$$k(R) := \frac{d^2\lambda\circ\rho/dR^2}{\Big(1+(d\lambda\circ\rho/dR)^2\Big)^{3/2}}$$

goes to 0. We can therefore extend $t = \lambda\circ\rho$ to the interval $(-1, 1)$ by odd reflection. Hence we get a complete curve symmetric about the origin. We obtain a complete properly immersed H–surface that contains the vertical axis t. Clearly, in view of (23), as $\ell \to 0$ the screw motion family of surfaces converge to two rotational surfaces, symmetric about the xy plane and fitting together at the origin.

Next, we prove Assertion B. According to (23) and (24), in view of (17), we observe that if $1 - 4H^2 < 0$, then using the inequality

$$\begin{aligned} 4H^2\,\ell^2 + &(1-4H^2)(1+\cosh\rho)\sinh^2\rho + 8H^2\sinh^2\rho + (1+\cosh\rho)^2 \\ &\geqslant 4H^2\ell^2 + (1+\cosh\rho)^2, \end{aligned}$$

we deduce once again that $t = \lambda\circ\rho$ is strictly increasing and strictly convex for $0 < R < 1/(2H)$, and is vertical at $1/(2H)$.

As before, we see that the graph can be extended to a complete periodic curve in the t direction, by vertical reflection at $1/(2H)$ and odd reflection at the origin. We obtain a complete immersed H–surface that contains the vertical axis t. Again, we see easily that the tangent to the curve at the origin forms an angle α with the x axis such that $\tan\alpha = 2H\ell$, and as $\ell \to 0$ the screw motion family of surfaces converges to a family of closed embedded rotational (genus 0) surfaces fitting together along the vertical axis.

This completes the proof of the first part of the statement.

Observe now that, on account of (23) and the geometric construction performed in the first part of the statement, the period $2\mathcal{P}$ is given by (with $\sinh(a) = 4H/(4H^2-1)$)

$$2\mathcal{P} = 4H\int_0^a \frac{\sqrt{\ell^2+\sinh^2\rho}}{\cosh(\rho/2)\sqrt{\cosh^2(\rho/2)\,(1-4H^2)+4H^2}}\,d\rho. \tag{25}$$

Hence

$$4H\ell\,A \leqslant 2\mathcal{P} \leqslant 4HA\sqrt{\ell^2 + [4H/(4H^2-1)]^2}, \tag{26}$$

where

$$A := \int_0^a \frac{1}{\cosh(\rho/2)\sqrt{\cosh^2(\rho/2)\,(1-4H^2)+4H^2}}\,d\rho.$$

We compute A by making the natural change of coordinates $R = \tanh(\rho/2)$, and obtain

$$
\begin{aligned}
A &= 2\int_0^{1/2H} \frac{1}{\sqrt{1-4R^2H^2}}dR \\
&= \frac{1}{H}\int_0^1 \frac{d\varsigma}{\sqrt{1-\varsigma^2}} \quad (\varsigma = 2HR) \\
&= \frac{\pi}{2H}.
\end{aligned} \tag{27}
$$

Finally, putting together (26) and (27), we derive the estimate (22) in the second part of the statement. Once again, as in Theorem 16, we see that there are two possibilities: Either the period $2\mathcal{P}$ and $2\pi\ell$ are linearly dependent over $\mathbb{Q}$, or else

$$\Lambda := \{2\mathcal{P}k + 2\pi\ell n; k, n \in \mathbb{Z}\}$$

is dense in $\mathbb{R}$. In the first case the immersion is proper, and in the second case it is dense in the region bounded by the cylinder, as asserted in the statement. Of course, any such rotational surface is embedded. This completes the proof of the Theorem. $\square$

Generating curve of H–screw motion surface in $\mathbb{H}^2 \times \mathbb{R}$:
$1 - 4H^2 \geqslant 0$, $d = -2H$, $\ell = 1$

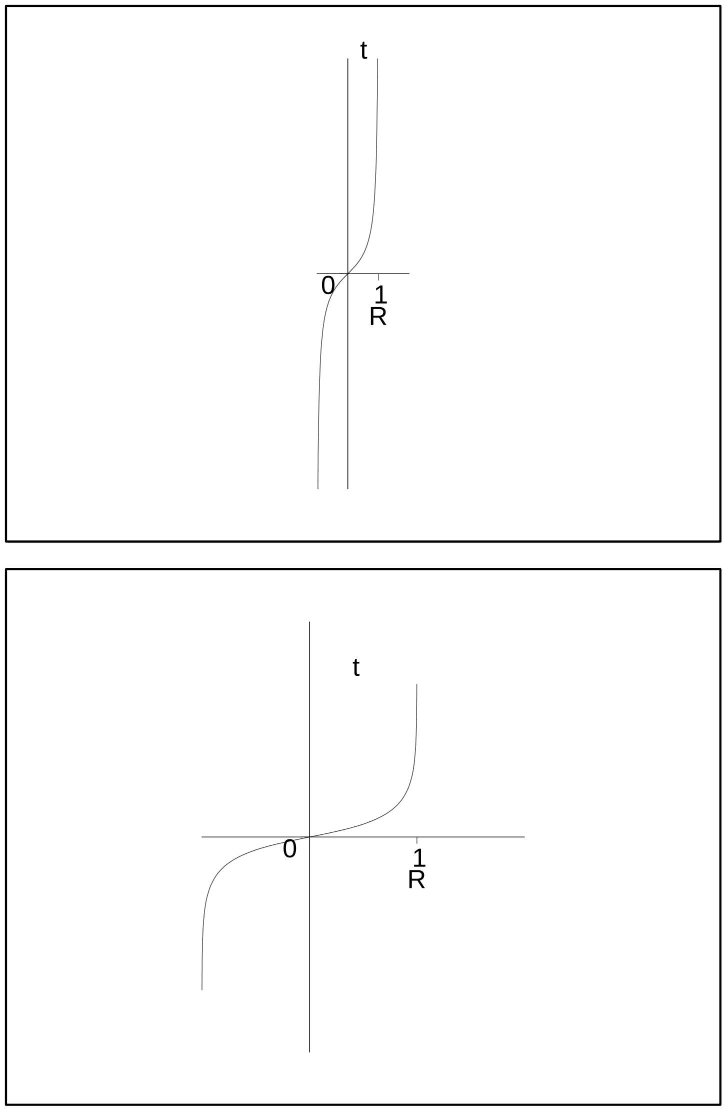

FIGURE 6. $H = 1/2$ and $H = 1/10$ (resp.)

