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ON THE DIMENSION OF THE STABILITY GROUP FOR A LEVI NON-DEGENERATE HYPERSURFACE

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Abstract. We classify locally defined Levi non-degenerate non-spherical real-analytic hypersurfaces in complex space for which the dimension of the group of local CR-automorphisms has the second largest positive value.

1. Introduction

Let M be a real-analytic hypersurface in $\mathbb{C}^{n+1}$ passing through the origin. Assume that the Levi form of M at 0 is non-degenerate and has signature $(n-m,m)$ with $n \geq 2m$. Then in some local holomorphic coordinates $z = (z_1,\dots,z_n)$, $w = u+iv$ in a neighborhood of the origin, M can be written in the Chern-Moser normal form (see [CM]), that is, given by an equation

$$v = \langle z,z\rangle + \sum_{k,\bar{l}\geq 2} F_{k\bar{l}}(z,\overline{z},u),$$

where $\langle z,z\rangle = \sum_{\alpha,\beta=1}^{n} h_{\alpha\beta} z_\alpha \overline{z_\beta}$ is a non-degenerate Hermitian form with signature $(n-m,m)$, and $F_{k\bar{l}}(z,\overline{z},u)$ are polynomials of degree k in z and $\bar{l}$ in $\overline{z}$ whose coefficients are analytic functions of u such that the following conditions hold:

$$\begin{array}{rcl} \mathrm{tr}\, F_{2\bar{2}} & \equiv & 0, \\ \mathrm{tr}^2\, F_{2\bar{3}} & \equiv & 0, \\ \mathrm{tr}^3\, F_{3\bar{3}} & \equiv & 0. \end{array} \tag{1.1}$$

Here the operator tr is defined as

$$\mathrm{tr} := \sum_{\alpha,\beta=1}^{n} \hat{h}_{\alpha\beta} \frac{\partial^2}{\partial z_\alpha \partial \overline{z}_\beta},$$

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where $(\hat{h}_{\alpha\beta})$ is the matrix inverse to $H := (h_{\alpha\beta})$. Everywhere below we assume that M is given in the normal form.

Let $\mathrm{Aut}_0(M)$ denote the group of all local CR-automorphisms of M defined near 0 and preserving 0. To avoid confusion with the term "isotropy group of M at 0" usually reserved for global CR-automorphisms of M preserving the origin, this group is often called the *stability group* of M at 0. Every element φ of $\mathrm{Aut}_0(M)$ extends to a biholomorphic mapping defined in a neighborhood of the origin in $\mathbb{C}^{n+1}$ and therefore can be written as

$$\begin{array}{rcl} z & \mapsto & f_\varphi(z,w), \\ w & \mapsto & g_\varphi(z,w), \end{array}$$

where f_φ and g_φ are holomorphic. We equip $\mathrm{Aut}_0(M)$ with the topology of uniform convergence of the partial derivatives of all orders of the component functions on a neighborhood of 0 in M. The group $\mathrm{Aut}_0(M)$ with this topology is a topological group.

It follows from [CM] that every element $\varphi = (f_\varphi, g_\varphi)$ of $\mathrm{Aut}_0(M)$ is uniquely determined by a set of parameters $(U_\varphi, a_\varphi, \lambda_\varphi, \sigma_\varphi, r_\varphi)$, where $\sigma_\varphi = \pm 1$, U_φ is an $n \times n$-matrix such that $\langle U_\varphi z, U_\varphi z\rangle = \sigma_\varphi \langle z, z\rangle$ for all $z \in \mathbb{C}^n$, $a_\varphi \in \mathbb{C}^n$, $\lambda_\varphi > 0$, $r_\varphi \in \mathbb{R}$ (note that σ_φ can be equal to -1 only for $n = 2m$). These parameters are determined by the following relations:

$$\frac{\partial f_\varphi}{\partial z}(0) = \lambda_\varphi U_\varphi, \quad \frac{\partial f_\varphi}{\partial w}(0) = \lambda_\varphi U_\varphi a_\varphi,$$

$$\frac{\partial g_\varphi}{\partial w}(0) = \sigma_\varphi \lambda_\varphi^2, \quad \mathrm{Re}\, \frac{\partial^2 g_\varphi}{\partial^2 w}(0) = 2\sigma_\varphi \lambda_\varphi^2 r_\varphi.$$

For results on the dependence of local CR-mappings on their jets in more general settings see [BER1], [BER2], [Eb], [Z].

We assume that M is *non-spherical at the origin*, i.e., that M in a neighborhood of the origin is not CR-equivalent to an open subset of the hyperquadric given by the equation $v = \langle z, z\rangle$. In this case for every element $\varphi = (f_\varphi, g_\varphi)$ of $\mathrm{Aut}_0(M)$ the parameters $a_\varphi, \lambda_\varphi, \sigma_\varphi, r_\varphi$ are uniquely determined by the matrix U_φ, and the mapping

$$\Phi :\ \mathrm{Aut}_0(M) \to GL_n(\mathbb{C}), \qquad \Phi :\ \varphi \mapsto U_\varphi$$

is a topological group isomorphism between $\mathrm{Aut}_0(M)$ and $G_0 := \Phi(\mathrm{Aut}_0(M))$ with G_0 being a real algebraic subgroup of $GL_n(\mathbb{C})$; in addition the mapping

$$\Lambda :\ G_0(M) \to \mathbb{R}_+, \qquad \Lambda :\ U_\varphi \mapsto \lambda_\varphi \tag{1.2}$$

is a Lie group homomorphism with the property $\Lambda(U_\varphi) = 1$ if all eigenvalues of U_φ are unimodular, where $\mathbb{R}_+$ is the group of positive real numbers with respect to multiplication (see [CM], [B], [L1], [BV], [VK]). Since $G_0(M)$ is a closed subgroup of $GL_n(\mathbb{C})$, we can pull back its Lie group structure to $\mathrm{Aut}_0(M)$ by means of Φ (note that the pulled back topology is identical to that of $\mathrm{Aut}_0(M)$). Let $d_0(M)$ denote the dimension of $\mathrm{Aut}_0(M)$. We are interested in characterizing hypersurfaces for which $d_0(M)$ is large.

