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ON CONVERGENCE TO THE DENJOY-WOLFF POINT

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Abstract. For holomorphic selfmaps of the open unit disc $\mathbb{U}$ that are not elliptic automorphisms, the Schwarz Lemma and the Denjoy-Wolff Theorem combine to yield a remarkable result: each such map φ has a (necessarily unique) "Denjoy-Wolff point" ω in the closed unit disc that attracts every orbit in the sense that the iterate sequence $(\varphi^{[n]})$ converges to ω uniformly on compact subsets of $\mathbb{U}$. In this paper we prove that, except for the obvious counterexamples—inner functions having $\omega \in \mathbb{U}$—the iterate sequence exhibits an even stronger affinity for the Denjoy-Wolff point; $\varphi^{[n]} \to \omega$ in the norm of the Hardy space H^p for $1 \leq p < \infty$. For each such map, some subsequence of iterates converges to ω almost everywhere on $\partial\mathbb{U}$, and this leads us to investigate the question of almost-everywhere convergence of the entire iterate sequence. Here our work makes natural connections with two important aspects of the study of holomorphic selfmaps of the unit disc: linear-fractional models and ergodic properties of inner functions.

1. Introduction

We study the dynamics of holomorphic selfmaps φ of the open unit disc $\mathbb{U}$ of the complex plane, i.e., properties of the *iterate sequence* $(\varphi^{[n]})$, where $\varphi^{[n]}$ denotes the composition of φ with itself n times $(n = 1, 2, \dots)$. If φ has a fixed point $\omega \in \mathbb{U}$ and is not an automorphism of $\mathbb{U}$ then an argument based on the Schwarz Lemma shows that the sequence $(\varphi^{[n]})$ converges, uniformly on compact subsets of $\mathbb{U}$, to the constant function ω. A much deeper result establishes that $(\varphi^{[n]})$ behaves similarly when φ has no fixed point in $\mathbb{U}$. This is the

Denjoy-Wolff Theorem. *Suppose that φ is a holomorphic selfmap of $\mathbb{U}$ that fixes no point in $\mathbb{U}$. Then there is a point $\omega \in \partial\mathbb{U}$ such that $(\varphi^{[n]})$ converges to ω uniformly on compact subsets of $\mathbb{U}$.*

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The (necessarily unique) point ω to which the iterate sequence of φ converges—whether in $\mathbb{U}$ or on $\partial\mathbb{U}$—is called the *Denjoy-Wolff point* of φ.

In Section 3 we show that, except for the obvious class of counter-examples (inner functions with Denjoy-Wolff point in $\mathbb{U}$), the iterate sequence of φ exhibits a stronger form of convergence to the Denjoy-Wolff point: convergence in the norm of the Hardy space H^p for $1 \leq p < \infty$ (Theorem 3.1). In this case the iterate sequence has a subsequence that converges almost everywhere on $\partial\mathbb{U}$ with respect to Lebesgue measure, and this raises the question, treated in Section 4, of a.e. convergence of the entire sequence on $\partial\mathbb{U}$ (which, by the Dominated Convergence Theorem, is even stronger than convergence in H^p). This in turn leads to interesting connections with established work on linear-fractional models for holomorphic selfmaps of the unit disc. We show that for a selfmap φ with Denjoy-Wolff point ω the iterate sequence converges a.e. to ω whenever

(a) $\omega \in \mathbb{U}$ and φ is not inner, or
(b) $\omega \in \partial\mathbb{U}$ and φ is of either hyperbolic or parabolic automorphic type.

The remaining case, $\omega \in \partial\mathbb{U}$ and φ of parabolic non-automorphic type, is more complicated. Even within the subclass of inner functions there are examples of maps whose iterates converge a.e. to the Denjoy-Wolff point, and other examples where they do not. We explore this issue in Sections 4 and 5.

It is known that inner functions of parabolic nonautomorphic type act *ergodically* on $\partial\mathbb{U}$, and we discuss in Section 5 the connections our work has with this area of investigation. In particular, we are able to complete work of Kim and Kim [15] on the ergodic behavior of atomic singular inner functions, and we modify ideas of Aaronson [2] to produce ergodic singular inner functions with iterate sequences converging a.e. to the Denjoy-Wolff point.

In the next section we outline, for the convenience of the reader, the prerequisites needed to understand what follows.

2. Background material

This section consists entirely of reference material intended to be consulted as needed.