H–screw motion surface in $\mathbb{H}^2 \times \mathbb{R} : 1 - 4H^2 \geqslant 0$, $d = -2H$, $\ell = 1$

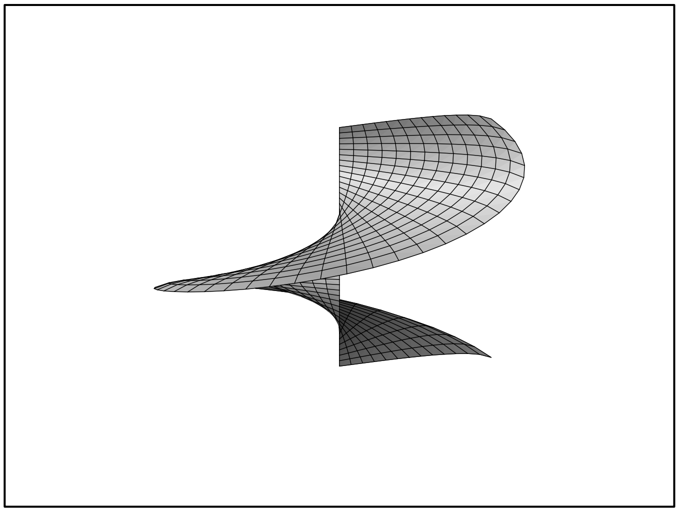

FIGURE 7. $H = 1/10$

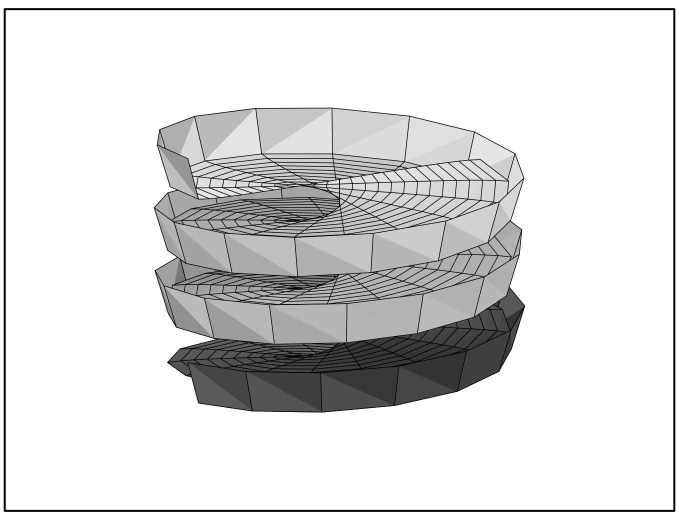

FIGURE 8. $H = 1/10$

Generating curve of H–screw motion surface in $\mathbb{H}^2 \times \mathbb{R}$:
$1 - 4H^2 < 0$, $d = -2H$, $\ell = 1$

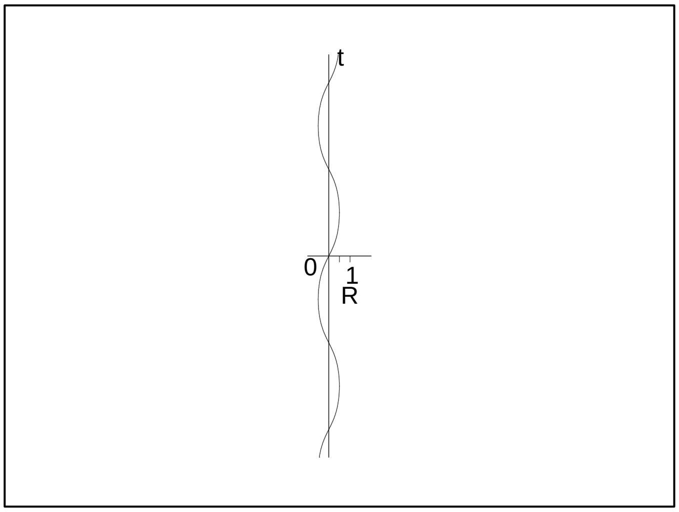

FIGURE 9. $H = 1$

H**–screw motion surface in** $\mathbb{H}^2 \times \mathbb{R} : 1 - 4H^2 < 0$, $d = -2H, \ell = 1$

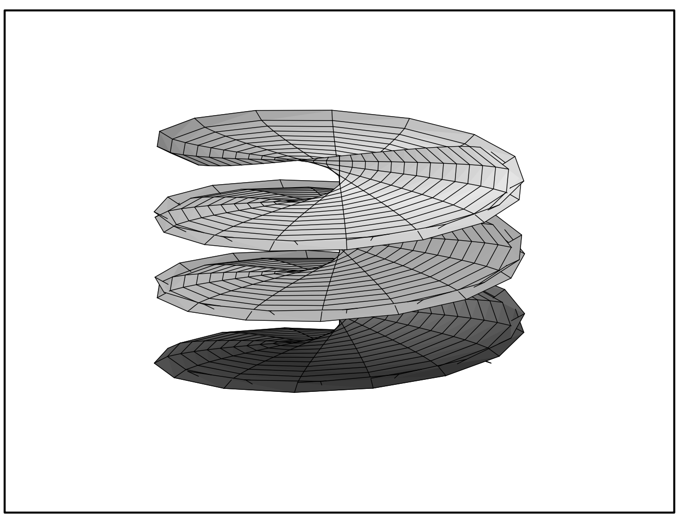

FIGURE 10. $H = 1$

Generating curve of H–screw motion surface in $\mathbb{H}^2 \times \mathbb{R}$:
$1 - 4H^2 > 0$, $d = 0$, $\ell = 1$