If $n > 2m$, $G_0(M)$ is a closed subgroup of the pseudounitary group $U(n-m, m)$ of all matrices U such that

$$U^t H \overline{U} = H,$$

(recall that H is the matrix of the Hermitian form $\langle z, z\rangle$). The group $U(n,0)$ is the unitary group $U(n)$. If $n = 2m$, G_0 is a closed subgroup of the group $U'(m,m)$ of all matrices U such that

$$U^t H \overline{U} = \pm H,$$

that has two connected components. In particular, we always have $d_0(M) \leq n^2$. If $d_0(M) = n^2$ and $n > 2m$, then $G_0(M) = U(n-m, m)$. If $d_0(M) = n^2$ and $n = 2m$, then we have either $G_0(M) = U(m,m)$, or $G_0(M) = U'(m,m)$.

We will say that the group $\mathrm{Aut}_0(M)$ is *linearizable*, if in some coordinates every $\varphi \in \mathrm{Aut}_0(M)$ can be written in the form

$$\begin{array}{rcl} z & \mapsto & \lambda U z, \\ w & \mapsto & \sigma \lambda^2 w. \end{array} \tag{1.3}$$

Clearly, in the above formula $U = U_\varphi$, $\lambda = \lambda_\varphi$, $\sigma = \sigma_\varphi$. The group $\mathrm{Aut}_0(M)$ is known to be linearizable, for example, for $m = 0$ (see [KL]) and for $m = 1$ (see [Ezh1], [Ezh2]). If all elements of $\mathrm{Aut}_0(M)$ in some coordinates have the form (1.3), we say that $\mathrm{Aut}_0(M)$ is *linear* in these coordinates. It is shown in Lemma 3 of [Ezh3] that if $\mathrm{Aut}_0(M)$ is linear in some coordinates, it is linear in some normal coordinates as well.

We will first discuss the case when $d_0(M)$ takes the largest possible value, that is, when $d_0(M) = n^2$. Observe that in this case $\mathrm{Aut}_0(M)$ is linearizable for any m. Indeed, if $d_0(M) = n^2$, the group $G_0(M)$ contains $U(n-m, m)$. Hence $G_0(M)$ contains the subgroup $Q := \{e^{it} \cdot E_n,\ t \in \mathbb{R}\}$, where E_n is the $n \times n$ identity matrix. Let $\hat{Q} = \Phi^{-1}(Q) \subset \mathrm{Aut}_0(M)$. The subgroup $\hat{Q}$ is compact, and the argument in [KL] (see also [VK]) yields that in some normal coordinates every element of $\hat{Q}$ can be written in the form (1.3). For every $\varphi \in \hat{Q}$ we clearly have $\sigma_\varphi = 1$. Further, since Q is compact, there are no non-trivial homomorphisms from Q into $\mathbb{R}_+$, and therefore $\lambda_\varphi = 1$ for every $\varphi \in \hat{Q}$. Hence, in these coordinates the function

$$F(z, \overline{z}, u) := \sum_{k, \bar{l} \geq 2} F_{k\bar{l}}(z, \overline{z}, u)$$

is invariant under all linear transformations from Q and thus $F_{k\bar{l}} \equiv 0$, if $k \neq \bar{l}$. We will now show that $\mathrm{Aut}_0(M)$ is linearizable. Since linearizability arguments of this kind will occur several times throughout the paper, we give some details on the linearizability of $\mathrm{Aut}_0(M)$ for general hypersurfaces.

Suppose that M is given in the Chern-Moser normal form near the origin. The main step in showing that $\mathrm{Aut}_0(M)$ is linearizable is to prove that in some normal coordinates for every $\varphi \in \mathrm{Aut}_0(M)$, we have $a_\varphi = 0$. Indeed, if $a_\varphi = 0$, it follows from [CM] that φ in the given coordinates is a fractional linear transformation that becomes linear if $r_\varphi = 0$. It is shown in the proof of Proposition 3 of [L2] that $a_\varphi = 0$ implies that $r_\varphi = 0$, provided $\lambda_\varphi = 1$. Furthermore, if for every $\varphi \in \mathrm{Aut}_0(M)$ we have $a_\varphi = 0$ and there exists $\varphi_0 \in \mathrm{Aut}_0(M)$ with $\lambda_{\varphi_0} \neq 1$, the group $\mathrm{Aut}_0(M)$ becomes linear after applying a transformation of the form

$$\begin{aligned} z &\mapsto \frac{z}{1+qw}, \\ w &\mapsto \frac{w}{1+qw}, \end{aligned} \tag{1.4}$$

for some $q \in \mathbb{R}$.