2.1. Hardy Spaces ([13, Chapter 2], [24, Chapter 17]). For $1 \leq p < \infty$, the Hardy space $H^p(\mathbb{U})$ consists of those functions f holomorphic on $\mathbb{U}$ that satisfy the growth condition

$$\|f\|_p := \left(\sup_{0 \leq r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p \, d\theta \right)^{1/p} < \infty, \tag{2.1}$$

where $d\theta$ denotes Lebesgue measure. The norm $\| \cdot \|_p$ makes $H^p(\mathbb{U})$ into a Banach space. It is a simple matter to check that for $p = 2$ and $f(z) =$

$\sum_0^\infty \hat{f}(n)z^n$ definition (2.1) can be rewritten as

$$\|f\|_2^2 = \sum_0^\infty |\hat{f}(n)|^2 < \infty,$$

hence the norm $\|\cdot\|_2$ makes H^2 into a Hilbert space.

Each Hardy-space function f has a radial limit

$$f^*(\zeta) = \lim_{r\to 1^-} f(r\zeta),$$

at a.e. $\zeta \in \partial\mathbb{U}$. The Hardy norm may be computed by a boundary integral involving this limit function:

$$\|f\|_p^p = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f^*(e^{i\theta})|^p \, d\theta \tag{2.2}$$

(see, e.g., [13, Theorem 2.2, page 17] or [24, Theorem 17.11, page 340]). From this point on we will drop the superscript "$*$" used to denote the radial limit function, so that given $\zeta \in \partial\mathbb{U}$, $f(\zeta)$ denotes the radial limit of f at ζ (provided the limit exists).

As a companion to the boundary calculation of the H^p norm, each function in H^p can be represented as both a Poisson and a Cauchy integral; either of these quickly shows that the H^p norm is "natural" in the sense that convergence in H^p implies uniform convergence on compact subsets of $\mathbb{U}$. This in turn provides the crucial step in showing that H^p is a Banach space (see, e.g., [13, §3.1–3.3]).

2.2. Inner functions. Holomorphic functions that are bounded on $\mathbb{U}$ belong to every H^p space, so each has a radial limit at a.e. point of $\partial\mathbb{U}$. If this radial limit function has modulus one a.e. on $\partial\mathbb{U}$, the holomorphic function in question is called an *inner function.* Two fundamental examples of inner functions are:

(a) "Standard" conformal automorphisms of the disc. These are functions of the form

$$\alpha_b(z) = \frac{b-z}{1-\bar{b}z} \qquad (z \in \mathbb{C}\backslash\{1/\bar{b}\}),$$

for $b \in \mathbb{U}$, and

(b) The "unit singular function:"

$$S(z) = \exp\left\{-\frac{1+z}{1-z}\right\} \qquad (z \in \mathbb{C}\backslash\{1\}).$$

α_b is a conformal automorphism of $\mathbb{U}$ (i.e., a 1-1 holomorphic map of $\mathbb{U}$ onto itself) that takes the value zero at b, and is its own inverse: $\alpha_b \circ \alpha_b(z) \equiv z$ for all $z \in \mathbb{U}$. Every conformal automorphism of $\mathbb{U}$ is a unimodular multiple of some α_b. The unit singular function, which is highly non-univalent, is an example of an inner function that never takes the value zero. More generally, each non-constant inner function on $\mathbb{U}$ with no zero has the form

$$S_\mu(z) = \exp\left\{-\int \frac{\zeta + z}{\zeta - z}\, d\mu(\zeta)\right\} \qquad (z \in \mathbb{U})$$

where μ is a finite positive Borel measure on $\partial\mathbb{U}$ singular with respect to Lebesgue measure. Inner functions with zeros can be formed as (finite or infinite) products of appropriate conformal automorphisms—the so called *Blaschke products*

$$B(z) = \prod_{n\geq 1} \omega_n \alpha_{b_n}(z) \qquad (z \in \mathbb{U}),$$

where $\omega_n \in \partial\mathbb{U}$ is chosen to make the value of the n-th factor positive at the origin (important only if there are infinitely many factors), and where the zero-sequence (b_n) satisfies the *Blaschke condition*

$$\sum_{n\geq 1}(1 - |b_n|) < \infty. \tag{2.3}$$

The Blaschke condition insures that the product converges uniformly on compact subsets of $\mathbb{U}$ to a non-constant holomorphic function with zeros only at the points b_n (with multiplicity equal to the number of times α_{b_n} is repeated in the product). See [13, §2.1–2.4] or [24, §15.21–15.24] for the details.

A remarkable fact about inner functions is that—up to multiplication by a unimodular constant—each one that vanishes nowhere in $\mathbb{U}$ is singular, and each one that has zeros in $\mathbb{U}$ is either a Blaschke product or a Blaschke product times a singular inner function. In particular, if (b_n) is the zero sequence of any inner function, then (b_n) satisfies the Blaschke condition [24, Theorem 17.15, page 342]. More generally, this is true of the zero sequence of any H^p-function [24, Theorem 17.10, page 339].