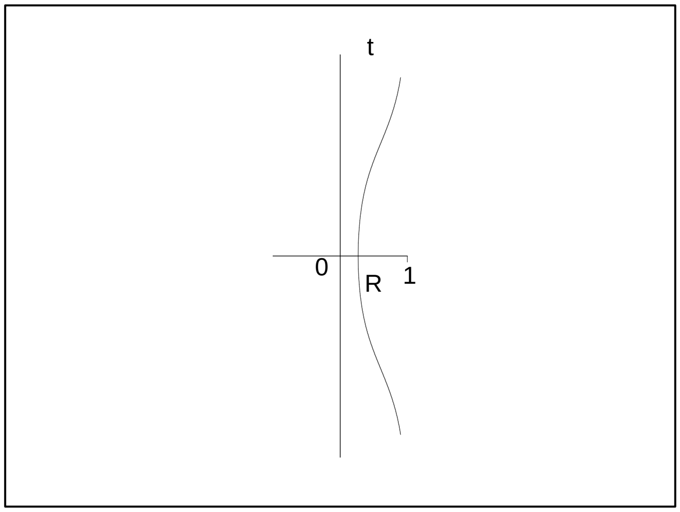

FIGURE 11. $H = 1/4$

H–screw motion surface in $\mathbb{H}^2 \times \mathbb{R} : 1 - 4H^2 > 0$, $d = 0$, $\ell = 1$

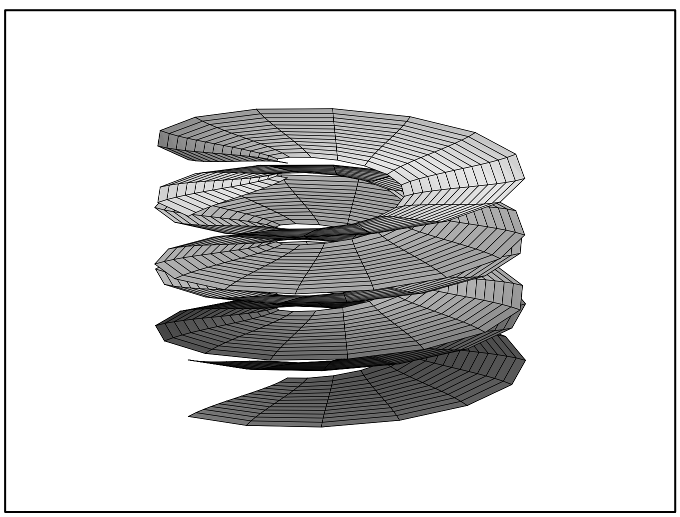

FIGURE 12. $H = 1/4$

On account of Lemma 14, we can give an analogous geometric description for H–screw motion surfaces ($H \neq 0$) immersed in $\mathbb{S}^2 \times \mathbb{R}$.

Again, recall that, up to a vertical translation or a symmetry about the xy plane, the first integral $(**)$ yields

$$\lambda(\rho) = \int_*^\rho \frac{(d - 2\cos r\, H)\sqrt{\ell^2 + \sin^2 r}}{\sin r \sqrt{\sin^2 r - (d - 2\cos r\, H)^2}} dr. \tag{28}$$

THEOREM 18.

A. *Assume $2H < d < 1, H \geqslant 0$. Then for any pitch ℓ, up to a vertical translation or a vertical reflection, the graph of $t = \lambda \circ \rho$ is strictly increasing for $R_1 := \tan(\rho_1/2) < R < R_2 := \tan(\rho_2/2)$, where*

$$\rho_1 = \arccos\left(\frac{\sqrt{1+4H^2-d^2}}{1+4H^2} + \frac{2dH}{1+4H^2}\right),$$

$$\rho_2 = \pi - \arccos\left(\frac{\sqrt{1+4H^2-d^2}}{1+4H^2} - \frac{2dH}{1+4H^2}\right),$$

and is vertical at R_1 and R_2. By extending the graph by vertical symmetry we obtain a complete embedded curve (that is periodic in the t direction). We obtain a complete immersed H–screw motion surface in $\mathbb{S}^2 \times \mathbb{R}$. Any such rotational H–surface is an embedded annulus. The period $2\mathcal{P}$ of any such surface is given by

$$2\mathcal{P} = \int_{\rho_1}^{\rho_2} 2\frac{(d - 2\cos r\, H)\sqrt{\ell^2 + \sin^2 r}}{\sin r \sqrt{\sin^2 r - (d - 2\cos r\, H)^2}} dr. \tag{29}$$

B. *Assume $d = 2H, H > 0$. Then for any pitch ℓ, up to a vertical translation or a vertical reflection, the graph $t = \lambda \circ \rho$ is strictly increasing for $-1/(2H) < R < 1/(2H)$, and vertical at $\pm 1/(2H)$. By extending the graph by vertical symmetry we obtain a complete embedded curve (periodic in the t direction). We obtain a complete immersed H–screw motion surface in $\mathbb{S}^2 \times \mathbb{R}$. Any such rotational H–surface is closed embedded (fitting together along the vertical axis).*

In addition, if the period $2\mathcal{P}$ and $2\pi\ell$ are linearly dependent over $\mathbb{Q}$, then the immersion is proper.