To prove that $a_\varphi = 0$ for a fixed $\varphi = (f_\varphi, g_\varphi)$ in the given coordinates, we introduce weights as follows. Let each of $z_1, \dots, z_n$, $\overline{z_1}, \dots, \overline{z_n}$ be of weight 1 and u be of weight 2. Then we can write a weight decomposition for the function F as

$$F(z, \overline{z}, u) = \sum_{j=\gamma}^{\infty} F_j,$$

where F_j is the component of F of weight j, and $F_\gamma \not\equiv 0$. Next, since φ is a local automorphism of M, we have

$$\mathrm{Im}\, g_\varphi = \langle f_\varphi, f_\varphi \rangle + F\left(f_\varphi, \overline{f_\varphi}, \mathrm{Re}\, g_\varphi\right), \tag{1.5}$$

where we set $v = \langle z, z \rangle + F(z, \overline{z}, u)$. Extracting all terms of weight $\gamma + 1$ from identity (1.5), we obtain the following identity (see [B], [L1], [L2])

$$\begin{aligned} &\mathrm{Re}\left(i\tilde{g}_{\gamma+1} + 2\langle \lambda_\varphi^{-1} U_\varphi^{-1} \tilde{f}_\gamma, z \rangle\right)|_{v=\langle z,z\rangle} + T(F_\gamma, a_\varphi) \\ &\qquad = F_{\gamma+1}(z, \overline{z}, u) - \frac{1}{\lambda_\varphi^2} F_{\gamma+1}(\lambda_\varphi U_\varphi z, \overline{\lambda_\varphi U_\varphi z}, \lambda_\varphi^2 u). \end{aligned} \tag{1.6}$$

Here $\left(\sum_{j=1}^{\infty} \tilde{f}_j, \sum_{j=1}^{\infty} \tilde{g}_j\right)$ is the weight decomposition for the map $(\tilde{f}, \tilde{g}) := (f_\varphi - f_\varphi^Q, g_\varphi - g_\varphi^Q)$, where $\varphi^Q = (f_\varphi^Q, g_\varphi^Q)$ is the following local automorphism of the hyperquadric given by the equation $v = \langle z, z \rangle$:

$$\begin{aligned} z &\mapsto \frac{\lambda_\varphi U_\varphi (z + a_\varphi w)}{1 - 2i\langle z, a\rangle - (r_\varphi + i\langle a, a\rangle) w}, \\ w &\mapsto \frac{\sigma_\varphi \lambda_\varphi^2 w}{1 - 2i\langle z, a\rangle - (r_\varphi + i\langle a, a\rangle) w}, \end{aligned}$$

and

$$T(F_\gamma, a_\varphi) := 2\,\mathrm{Re}\Big(-2i\langle z, a_\varphi\rangle F_\gamma + (u + i\langle z,z\rangle)\sum_{j=1}^{n} a_j \frac{\partial F_\gamma}{\partial z_j} + 2i\langle z, a\rangle \sum_{j=1}^{n} z_j \frac{\partial F_\gamma}{\partial z_j} + i\langle z, a_\varphi\rangle (u + i\langle z,z\rangle)\frac{\partial F_\gamma}{\partial u}\Big),$$

where $a_1, \dots, a_n$ denote the components of the vector a_φ.

If $F_{\gamma+1} = 0$, the right-hand side of (1.6) vanishes, and the proof of Proposition 1 in [L1] shows that the resulting homogeneous identity can only hold if $a_\varphi = 0$. Clearly, if $F_{k\bar{l}} \equiv 0$ for $k \neq \bar{l}$, then the weight decomposition for F contains only terms of even weights, and, in particular, we have $F_{\gamma+1} = 0$. Thus, we have shown that $\mathrm{Aut}_0(M)$ is linearizable if $d_0(M) = n^2$.

Observe further that if $d_0(M) = n^2$, the mapping Λ defined in (1.2) is constant, that is, $\lambda_\varphi = 1$ for all $\varphi \in \mathrm{Aut}_0(M)$. Indeed, consider the restriction of Λ to $U(n-m,m)$. Every element $U \in U(n-m,m)$ can be represented as $U = e^{i\psi} \cdot V$ with $\psi \in \mathbb{R}$ and $V \in SU(n-m,m)$. Note that there are no non-trivial homomorphisms from the unit circle into $\mathbb{R}_+$ since $\mathbb{R}_+$ has no non-trivial compact subgroups. Also, there are no non-trivial homomorphisms from $SU(n-m,m)$ into $\mathbb{R}_+$ since the kernel of any such homomorphism is a proper normal subgroup of $SU(n-m,m)$ of positive dimension, and $SU(n-m,m)$ is a simple group. Thus, Λ is constant on $U(n-m,m)$ and hence on all of $G_0(M)$. It then follows that, in coordinates in which $\mathrm{Aut}_0(M)$ is linear, the function F is invariant under all linear transformations of the z-variables from $U(n-m,m)$ and therefore depends only on $\langle z,z\rangle$ and u. Conditions (1.1) imply that $F_{2\bar{2}} \equiv 0$, $F_{3\bar{3}} \equiv 0$. Thus, F has the form

$$F(z,\bar{z},u) = \sum_{k=4}^{\infty} C_k(u)\langle z,z\rangle^k, \tag{1.7}$$

where $C_k(u)$ are real-valued analytic functions of u, and for some k we have $C_k(u) \not\equiv 0$. Note, in particular, that if $d_0(M) = n^2$, then 0 is an umbilic point of M.

Conversely, if M is given in the normal form by an equation

$$v = \langle z,z\rangle + F(z,\bar{z},u),$$

with $F \not\equiv 0$ of the form (1.7), then $\mathrm{Aut}_0(M)$ contains all linear transformations (1.3) with $U \in U(n-m,m)$, $\lambda = 1$ and $\sigma = 1$, and therefore $d_0(M) = n^2$. For $n > 2m$ and for $n = 2m$ with $G_0(M) = U(m,m)$, $\mathrm{Aut}_0(M)$ clearly coincides with the group of all transformations of the form

$$\begin{aligned} z &\mapsto Uz, \\ w &\mapsto w. \end{aligned} \tag{1.8}$$

where $U \in U(n-m,m)$. If $n = 2m$ and $G_0(M) = U'(m,m)$, then $\mathrm{Aut}_0(M)$ consists of all mappings

$$\begin{array}{rcl} z & \mapsto & Uz, \\ w & \mapsto & \sigma w, \end{array}$$

where $U \in U'(m,m)$, $\langle Uz, Uz\rangle = \sigma\langle z, z\rangle$, $\sigma = \pm 1$.