Crucial for our purposes is the fact that the composition of two inner functions is again inner. While this might at first glance seem trivial, it does require some proof; for example it follows readily from Lindelöf's theorem [25, page 163, Lemma 3] that if f and g are bounded holomorphic functions, then the radial limit function of $f \circ g$ coincides a.e. on $\partial\mathbb{U}$ with the corresponding composition of radial limit functions on $\partial\mathbb{U}$.

2.3. The Denjoy-Wolff Theorem (see, e.g., [25, Chapters 4 and 5])**.** If a holomorphic selfmap φ of $\mathbb{U}$ has a fixed point ω in $\mathbb{U}$ and is not an automorphism, then an argument based on the Schwarz Lemma shows that this point attracts orbits, in the sense that the iterate sequence $(\varphi^{[n]})$ converges to ω uniformly on compact subsets of $\mathbb{U}$. The Denjoy-Wolff Theorem makes the striking assertion that, even if φ has no fixed point in $\mathbb{U}$, there is still a (necessarily unique) point ω, this time on $\partial\mathbb{U}$, that attracts orbits ($\varphi^{[n]} \to \omega$ uniformly on compact sets of $\mathbb{U}$) and behaves like a fixed point in that φ has radial (in fact non-tangential) limit ω at ω. In either case we call this unique attracting point the *Denjoy-Wolff point* of φ.

Fundamental to the proof of the Denjoy-Wolff Theorem is *Julia's Lemma*, a sort of "boundary Schwarz lemma" which, assuming φ has no fixed point in $\mathbb{U}$, asserts the existence of a point $\omega \in \partial\mathbb{U}$ such that each disc in $\mathbb{U}$ with boundary tangent to $\partial\mathbb{U}$ at ω is taken into itself by φ. This point turns out to be the Denjoy-Wolff point; in addition to attracting orbits and being a "radial limit fixed point" it is also a "point of conformality" for φ in the sense that all of the radial limits below exist and are equal, with their common value lying in the positive interval $(0,1]$:

$$\lim_{r\to 1-}\frac{1-|\varphi(r\omega)|}{1-r}, \qquad \lim_{r\to 1-}\frac{\omega-\varphi(r)}{\omega-r\omega}, \qquad \lim_{r\to 1-}\varphi'(r\omega).$$

This follows from the *Julia-Carathéodory Theorem* (see [25, Chapter 4 and §5.5]), which implies that if ω is any radial-limit fixed point of φ for which one of the above limits exists, then they all exist, and their values are the same. The common value of these limits is denoted by $\varphi'(\omega)$, and called the *angular derivative of* φ *at* ω.

Julia's Lemma can be stated more generally in terms of the angular derivative. For ω any point of $\partial\mathbb{U}$ and $0<\lambda<\infty$, let

$$H(\omega,\lambda) := \{z\in\mathbb{C} : |\omega - z|^2 < \lambda(1-|z|^2)\},$$

so $H(\omega,\lambda)$ is the Euclidean disc centered at $\omega/(1+\lambda)$ with radius $\lambda/(1+\lambda)$. The resulting discs lie in $\mathbb{U}$, have boundaries tangent to $\partial\mathbb{U}$ at ω, and expand to exhaust $\mathbb{U}$ as $\lambda \nearrow \infty$. With this notation in hand, Julia's Lemma asserts that if φ has a finite angular derivative at a radial-limit fixed point ω, then:

$$\varphi(H(\omega,\lambda)) \subset H(\omega,\varphi'(\omega)\lambda) \qquad (0<\lambda<\infty) \tag{2.4}$$

(see, e.g., [25, §4.4, page 63]).

A particularly pleasing restatement of this result follows upon mapping the unit disc to the right half-plane via the linear-fractional map $w=(\omega+z)/(\omega-z)$, which sends the "H-discs" to half-planes $P(\delta)=\{w\in\mathbb{C} : \operatorname{Re} w > \delta\}$ where $\delta>0$. In this setting, with Φ the selfmap of the right half-plane corresponding to φ, statement (2.4) simplifies to:

$$\Phi(P(\delta)) \subset P(\delta/\varphi'(\omega)) \qquad\qquad (\delta>0).$$

Note that if $\varphi'(\omega)\le 1$ then $\varphi(H(\omega,\lambda))\subset H(\omega,\lambda)$; this implies that ω is the Denjoy-Wolff point of φ (see [25, §5.5], for example). To summarize:

2.4. Proposition. *A point $\omega\in\partial\mathbb{U}$ is the Denjoy-Wolff point of φ iff:*

(i) *ω is a radial-limit fixed point of φ,*
(ii) *the angular derivative of φ exists at ω, and*
(iii) $\varphi'(\omega)\le 1$.