Proof. Let us first proceed to prove Assertion A. We begin with the following computations:

$$\sin^2\rho - (d - 2\cos\rho H)^2$$
$$= (1+4H^2)\left[\left(\frac{\sqrt{1+4H^2-d^2}}{1+4H^2}\right)^2 - \left(\cos\rho - \frac{2dH}{1+4H^2}\right)^2\right].$$

Thus $\sin^2\rho - (d - 2\cos\rho H)^2 = 0$ if

$$\cos\rho = \pm\frac{\sqrt{1+4H^2-d^2}}{1+4H^2} + \frac{2dH}{1+4H^2},$$

and $\sin^2\rho - (d - 2\cos\rho H)^2 > 0$ if

$$-\frac{\sqrt{1+4H^2-d^2}}{1+4H^2} + \frac{2dH}{1+4H^2} < \cos\rho < \frac{\sqrt{1+4H^2-d^2}}{1+4H^2} + \frac{2dH}{1+4H^2}.$$

Notice also that our assumptions imply

$$-1 < -\frac{\sqrt{1+4H^2-d^2}}{1+4H^2} + \frac{2dH}{1+4H^2} < 0,$$
$$\frac{\sqrt{1+4H^2-d^2}}{1+4H^2} + \frac{2dH}{1+4H^2} < 1,$$

and $d - 2\cos\rho H > 0$.

Now, in view of (28), we obtain under the assumptions $2H < d < 1, H \geqslant 0$, the height of $t = \lambda(\rho)$, up to a vertical translation or vertical reflection, is defined for

$$\rho_1 := \arccos\left(\frac{\sqrt{1+4H^2-d^2}}{1+4H^2} + \frac{2dH}{1+4H^2}\right) < \rho < \pi - \arccos\left(\frac{\sqrt{1+4H^2-d^2}}{1+4H^2} - \frac{2dH}{1+4H^2}\right) := \rho_2$$

by

$$\lambda(\rho) = \int_{\rho_1}^{\rho} \frac{(d - 2\cos r\, H)\sqrt{\ell^2 + \sin^2 r}}{\sin r\sqrt{\sin^2 r - (d - 2\cos r\, H)^2}}dr. \tag{30}$$

Of course, the graph is given by an increasing function λ in the interval (ρ_1, ρ_2), departing vertically at $\rho = \rho_1$ and arriving vertically at $\rho = \rho_2$. Again, by applying vertical symmetry, we can extend the graph to obtain a complete embedded curve (periodic in the t direction). We obtain a complete immersed H–screw motion surface in $\mathbb{S}^2 \times \mathbb{R}$. This completes the proof of the first part of the statement. Of course, each rotational H–surface constructed in the first part is embedded.

Now, in light of (30), we deduce that the period $2\mathcal{P}$ of the generating curve is given by

$$2\mathcal{P} = \int_{\rho_1}^{\rho_2} 2\frac{(d - 2\cos r\, H)\sqrt{\ell^2 + \sin^2 r}}{\sin r\sqrt{\sin^2 r - (d - 2\cos r\, H)^2}}dr. \tag{31}$$

This completes the proof of Assertion A.

Let us now prove Assertion B. In light of (28) we see that if $d = 2H$, the height λ up to a vertical translation is defined in the interval

$$-2\arccos\bigl(2H/\sqrt{1+4H^2}\bigr) < \rho < 2\arccos\bigl(2H/\sqrt{1+4H^2}\bigr)$$

by
(32)

$$\lambda(\rho) = \int_{-2\arccos\left(2H/\sqrt{1+4H^2}\right)}^{\rho} H \frac{\sqrt{\ell^2 + \sin^2 r}}{\cos(r/2)\sqrt{\cos^2(r/2)(1+4H^2) - 4H^2}} dr.$$

Hence the graph is vertical at $\rho = -2\arccos\bigl(2H/\sqrt{1+4H^2}\bigr)$ and at $\rho = 2\arccos\bigl(2H/\sqrt{1+4H^2}\bigr)$ and is increasing in the interval between these two points. As in our earlier constructions, we apply vertical symmetry to build a complete embedded curve (periodic in the t direction). We obtain a complete immersed H–screw motion surface in $\mathbb{S}^2 \times \mathbb{R}$. Of course, any such rotational surface is embedded. Once again, as in Theorem 16, if the period $2\mathcal{P}$ and $2\pi\ell$ are linearly dependent over $\mathbb{Q}$, then the immersion is proper. This completes the proof of the theorem. $\square$

Generating curve of H–screw motion surface in $\mathbb{S}^2 \times \mathbb{R}$:
$\ell = 1$, $H = 1/20$, $d = 1/2$.

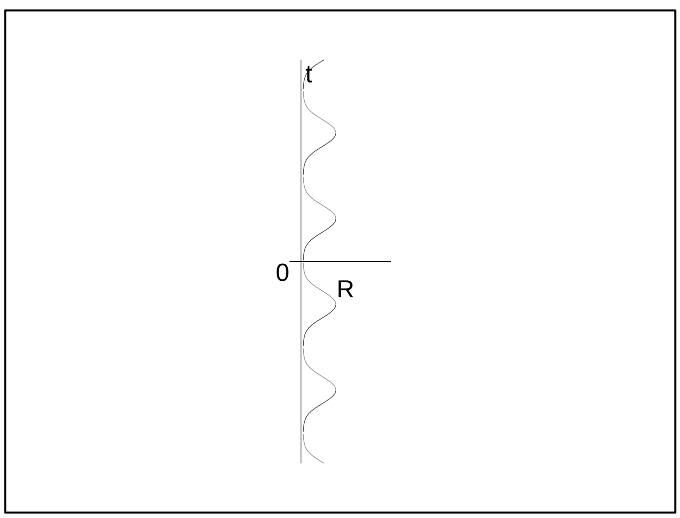

Figure 13

H–screw motion surface in $\mathbb{S}^2 \times \mathbb{R} : \ell = 1,\ H = 1/20,\ d = 1/2$