We will now concentrate on the case $0 < d_0(M) < n^2$ (hence assuming that $n \geq 2$). For the strongly pseudoconvex case we obtain:

THEOREM 1.1. *Let M be a strongly pseudoconvex real-analytic non-spherical hypersurface in $\mathbb{C}^{n+1}$ with $n \geq 2$ (here $m = 0$) given in normal coordinates in which $\mathrm{Aut}_0(M)$ is linear. Then the following holds:*

(i) *$d_0(M) \geq n^2 - 2n + 3$ implies $d_0(M) = n^2$.*

(ii) *If $d_0(M) = n^2 - 2n + 2$, after a linear change of the z-coordinates the equation of M takes the form*

$$v = \sum_{\alpha=1}^{n} |z_\alpha|^2 + F(z, \overline{z}, u), \tag{1.9}$$

where F is a function of $|z_1|^2$, $\langle z, z\rangle := \sum_{\alpha=1}^{n} |z_\alpha|^2$ and u:

$$F(z, \overline{z}, u) = \sum_{p+q\geq 4} C_{pq}(u)|z_1|^{2p}\langle z, z\rangle^q, \tag{1.10}$$

where $C_{pq}(u)$ are real-valued analytic functions of u, and $C_{pq}(u) \not\equiv 0$ for some p, q with $p > 0$.

(iii) *If a hypersurface M is given in the form described in (ii) (without assuming the linearity of $\mathrm{Aut}_0(M)$ a priori), the group $\mathrm{Aut}_0(M)$ coincides with the group of all mappings of the form (1.8), where $U \in U(1) \times U(n-1)$ (with $U(1) \times U(n-1)$ realized as a group of block-diagonal matrices in the standard way).*

COROLLARY 1.2. *If M is a strongly pseudoconvex real-analytic hypersurface in $\mathbb{C}^{n+1}$, and the dimension of $\mathrm{Aut}_0(M)$ is greater than or equal to $n^2 - 2n + 2$, then the origin is an umbilic point of M.*

For the case $m \geq 1$ we prove:

THEOREM 1.3. *Let M be a Levi non-degenerate real-analytic non-spherical hypersurface in $\mathbb{C}^{n+1}$ with $m \geq 1$. Then the following holds:*

(i) *$d_0(M) \geq n^2 - 2n + 4$ implies $d_0(M) = n^2$.*

(ii) *If $d_0(M) = n^2 - 2n + 3$, the group $\mathrm{Aut}_0(M)$ is linearizable and in some normal coordinates in which $\mathrm{Aut}_0(M)$ is linear, the equation of M takes the* *form*

$$v = 2\,\mathrm{Re}\, z_1\overline{z}_n + 2\,\mathrm{Re}\, z_2\overline{z}_{n-1} + \cdots + 2\,\mathrm{Re}\, z_m\overline{z}_{n-m+1} + \sum_{\alpha=m+1}^{n-m} |z_\alpha|^2 + F(z,\overline{z},u), \tag{1.11}$$

where F is a function of $|z_n|^2$,

$$\langle z, z\rangle := 2\,\mathrm{Re}\, z_1\overline{z}_n + 2\,\mathrm{Re}\, z_2\overline{z}_{n-1} + \cdots + 2\,\mathrm{Re}\, z_m\overline{z}_{n-m+1} + \sum_{\alpha=m+1}^{n-m} |z_\alpha|^2$$

and u:

$$F(z,\overline{z},u) = \sum C_{rpq} u^r |z_n|^{2p} \langle z, z\rangle^q, \tag{1.12}$$

where at least one of $C_{rpq} \in \mathbb{R}$ is non-zero, the summation is taken over $p \geq 1$, $q \geq 0$, $r \geq 0$ such that $(r+q-1)/p = s$ with $s \geq -1/2$ being a fixed rational number, and

$$F(z,\overline{z},u) = \sum_{k,\bar{l}\geq 2} F_{k\bar{l}}(z,\overline{z},u),$$

where $F_{2\overline{3}} = 0$ and identities (1.1) hold for $F_{2\overline{2}}$ and $F_{3\overline{3}}$.

(iii) *If a hypersurface M is given in the form described in (ii) (without assuming the linearity of* $\mathrm{Aut}_0(M)$ *a priori), the group* $\mathrm{Aut}_0(M)$ *coincides with the group of all mappings of the form*

$$\begin{array}{rcl} z & \mapsto & |\mu|^{1/(s+1)} U z, \\ w & \mapsto & |\mu|^{2/(s+1)} w, \end{array} \tag{1.13}$$

with $U \in S$, where S is the group introduced in Lemma 3.1 below, and μ is a parameter in this group (see formula (3.2)).

COROLLARY 1.4. *Let M be a Levi non-degenerate real-analytic hypersurface in $\mathbb{C}^{n+1}$, with $n \geq 2$ and $m \geq 1$, and assume that the dimension of* $\mathrm{Aut}_0(M)$ *is greater than or equal to n^2-2n+3. If the origin is a non-umbilic point of M, then in some normal coordinates the equation of M takes the form*

$$v = 2\,\mathrm{Re}\, z_1\overline{z}_n + 2\,\mathrm{Re}\, z_2\overline{z}_{n-1} + \dots + 2\,\mathrm{Re}\, z_m\overline{z}_{n-m+1} + \sum_{\alpha=m+1}^{n-m} |z_\alpha|^2 \pm |z_n|^4. \tag{1.14}$$

We remark that hypersurfaces (1.14) occur in [P] in connection with studying unbounded homogeneous domains in complex space.

The proofs of Theorems 1.1 and 1.3 are given in Sections 2 and 3 respectively.

2. The strongly pseudoconvex case

First of all, we note that in this case the mapping Λ defined in (1.2) is constant, that is, $\lambda_\varphi = 1$ for all $\varphi \in \mathrm{Aut}_0(M)$. This follows from the fact that all eigenvalues of U_φ are unimodular, or, alternatively, from the compactness of $G_0(M)$ and the observation that $\mathbb{R}_+$ does not have non-trivial compact subgroups. Next, by a linear change of the z-coordinates the matrix H can be transformed into the identity matrix E_n, and for the remainder of this section we assume that $H = E_n$. Hence we assume that the equation of M is written in the form (1.9), where the function F satisfies the normal form conditions.

It is shown in Lemma 2.1 of [IK] (see also [K]) that any closed connected subgroup of the unitary group $U(n)$ of dimension n^2-2n+3 or larger is either $SU(n)$ or $U(n)$ itself. Hence, if $d_0(M) \geq n^2-2n+3$, we have $G_0(M) \supset SU(n)$, and therefore $F(z, \overline{z}, u)$ is invariant under all linear transformations of the z-variables from $SU(n)$. This implies that $F(z, \overline{z}, u)$ is a function of $\langle z, z\rangle$ and u, which gives that $F(z, \overline{z}, u)$ is invariant under the action of the full unitary group $U(n)$ and thus $d_0(M) = n^2$, as stated in (i).

The proof of part (ii) of the theorem is also based on Lemma 2.1 of [IK] (see also [K]). For the case $d_0(M) = n^2 - 2n + 2$ the lemma gives that the connected identity component G_0^c of G_0 is either conjugate in $U(n)$ to the subgroup $U(1) \times U(n-1)$ realized as block-diagonal matrices, or, for $n = 4$, contains a subgroup conjugate in $U(n)$ to $Sp_{2,0}$. If the latter is the case, then, since $Sp_{2,0}$ acts transitively on the sphere of dimension 7 in $\mathbb{C}^4$, $F(z, \overline{z}, u)$ is a function of $\langle z, z\rangle$ and u, which implies that $F(z, \overline{z}, u)$ is invariant under the action of the full unitary group $U(4)$ and thus $d_0(M) = 16$, which is impossible. Hence G_0^c is conjugate to $U(1) \times U(n-1)$, and therefore, after a unitary change of the z-coordinates, the equation of M can be written in the form (1.9), where the function F depends on $|z_1|^2$, $\langle z, z\rangle' := \sum_{\alpha=2}^n |z_\alpha|^2$ and u. Clearly, $\langle z, z\rangle' = \langle z, z\rangle - |z_1|^2$, and F can be written as a function of $|z_1|^2$, $\langle z, z\rangle$ and u as in (1.10). Next, conditions (1.1) imply that $F_{2\overline{2}} \equiv 0$, $F_{3\overline{3}} \equiv 0$, and thus the summation in (1.10) is taken over p, q such that $p + q \geq 4$. Further, if $C_{pq} \equiv 0$ for all $p > 0$, F has the form (1.7) and therefore $G_0 = U(n)$, which is impossible. Thus for some p, q with $p > 0$ we have $C_{pq} \not\equiv 0$, and (ii) is established.

If M is given in the normal form and is written as in (1.9), (1.10), $\mathrm{Aut}_0(M)$ clearly contains all maps of the form (1.8) with $U \in U(1) \times U(n-1)$. Hence $d_0(M) \geq n^2 - 2n + 2$. If $d_0(M) > n^2 - 2n + 2$, then by part (i) of the theorem, $d_0(M) = n^2$ and hence $G_0(M) = U(n)$. Then F has the form (1.7), which is impossible because for some p, q with $p > 0$ the function C_{pq} does not vanish identically. Thus $d_0(M) = n^2 - 2n + 2$, and hence $G_0^c(M) = U(1) \times U(n-1)$. It is not hard to show that $G_0(M)$ is connected (note, for example, that by an argument given in the introduction, $\mathrm{Aut}_0(M)$ is linear in these coordinates), and therefore $\mathrm{Aut}_0(M)$ coincides with the group of all mappings of the form (1.8), where $U \in U(1) \times U(n-1)$.

Thus, (iii) is established, and the theorem is proved. □

3. The case of $m \geq 1$

We start with the following algebraic lemma.

LEMMA 3.1. *Let $G \subset U(n-m,m)$ be a connected real algebraic subgroup of $GL_n(\mathbb{C})$, $n \geq 2m$, $m \geq 1$, with Hermitian form preserved by $U(n-m,m)$ written as*

$$\begin{pmatrix} & & & & & & 1 \\ & \mathit{0} & & & & \cdot^{\cdot^{\cdot}} & \\ & & & & 1 & & \\ & & & E_{n-2m} & & & \\ & & 1 & & & & \\ & \cdot^{\cdot^{\cdot}} & & & & \mathit{0} & \\ 1 & & & & & & \end{pmatrix}, \tag{3.1}$$

where E_{n-2m} is the $(n-2m) \times (n-2m)$ identity matrix, and the number of 1's on each side of E_{n-2m} is m. Then the following holds:

(a) *If $\dim G \geq n^2 - 2n + 4$, we have either $G = SU(n-m,m)$, or $G = U(n-m,m)$.*