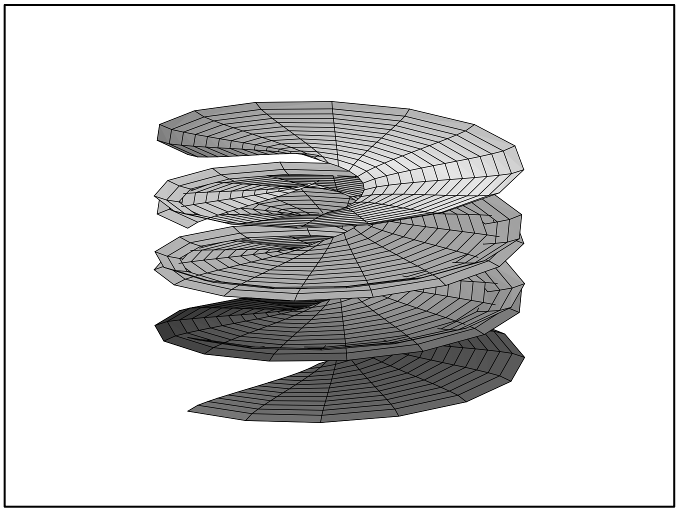

FIGURE 14

REMARK. Observe that the period of the H–screw motion surface encountered in Theorem 18 (**B**) is given by

$$2\mathcal{P} = \int_{-2\arccos\left(2H/\sqrt{1+4H^2}\right)}^{2\arccos\left(2H/\sqrt{1+4H^2}\right)} 2H\,\frac{\sqrt{\ell^2+\sin^2\rho}}{\cos(\rho/2)\sqrt{\cos^2(\rho/2)(1+4H^2)-4H^2}}d\rho.$$

6. Isometric screw motion immersions in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$ of constant mean or Gauss curvature

We first examine screw motion surfaces immersed in $\mathbb{H}^2 \times \mathbb{R}$. Notice that the induced metric $d\mu^2$, in cylindrical coordinates, is given by

$$d\mu^2 = (1+\lambda'^2)d\rho^2 + 2\ell\lambda' d\rho d\varphi + (\ell^2+\sinh^2\rho)d\varphi^2.$$

The idea, originating with Bour in the 19th century [3], improved by Do Carmo-Dajczer [4] in Euclidean space, and developed by Ordóñes in space form in his doctoral thesis at PUC-Rio [10], works nicely in our context. We will find natural coordinates s, τ, where s is the arc length of a geodesic in the surface, orthogonal to the helices (orbits), such that the metric is given by

$$d\mu^2 = ds^2 + U^2(s)d\tau^2, \tag{33}$$

where $U(s) > 0$ is a smooth function. More precisely, we have the following result.

THEOREM 19 (Bour's Lemma in $\mathbb{H}^2 \times \mathbb{R}$). *Any surface that is invariant by screw motions can be parametrized locally by natural coordinates s, τ. Let S be such a screw motion surface with pitch $\ell \neq 0$ in $\mathbb{H}^2 \times \mathbb{R}$. Then there exists a two parameter family $\mathcal{F}(m, \ell), m \neq 0$, of screw motion surfaces isometric to S, with contains a surface of revolution, given by*

$$m^2 U^2(s) = \ell^2 + \sinh^2 \rho(s), \tag{34}$$

$$1 - \rho'^2(s) = \frac{\lambda'^2 \sinh^2 \rho}{\ell^2 + \sinh^2 \rho + \lambda'^2 \sinh^2 \rho}, \tag{35}$$

$$\rho(s) = \int \sqrt{\frac{m^4 U^2 U'^2}{(m^2 U^2 - \ell^2)(m^2 U^2 - \ell^2 + 1)}}\, ds, \tag{36}$$

$$\lambda \circ \rho(s) = \int \sqrt{\frac{(m^2U^2 - \ell^2 + 1)(m^2U^2 - \ell^2) - m^4 U^2 U'^2}{m^2 U^2 - \ell^2 + 1}} \times \times \frac{mU}{m^2U^2 - \ell^2}\, ds, \tag{37}$$

$$\varphi(s, \tau) = \frac{\tau}{m} - \ell \int \sqrt{\frac{(m^2U^2 - \ell^2 + 1)(m^2U^2 - \ell^2) - m^4 U^2 U'^2}{m^2 U^2 - \ell^2 + 1}} \times \times \frac{1}{mU(m^2U^2 - \ell^2)}\, ds. \tag{38}$$

In addition, given $a^2 \leqslant 1$, if $m^2 a^2 > \ell^2$ and $\ell^2 \leqslant 1$, there exists a family of complete isometric screw motion immersions with pitch ℓ and constant Gauss curvature $K = -a^2$.

Proof. In view of (2) the induced metric $d\mu^2$ of a given screw motion surface, say S, immersed in $\mathbb{H}^2 \times \mathbb{R}$, is given by

$$d\mu^2 = \left(1 + \lambda'^2\right) d\rho^2 + 2\ell\, \lambda' d\rho d\varphi + \left(\ell^2 + \sinh^2 \rho\right) d\varphi^2. \tag{39}$$

Write (39) in the form

$$d\mu^2 = \underbrace{\left(1 + \frac{\lambda'^2 \sinh^2 \rho}{\ell^2 + \sinh^2 \rho}\right) d\rho^2}_{ds^2} + \underbrace{\left(\ell^2 + \sinh^2 \rho\right)}_{U^2} \underbrace{\left(d\varphi + \frac{\ell}{\ell^2 + \sinh^2 \rho} d\lambda \circ \rho\right)^2}_{d\tau^2}. \tag{40}$$

We can therefore define (locally) natural coordinates s, τ with the aid of the Inverse Function Theorem by the following equations:

$$ds = \sqrt{1 + \frac{\lambda'^2 \sinh^2 \rho}{\ell^2 + \sinh^2 \rho}}\, d\rho, \tag{41}$$

$$d\tau = d\varphi + \frac{\ell \lambda'}{\ell^2 + \sinh^2 \rho} d\rho. \tag{42}$$

Next, we search for an explicit parametrization of an arbitrary screw motion surface (with pitch denoted by ℓ for convenience) isometric to S, by natural coordinates s, τ, involving a simple expression in terms of U and the parameters ℓ, m as in the statement. We recall that the metric is given by (33). Notice that we may suppose $U > 0$, since we assume for the moment $\rho > 0$ (the extension to $\rho = 0$ or to negative values requires an additional argument).