(b) *If $\dim G = n^2 - 2n + 3$, the group G either is conjugate in $U(n-m,m)$ to the group S that consists of all matrices of the form*

$$\begin{pmatrix} \mu & -\mu \overline{x}^T H' A & c \\ 0 & A & x \\ 0 & 0 & 1/\overline{\mu} \end{pmatrix}, \tag{3.2}$$

where $\mu, c \in \mathbb{C}$, $\mu \neq 0$, $x \in \mathbb{C}^{n-2}$, $A \in U(n-m-1, m-1)$ (i.e., A is an $(n-2) \times (n-2)$-matrix with complex elements such that $A^T H' \overline{A} = H'$ with H' obtained from matrix (3.1) by removing the first and the last columns and rows), and we have

$$2 \,\mathrm{Re}\, \frac{c}{\mu} + x^T H' \overline{x} = 0,$$

or, if $n = 4$ and $m = 2$, coincides with $e^{i\mathbb{R}}\Big(Sp_4(B,\mathbb{C}) \cap SU(2,2)\Big)$, or, if $n = 2$ and $m = 1$, coincides with $SU(1,1)$. Here the subgroup $Sp_4(B,\mathbb{C}) \subset GL_4(\mathbb{C})$ consists of matrices preserving a non-degenerate skew-symmetric bilinear form B equivalent to the form given by the matrix

$$B_0 := \begin{pmatrix} 0 & E_2 \\ -E_2 & 0 \end{pmatrix}, \tag{3.3}$$

where E_2 is the 2×2 identity matrix.

Proof. The lemma can be derived from the classification of maximal subalgebras of the classical Lie algebras obtained in [K]. Below we sketch a proof that does not rely on [K].

Let $V \subset U(n-m,m)$ be a real algebraic subgroup of $GL_n(\mathbb{C})$ such that $\dim V \geq n^2-2n+3$. Consider $V_1 := V \cap SU(n-m,m)$. Clearly, $\dim V_1 \geq n^2-2n+2$. Let $V_1^{\mathbb{C}} \subset SL_n(\mathbb{C})$ be the complexification of V_1. We have $\dim_{\mathbb{C}} V_1^{\mathbb{C}} \geq n^2-2n+2$. Consider the maximal complex closed subgroup $W(V) \subset SL_n(\mathbb{C})$ that contains $V_1^{\mathbb{C}}$. Clearly, $\dim_{\mathbb{C}} W(V) \geq n^2-2n+2$. All closed maximal subgroups of $SL_n(\mathbb{C})$ were determined in [D], and the lower bound on the dimension of $W(V)$ gives that either $W(V) = SL_n(\mathbb{C})$, or $W(V)$ is conjugate to one of the parabolic subgroups

$$P_1^1 := \left\{\begin{pmatrix} 1/\det C & * \\ 0 & C \end{pmatrix}, C \in GL_{n-1}(\mathbb{C})\right\},$$

$$P_1^2 := \left\{\begin{pmatrix} C & * \\ 0 & 1/\det C \end{pmatrix}, C \in GL_{n-1}(\mathbb{C})\right\},$$

$$P_2^1 := \left\{\begin{pmatrix} A & * \\ 0 & C \end{pmatrix}, A \in GL_2(\mathbb{C}),\ C \in GL_{n-2}(\mathbb{C}), \det A = 1/\det C\right\},\ n \geq 4,$$

$$P_2^2 := \left\{\begin{pmatrix} C & * \\ 0 & A \end{pmatrix}, A \in GL_2(\mathbb{C}),\ C \in GL_{n-2}(\mathbb{C}), \det A = 1/\det C\right\},\ n \geq 4,$$

$$P_3 := \left\{\begin{pmatrix} A & * \\ 0 & C \end{pmatrix}, A, C \in GL_3(\mathbb{C}),\ \det A = 1/\det C\right\} \text{ (here } n = 6\text{)},$$

or, for $n = 4$, $W(V)$ is conjugate to $Sp_4(\mathbb{C})$.

Suppose that for some $g \in SL_n(\mathbb{C})$ and $j \in \{1,2\}$ we have $g^{-1}W(V)g = P_1^j$. It is not hard to show that, due to the lower bound on the dimension of $W(V)$, g can be chosen to belong to $SU(n-m,m)$. Then $g^{-1}V_1 g \subset P_1^j \cap SU(n-m,m)$. It is easy to compute the intersections $P_1^j \cap SU(n-m,m)$ for $j = 1,2$ and see that they are equal and coincide with the group S_1 of matrices of the form (3.2) with determinant 1. Clearly, $\dim S_1 = n^2-2n+2 \leq \dim V_1$ and therefore V_1 is conjugate to S_1 in $SU(n-m,m)$.

Further, $W(V)$ cannot in fact be conjugate to any of P_2^j, P_3. Indeed, it is straightforward to observe that for any $g \in SL_n(\mathbb{C})$ the intersections of each of these groups with $g^{-1}SU(n-m,n)g$ have dimension less than n^2-2n+2.

Suppose now that $n=4$ and for some $g \in SL_4(\mathbb{C})$ we have $g^{-1}W(V)g = Sp_4(\mathbb{C})$. In particular, $g^{-1}V_1 g \subset Sp_4(\mathbb{C}) \cap g^{-1}SU(4-m,m)g$ (here we have either $m=1$, or $m=2$). It can be shown that dim $Sp_4(\mathbb{C}) \cap g^{-1}SU(3,1)g \leq 6$ for all $g \in SL_4(\mathbb{C})$. At the same time we have dim $V_1 \geq 10$. Hence $W(V)$ in fact cannot be conjugate to $Sp_4(\mathbb{C})$, if $m=1$. Therefore, $m=2$, and $V_1 \subset gSp_4(\mathbb{C})g^{-1} \cap SU(2,2) = Sp_4(B,\mathbb{C}) \cap SU(2,2)$, where B is some non-degenerate skew-symmetric bilinear form. It is straightforward to show that $Sp_4(B,\mathbb{C}) \cap SU(2,2)$ is connected and dim $Sp_4(B,\mathbb{C}) \cap SU(2,2) \leq 10$. Therefore $V_1 = Sp_4(B,\mathbb{C}) \cap SU(2,2)$.

Suppose now that dim $G \geq n^2-2n+4$. Then dim $G_1 \geq n^2-2n+3$, and the above considerations give that $W(G) = SL_n(\mathbb{C})$. Hence $G_1 = SU(n-m,m)$, which implies that either $G = SU(n-m,m)$, or $G = U(n-m,m)$, thus proving (a).

Let dim $G = n^2-2n+3$. In this case we have either dim $G_1 = n^2-2n+2$, or $G = SU(1,1)$, if $n=2$, $m=1$. In the first case we obtain that G_1 either is conjugate to S_1 in $SU(n-m,m)$, or, for $n=4$ and $m=2$, coincides with $Sp_4(B,\mathbb{C}) \cap SU(2,2)$ for some non-degenerate skew-symmetric bilinear form B equivalent to the form B_0 defined in (3.3). This gives that G in the first case either is conjugate to S in $U(n-m,m)$, or for $n=4$ and $m=2$ coincides with $e^{i\mathbb{R}}\Big(Sp_4(B,\mathbb{C}) \cap SU(2,2)\Big)$, and (b) is established.

The lemma is proved. $\square$

We will now prove Theorem 1.3. Suppose first that $d_0(M) \geq n^2-2n+4$ and assume that H is written in the diagonal form with 1's in the first $n-m$ positions and -1's in the last m positions on the diagonal. Lemma 3.1 gives that either $G_0^c(M) = SU(n-m,m)$ (in which case $n \geq 3$), or $G_0(M) = U(n-m,m)$, or, for $n=2m$, $G_0(M) = U'(m,m)$. If $G_0(M) \supset U(n-m,m)$, then $d_0(M) = n^2$, and (i) is established. Assume that $G_0(M) \supset SU(n-m,m)$. Suppose that $m \geq 2$. Then G_0 contains the product $R := SU(n-m) \times SU(m)$ realized as block-diagonal matrices. Arguing as in the introduction, we obtain that in some normal coordinates all elements of the compact group $\hat{R} := \Phi^{-1}(R)$ can be written in the form (1.8) and thus F is a function of $\langle z,z\rangle_+ := \sum_{j=1}^{n-m}|z_j|^2$, $\langle z,z\rangle_- := \sum_{j=n-m+1}^{n}|z_j|^2$, and u. Hence all elements of odd weight in the weight decomposition for F are zero. This shows that $F_{\gamma+1} \equiv 0$, and identity (1.6) again implies that $\mathrm{Aut}_0(M)$ becomes linear after a change of coordinates of the form (1.4). If $m=1$, $\mathrm{Aut}_0(M)$ is linearizable by [Ezh1], [Ezh2].

Therefore, there exist normal coordinates where the corresponding function F is invariant under all linear transformations of the z-variables from

$SU(n-m,m)$. This implies that F is in fact invariant under all linear transformations of the z-variables from $U(n-m,m)$. Hence $d_0(M)=n^2$, and (i) is established.

Suppose now that $d_0(M)=n^2-2n+3$. By a linear change of the z-coordinates the matrix H can be transformed into matrix (3.1), and from now on we assume that H is given in this form. Hence the equation of M is written as in (1.11), where the function F satisfies the normal form conditions. Arguing as in the preceding paragraph, we see that for $n=2$, $m=1$, the group G_0^c cannot coincide with $SU(1,1)$. Assume first that after a linear change of the z-coordinates preserving the form H the group $G_0^c(M)$ coincides with S. Then $G_0(M)$ contains the compact subgroup $Q=\{e^{it}\cdot E_n,\ t\in\mathbb{R}\}$, where E_n is the $n\times n$ identity matrix. The argument based on identity (1.6) that we gave in the introduction, again yields that $\mathrm{Aut}_0(M)$ is linearizable. Passing to coordinates in which $\mathrm{Aut}_0(M)$ is linear, we obtain that for every $U\in S$ the equation of M is invariant under the linear transformation

$$
\begin{array}{rcl} z & \mapsto & \lambda_U U z,\\ w & \mapsto & \lambda_U^2 w, \end{array} \tag{3.4}
$$

where $\lambda_U=\Lambda(U)$. The group S contains $U(n-m-1,m-1)$ realized as the subgroup of all matrices of the form (3.2) with $\mu=1$, $c=0$, $x=0$. Since Λ is constant on $U(n-m-1,m-1)$, we have $\lambda_U=1$ for all $U\in U(n-m-1,m-1)$. Therefore, the function $F(z,\overline{z},u)$ depends on z_1, z_n, $\overline{z}_1$, $\overline{z}_n$,

$$
\langle z,z\rangle' := 2\,\mathrm{Re}\ z_2\overline{z}_{n-1}+\cdots+2\,\mathrm{Re}\ z_m\overline{z}_{n-m+1}+\sum_{\alpha=m+1}^{n-m}|z_\alpha|^2
$$

and u. Clearly, $\langle z,z\rangle'=\langle z,z\rangle-2\,\mathrm{Re}\ z_1\overline{z_n}$, and F can be written as

$$
F(z,\overline{z},u)=\sum_{r,q\geq 0} D_{rq}(z_1,z_n,\overline{z}_1,\overline{z}_n)u^r\langle z,z\rangle^q,
$$

where D_{rq} are real-analytic.