Clearly, in view of (41), any screw motion surface satisfies that ρ (and, of course, λ) does not depend on τ. Hence $\rho = \rho(s), U = U(s)$. Again, in view of (41), we infer (35) as claimed in the statement, which we rewrite here for future reference as follows:

$$\rho'^2 = \frac{\ell^2 + \sinh^2 \rho}{\ell^2 + \sinh^2 \rho + \lambda'^2 \sinh^2 \rho}. \tag{43}$$

Now, on account of (40), we must have

$$U d\tau = \pm\sqrt{\ell^2 + \sinh^2 \rho}\, \left(d\varphi + \frac{\ell}{\ell^2 + \sinh^2 \rho} d\lambda{\circ}\rho \right).$$

Consequently,

$$\frac{\partial \varphi}{\partial \tau} = \pm \frac{U}{\sqrt{\ell^2 + \sinh^2 \rho}}, \tag{44}$$

$$\frac{\partial \varphi}{\partial s} = \frac{-\ell}{\ell^2 + \sinh^2 \rho} \frac{d\lambda{\circ}\rho}{ds}. \tag{45}$$

In view of (44) and (45), we deduce that $\partial\varphi/\partial\tau$ does not depend on s. Hence we obtain the important formula

$$\pm \frac{U}{\sqrt{\ell^2 + \sinh^2 \rho}} = \frac{1}{m}, \quad m \neq 0. \tag{46}$$

This gives (34) in the statement.

Next, we differentiate (34) with respect to s and deduce, after some calculation,

$$\rho'^2 = \frac{m^4 U^2 U'^2}{(m^2U^2 - \ell^2)\,(m^2U^2 - \ell^2 + 1)}. \tag{47}$$

This proves (36) in the statement.

Employing (47), in view of (43), we infer, after some simple operations, a formula for $(\lambda{\circ}\rho)'(s)$ which proves (37) in the statement. Combining (44), (45) and (37) we finally obtain (38).

Now observe that if the formulas (36), (37) and (38) hold for some pitch $\ell \geqslant 0$, then they also hold for any pitch $\widetilde{\ell}$ in the interval $[0, \ell]$, since $(m^2U^2 - \ell^2 + 1)(m^2U^2 - \ell^2) - m^4U^2U'^2 \geqslant 0$. We obtain thus a family $\mathcal{F}(m, \ell), m \neq 0$, of surfaces isometric to S containing a surface of revolution ($\ell = 0$). This completes the proof of the first part of the statement.

Next, we proceed to prove the second part of the statement. Recalling our natural conformal coordinates $w = v + i\tau$, where $v = \int(1/U)ds$, we infer easily that the Gaussian curvature K of a screw motion immersion is given by

$$K = \frac{-U''}{U}. \tag{48}$$

An elementary calculation shows that if $K = -a^2, 0 < a^2 \leqslant 1$, taking a solution of (48) of the form $U = \cosh as$, if $m^2a^2 > \ell^2$ and $\ell \leqslant 1$, then $m^2U^2 - \ell^2 > 0$ and $(m^2U^2 - \ell^2 + 1)(m^2U^2 - \ell^2) - m^4U^2U'^2 > 0$ for all s. Therefore, we can define a complete screw motion surface with prescribed Gaussian curvature $K = -a^2$, with the aid of (36), (37) and (38). This completes the proof of the theorem. □

Once again, we can derive an analogous result for the case when the ambient space is $\mathbb{S}^2 \times \mathbb{R}$.

THEOREM 20 (Bour's Lemma in $\mathbb{S}^2 \times \mathbb{R}$). *Any surface that is invariant by screw motions can be parametrized locally by natural coordinates s, τ. Let S be a screw motion surface in $\mathbb{S}^2 \times \mathbb{R}$. Then there exists a two parameter family $\mathcal{F}(m, \ell), m \neq 0$, of screw motion surfaces isometric to S, given by*

$$\begin{aligned}
m^2U^2(s) &= \ell^2 + \sin^2\rho(s),\\
1 - \rho'^2(s) &= \frac{\lambda'^2 \sin^2\rho}{\ell^2 + \sin^2\rho + \lambda'^2\sin^2\rho},\\
\rho &= \int\sqrt{\frac{m^4U^2U'^2}{(m^2U^2 - \ell^2)(1 - m^2U^2 + \ell^2)}}\,ds,\\
\lambda\circ\rho(s) &= \int\sqrt{\frac{(1 - m^2U^2 + \ell^2)(m^2U^2 - \ell^2) - m^4U^2U'^2}{1 - m^2U^2 + \ell^2}}\times\\
&\quad\times\frac{mU}{m^2U^2 - \ell^2}\,ds,\\
\varphi(s,\tau) &= \frac{\tau}{m} - \ell\int\sqrt{\frac{(1 - m^2U^2 + \ell^2)(m^2U^2 - \ell^2) - m^4U^2U'^2}{1 - m^2U^2 + \ell^2}}\times\\
&\quad\times\frac{1}{mU(m^2U^2 - \ell^2)}\,ds.
\end{aligned}$$

Proof. The proof is analogous to that of Theorem 19, replacing $\sinh\rho$ by $\sin\rho$ and $\cosh\rho$ by $\cos\rho$. □

We now use Theorem 19 to determine all isometric immersions (with the same U) in $\mathbb{H}^2 \times \mathbb{R}$ that are invariant by screw motions and have the same constant mean curvature H.

Corollary 21. *Let S be a H–screw motion surface in $\mathbb{H}^2\times\mathbb{R}$ parametrized by natural coordinates s,τ.*

(i) *If $1-4H^2>0$, then*

$$(m^2U^2-\ell^2+1)^{1/2}=\frac{\sqrt{d^2+1-4H^2}}{1-4H^2}\cdot\cosh\left(\sqrt{1-4H^2}\,(s-s_0)\right)+\frac{2dH}{1-4H^2}. \tag{49}$$

(ii) *If $H=1/2$ (necessarily $d<0$), then*

$$(m^2U^2-\ell^2+1)^{1/2}=\left(\sqrt{\frac{-d}{2}}\,(s-s_0)\right)^2-\frac{d^2+1}{2d}. \tag{50}$$

(iii) *If $1-4H^2<0$ (necessarily $d^2+1-4H^2>0$), then*

$$(m^2U^2-\ell^2+1)^{1/2}=\frac{\sqrt{d^2+1-4H^2}}{4H^2-1}\cos\left(\sqrt{4H^2-1}\,(s-s_0)\right)+\frac{2dH}{1-4H^2}. \tag{51}$$

Proof. From the calculations in the proof of Theorem 19 we see that the data of a screw motion immersion in $\mathbb{H}^2\times\mathbb{R}$ can be written using elementary operations in terms of U,ℓ,m. Inserting these data in the first integral formula $(*)$, we deduce the following first order ordinary differential equation:

$$\frac{(m^2U^2-\ell^2+1)\,(m^2U^2-\ell^2)-m^4U^2U'^2}{m^2U^2-\ell^2+1}=(2\cosh\rho H+d)^2\,. \tag{52}$$

Changing variables $Q:=\left(m^2U^2-\ell^2+1\right)^{1/2}$, (52) becomes

$$Q'^2=Q^2(1-4H^2)-4dHQ-(d^2+1). \tag{53}$$

Now (53) can be solved separately by elementary differential equations techniques if $1-4H^2>0$, $H=1/2$ ($d<0$) and $1-4H^2<0$ ($d^2+1-4H^2>0$). This gives (49), (50), and (51), respectively, and completes the proof of the corollary. □

Next, we use Theorem 20 to describe all isometric immersions (with the same U) in $\mathbb{S}^2\times\mathbb{R}$ that are invariant by screw motions and have the same mean curvature H.

Corollary 22. *Let S be a screw motion immersion in $\mathbb{S}^2\times\mathbb{R}$ parametrized by natural coordinates s,τ. Then (necessarily $1+4H^2-d^2>0$)*

$$1-m^2U^2+\ell^2=\left(\frac{\sqrt{1+4H^2-d^2}}{1+4H^2}\cos\left(\sqrt{1+4H^2}\,(s-s_0)\right)+\frac{2dH}{1+4H^2}\right)^2. \tag{54}$$

Proof. Combining the first integral formula $(**)$ with the results derived in Theorem 20 we infer

$$\frac{\left(1 - m^2U^2 + \ell^2\right)\left(m^2U^2 - \ell^2\right) - m^4U^2U'^2}{1 - m^2U^2 + \ell^2} = \left(d - 2\cos\rho H\right)^2. \tag{55}$$

A change of variables as in the proof of Corollary 21 and elementary differential equations techniques yields (54). This completes the proof of the corollary. □

Recall that taking $\upsilon = \int (1/U)ds$ we have natural isothermal parameters $w = \upsilon + i\tau$ for surfaces that are invariant by screw motions. Hence we may compute the Hopf function ϕ for such minimal immersions. We have therefore:

PROPOSITION 23. *Let $X : \Omega \subset \mathbb{C} \looparrowright \mathbb{H}^2 \times \mathbb{R}$ be a screw motion minimal immersion parametrized by natural conformal coordinates $w = \upsilon + i\tau$. The holomorphic Hopf function ϕ satisfies*

$$4\,\Re\phi = \frac{\ell^2}{m^2} - \frac{1}{m^2} \cdot \frac{\sinh^4\rho\,\lambda'^2}{\ell^2 + \sinh^2\rho + \lambda'^2\sinh^2\rho}. \tag{56}$$

The imaginary part for a rotational surface is zero. If $\ell \neq 0$, then the holomorphic Hopf function satisfies

$$4\,\Re\phi = \frac{\ell^2}{m^2} - \frac{4m^2}{\ell^2}(\Im\phi)^2. \tag{57}$$

If $d \geqslant 0$, then the Hopf function is constant and is given by

$$4\phi = \frac{\ell^2 - d^2}{m^2} + i\frac{2\ell}{m^2}d. \tag{58}$$

Proof. A straightforward computation shows that the Hopf function ϕ (see (13)) is given by

$$4\phi = \rho'^2U^2 - \sinh^2\rho\left(\varphi_\tau + iU\varphi_s\right)^2. \tag{59}$$

Inserting (34), (43), (44), (45), (46) in the imaginary part of (59), and inserting in the real part of (59), we deduce

$$\begin{aligned} 4\left(\Im\phi\right)^2 &= \frac{\ell^2}{m^4} \cdot \frac{\sinh^4\rho\,\lambda'^2}{\ell^2 + \sinh^2\rho + \lambda'^2\sinh^2\rho}, \\ 4m^2\,\Re\phi &= \ell^2 - \frac{\lambda'^2\,\sinh^4\rho}{\ell^2 + \sinh^2\rho + \lambda'^2\,\sinh^2\rho}. \end{aligned} \tag{60}$$

In view of (60), by the first integral formula $(*)$ and since $H = 0$, we infer (56), (57) and(58). This completes the proof of the proposition. □

PROPOSITION 24. *Let $X : \Omega \subset \mathbb{C} \looparrowright \mathbb{S}^2 \times \mathbb{R}$ be a conformal screw motion minimal immersion. The holomorphic Hopf function ϕ satisfies*

$$4\,\Re\phi = \frac{\ell^2}{m^2} - \frac{1}{m^2} \cdot \frac{\sin^4\rho\,\lambda'^2}{\ell^2 + \sin^2\rho + \lambda'^2\sin^2\rho}.$$

The imaginary part for a rotational surface is zero. If $\ell \neq 0$, then the holomorphic Hopf function satisfies

$$4\,\Re\phi = \frac{\ell^2}{m^2} - \frac{4m^2}{\ell^2}(\Im\phi)^2.$$

If $d \geqslant 0$, then the Hopf function is constant and is given by

$$4\phi = \frac{\ell^2 - d^2}{m^2} + i\frac{2\ell}{m^2}d.$$

Proof. The proof is analogous to that of Proposition 23, using Theorem 20 and the first integral formula $(**)$ and replacing $\sinh\rho$ by $\sin\rho$ and $\cosh\rho$ by $\cos\rho$. □