We will now determine the form of the functions D_{rq}. The group S contains the subgroup I of all matrices as in (3.2) with $|\mu|=1$, $x=0$ and $A=E_{n-2}$, where E_{n-2} is the $(n-2)\times(n-2)$ identity matrix. Since every eigenvalue of any $U\in I$ is unimodular, we have $\lambda_U=1$ for all $U\in I$, and therefore D_{rq} is invariant under all linear transformations from I. It is straightforward to show (see also [Ezh2]) that any polynomial of z_1, z_n, $\overline{z_1}$, $\overline{z_n}$ invariant under all linear transformations from I is a function of $\mathrm{Re}\ z_1\overline{z}_n$ and $|z_n|^2$, and hence every D_{rq} has this property. Let further J be the subgroup of S given by the conditions $\mu=1$, $A=E_{n-2}$. For every $U\in J$ we also have $\lambda_U=1$, and hence D_{rq} is invariant under all linear transformations from J. It is then easy to see that D_{rq} has to be a function of $|z_n|^2$ alone. Thus, the function F has the form (1.12), and it remains to show that the summation in (1.12) is taken over $p \geq 1$, $q \geq 0$, $r \geq 0$ such that $(r+q-1)/p = s$, where $s \geq -1/2$ is a fixed rational number.

Let K be the 1-dimensional subgroup of S given by the conditions $\mu > 0$, $c = 0$, $x = 0$, $A = E_{n-2}$. It is straightforward to show that every homomorphism $\Psi : K \to \mathbb{R}_+$ has the form $U \mapsto \mu^\alpha$, where $\alpha \in \mathbb{R}$. Considering $\Psi = \Lambda|_K$ we obtain that there exists $\alpha \in \mathbb{R}$ such that for every $U \in K$ we have $\lambda_U = \mu^\alpha$. We will now prove that $\alpha \neq 0$. Indeed, otherwise F would be invariant under all linear transformations from K and therefore would be a function of $\langle z, z\rangle$ and u, which implies that $G_0(M) \supset U(n-m, m)$. This contradiction shows that $\alpha \neq 0$ and hence $\lambda_U \neq 1$ for every $U \in K$ with $\mu \neq 1$.

Plugging a mapping of the form (3.4) with $U \in K$, $\mu \neq 1$, into equation (1.11), where $F \not\equiv 0$ has the form (1.12), we obtain that, if $C_{rpq} \neq 0$, then

$$\lambda_U^{r+p+q-1} = \mu^p. \tag{3.5}$$

The equation of M is written in the normal form, hence $p + q \geq 2$ and $r + p + q - 1 \geq 1$. Since $\lambda_U \neq 1$, we obtain that $p \geq 1$. Further, (3.5) implies

$$\lambda_U^{(r+p+q-1)/p} = \mu,$$

and, since the right-hand side in the above identity does not depend on r, p, q, for all non-zero coefficients C_{rpq} the ratio $(r+q-1)/p$ must have the same value; we denote it by s. Clearly, s is a rational number and $s \geq -1/2$. We also remark that $\alpha = p/(r+p+q-1) = 1/(s+1)$.

Assume now that $n = 4$, $m = 2$ and $G_0^c(M)$ coincides with $e^{i\mathbb{R}}\Big(Sp_4(B,\mathbb{C}) \cap SU(2,2)\Big)$ for some non-degenerate skew-symmetric bilinear form B equivalent to the form B_0 defined in (3.3). Then $G_0(M)$ contains the compact subgroup $Q = \{e^{it} \cdot E_4,\, t \in \mathbb{R}\}$, where E_4 is the 4×4 identity matrix. Arguing as above, we obtain that $\mathrm{Aut}_0(M)$ is linearizable. Further, it is straightforward to prove that $Sp_4(B,\mathbb{C}) \cap SU(2,2)$ is a real form of $Sp_4(B,\mathbb{C})$ and therefore is simple. Hence there does not exist a non-trivial homomorphism from $Sp_4(B,\mathbb{C}) \cap SU(2,2)$ into $\mathbb{R}_+$. Further, since $\mathbb{R}_+$ does not have non-trivial compact subgroups, any homomorphism from the unit circle into $\mathbb{R}_+$ is constant. Hence Λ is constant on $G_0(M)$. This implies that F is invariant under all linear transformations from $Sp_4(B,\mathbb{C}) \cap SU(2,2)$. It can be shown that this group acts transitively on any pseudosphere in $\mathbb{C}^4$ given by the equation $\langle z, z\rangle = r$, which yields that F is a function of $\langle z, z\rangle$ and u and hence $d_0(M) = n^2$. This contradiction proves that in fact $G_0^c(M) \neq e^{i\mathbb{R}}\Big(Sp_4(B,\mathbb{C}) \cap SU(2,2)\Big)$ for $n = 4$, $m = 2$. Thus, (ii) is established.

Suppose that M is given in the normal form, written as in (1.11), (1.12), and the summation in (1.12) is taken over $p \geq 1$, $q \geq 0$, $r \geq 0$ such that $(r+q-$

$1)/p = s$, where $s \geq -1/2$ is a fixed rational number. Set $\alpha = 1/(s+1)$ and for every $U \in S$ define $\lambda_U = |\mu|^\alpha$. It is then straightforward to verify that every mapping of the form (3.4) with $U \in S$ is an automorphism of M. Therefore, $G_0(M)$ contains S and hence $d_0(M) \geq n^2 - 2n + 3$. If $d_0(M) > n^2 - 2n + 3$, then by part (i) of the theorem, $d_0(M) = n^2$ and hence $G_0(M) \supset U(n-m, m)$. Then F is a function of $\langle z, z\rangle$ and u, which is impossible since for every nonzero C_{rpq} we have $p \geq 1$. Hence $d_0(M) = n^2 - 2n + 3$ and hence $G_0^c(M) = S$. Finally, observe that by an argument given in the introduction, $\mathrm{Aut}_0(M)$ is linear in these coordinates. It is now straightforward to show that $\mathrm{Aut}_0(M)$ coincides with the group of all mappings of the form (1.13).

Thus, (iii) is established, and the theorem is proved. □

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