COROLLARY 25. *Any two isometric screw motion minimal immersions in $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{S}^2 \times \mathbb{R}$ are associate. When the ambient space is $\mathbb{H}^2 \times \mathbb{R}$, the catenoid is conjugate to a helicoid of pitch $\ell < 1$. A helicoid is conjugate to a catenoid if and only if $\ell < 1$.*

Proof. Observe that in view of Corollary 21 any family of minimal isometric screw motion surfaces in $\mathbb{H}^2 \times \mathbb{R}$ must satisfy

$$m^2U^2 = \ell^2 + d^2 - \frac{(d^2+1)}{2}\Big(1 - \cosh(2(s-s_0))\Big).$$

Each surface of such a family has the same absolute value of the Hopf function, and hence they are associate. Indeed, according to Proposition 23 the related Hopf function is constant and is given by

$$4\phi = \frac{\ell^2 - d^2}{m^2} + i\frac{2\ell}{m^2}d.$$

In view of these last two formulas, it is readily seen that the conjugate surface to a catenoid (surfaces of revolution) is a helicoid with pitch less than 1. But the helicoid with pitch greater than or equal 1 is not conjugate to a catenoid. On the other hand, owing to Corollary 22, we have that any family of isometric minimal screw motion surfaces in $\mathbb{S}^2 \times \mathbb{R}$ satisfies

$$m^2U^2 = \ell^2 + d^2 - \frac{(d^2-1)}{2}\Big(1 - \cos(2(s-s_0))\Big).$$

The related Hopf function is the same as the (constant) preceding function

$$4\phi = \frac{\ell^2 - d^2}{m^2} + i\frac{2\ell}{m^2}d.$$

Hence they are associate as well. □

REMARK. Notice that Proposition 23 and Proposition 24 suggest an alternative approach to the first integral formulas $(*)$ and $(**)$ for minimal immersions.

We now will give an explicit formula, in our context, for the Abresch-Rosenberg holomorphic quadratic differential Q (see [1]).

Proposition 26. *Let $X(\upsilon, \tau)$ be a screw motion H–surface immersed in $\mathbb{H}^2 \times \mathbb{R}$, where s, τ are natural coordinates given in Theorem 19 and $\upsilon = \int (1/U) ds$. Then the Abresch-Rosenberg holomorphic function satisfies*

$$\begin{aligned}(61)\qquad \frac{1}{2} \cdot \Big[Q(X_\upsilon, X_\upsilon) - Q(X_\tau, X_\tau) \Big] - iQ(X_\upsilon, X_\tau)& \\ = \frac{1}{m^2} \left(d^2 - 4H^2 + \ell^2(4H^2 - 1) \right) - i\frac{2\ell}{m^2} d.&\end{aligned}$$

Proof. After a lengthy computation, using Theorem 19 and (3), (4), (5), (6), we deduce

$$\begin{aligned}\frac{1}{2} \cdot \Big[Q(X_\upsilon, X_\upsilon) - Q(X_\tau, X_\tau) \Big] - iQ(X_\upsilon, X_\tau)& \\ = -\frac{2H}{mU} \frac{\lambda' \rho' \cosh\rho \sinh^2\rho}{m^2} - \frac{\ell^2}{m^2} + 2H\ell^2 \frac{\lambda'\rho'\cosh\rho}{m^3 U}(1 + \rho'^2)& \\ + \frac{2HU}{m} \lambda'' \rho'^3 \sinh\rho + (1 - \rho'^2) \frac{\sinh^2\rho}{m^2}& \\ - \frac{2\ell}{m^2} i \left(\sqrt{1 - \rho'^2} \sinh\rho - 2H\cosh\rho \right).&\end{aligned}$$

Now, inserting the value of λ'' given by the mean curvature equation (14) and using (43) and the first integral formula, we obtain the formula (61) in the statement. This completes the proof of the proposition. $\square$

Proposition 27. *Let $X(\upsilon, \tau)$ be a screw motion H–surface immersed in $\mathbb{S}^2 \times \mathbb{R}$, where s, τ are natural coordinates given in Theorem 19 and $\upsilon = \int (1/U) ds$. Then the Abresch-Rosenberg holomorphic function satisfies*

$$\begin{aligned}(62)\qquad \frac{1}{2} \cdot \Big[Q(X_\upsilon, X_\upsilon) - Q(X_\tau, X_\tau) \Big] - iQ(X_\upsilon, X_\tau)& \\ = \frac{1}{m^2} \left(4H^2 - d^2 + \ell^2(1 + 4H^2) \right) + i\frac{2\ell}{m^2} d.&\end{aligned}$$

Proof. Using Theorem 20, (8), (9), (10) and (11), we obtain

$$\begin{aligned}\frac{1}{2} \cdot \Big[Q(X_\upsilon, X_\upsilon) - Q(X_\tau, X_\tau) \Big] - iQ(X_\upsilon, X_\tau)& \\ = -\frac{2H}{mU} \frac{\lambda' \rho' \cos\rho \sin^2\rho}{m^2} + \frac{\ell^2}{m^2} + 2H\ell^2 \frac{\lambda'\rho'\cos\rho}{m^3 U}(1 + \rho'^2)& \\ + \frac{2HU}{m} \lambda'' \rho'^3 \sin\rho - (1 - \rho'^2) \frac{\sin^2\rho}{m^2}& \\ + \frac{2\ell}{m^2} i \left(\sqrt{1 - \rho'^2} \sin\rho + 2H\cos\rho \right).&\end{aligned}$$

Finally, observe that (62) can be obtained analogously to (61), with obvious modifications. □

Remark. Notice that Proposition 26 and Proposition 27 give an alternative approach to the first integral formulas $(*)$ and $(**)$ for H–surfaces.

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