2.5. The Linear-Fractional Model Theorem. This result asserts that every holomorphic selfmap of $\mathbb{U}$ without interior fixed point is modelled by a linear-fractional map. More precisely, let $\omega \in \partial\mathbb{U}$ be the Denjoy-Wolff point of the holomorphic selfmap φ of $\mathbb{U}$, and recall that the angular derivative $\varphi'(\omega)$ exists, and lies in the positive interval $(0, 1]$. Valiron proved in 1931 [27] that if $\varphi'(\omega) < 1$ then there is a holomorphic mapping σ on $\mathbb{U}$, with values in the right half-plane, such that $\sigma \circ \varphi = \varphi'(\omega)\sigma$ (see also [28]). This complemented an 1884 result of Koenigs [16] who proved that if φ has an interior fixed point $\omega \in \mathbb{U}$ with $\varphi'(\omega) \neq 0$, then there is a holomorphic function σ on $\mathbb{U}$ obeying the same functional equation.

Pommerenke and Baker showed in 1979 ([5], [21]) that for $\omega \in \partial\mathbb{U}$ the case $\varphi'(\omega) = 1$ separates into two subcases, distinguished by the behavior of orbits relative to the *pseudo-hyperbolic metric*, defined on $\mathbb{U}$ by

$$\rho(z, w) := |\alpha_w(z)| = \frac{|w - z|}{|1 - \overline{w}z|} \qquad (z, w \in \mathbb{U}). \tag{2.5}$$

They showed that there exists a holomorphic function σ on $\mathbb{U}$ such that either $\sigma \circ \varphi = \sigma + ib$ for some real $b \neq 0$, or $\sigma \circ \varphi = \sigma + 1$. The former case arises when the orbits of φ are *pseudo-hyperbolically separated* in the sense that

$$\inf_n \rho(\varphi^{[n+1]}(z), \varphi^{[n]}(z)) > 0,$$

and the latter arises when they are not:

$$\inf_n \rho(\varphi^{[n+1]}(z), \varphi^{[n]}(z)) = 0 \qquad (\forall z \in \mathbb{U}).$$

In both cases "$\inf_n$" is the same as "$\lim_n$" because φ decreases the pseudo-hyperbolic metric ρ. In the first case (separated orbits) the image of σ can be taken to lie in the right half-plane.

It turns out that this separation dichotomy is independent of the "base point:" if it holds for one $z \in U$ then it holds for all of them.

Given this result it seems natural to say that φ is of *hyperbolic type* if $\varphi'(\omega) < 1$ (since in that case φ is intertwined by σ with a hyperbolic automorphism of the right half-plane), and of *parabolic type* if $\varphi'(\omega) = 1$. The maps of parabolic type then fall into two subclasses: *automorphic type* if orbits are separated and *non-automorphic type* if they are not.

The distinction between automorphic and non-automorphic parabolic type is the most subtle aspect of the linear-fractional model theorem. We will need two known sufficient conditions for non-automorphic type; for completeness we present their proofs.

2.6. Lemma. *Suppose φ is a holomorphic selfmap of $\mathbb{U}$ that is of parabolic type. If either:*

(a) *φ takes the interval $(0, 1)$ into itself, or*
(b) $\sum_n (1 - |\varphi^{[n]}(0)|) = \infty$,

then φ is of non-automorphic type.

Proof. (a) The hypothesis here is that $\varphi(r) > 0$ whenever $0 < r < 1$. Let $r_n := \varphi^{[n]}(0)$. Then the sequence (r_n) converges to the Denjoy-Wolff point which, since φ is assumed to be of parabolic type, lies on $\partial\mathbb{U}$, and so must be at 1 (in fact, by Julia's Lemma, (r_n) must *increase* to 1). From the discussion of §2.3 we therefore have

$$\lim_{r\to 1-} \frac{1-\varphi(r)}{1-r} = 1,$$

so in particular

$$\lim_n \frac{1-r_{n+1}}{1-r_n} = 1. \tag{2.6}$$

With the notation $q_n := \frac{1-r_{n+1}}{1-r_n}$ the definition (2.5) of pseudo-hyperbolic metric yields after a little algebraic manipulation:

$$\rho(r_{n+1}, r_n) = \frac{1-q_n}{q_n + r_{n+1}}, \tag{2.7}$$

hence by (2.6), $\lim_n \rho(r_{n+1}, r_n) = 0$, as desired.

(b) We prove the contrapositive statement. Suppose φ is of parabolic automorphic type. The Linear-Fractional Model Theorem (§2.5) then provides a function σ holomorphic on $\mathbb{U}$ with values in the right half-plane such that $\sigma\circ\varphi = \sigma + ib$ for some real $b \neq 0$. Hence more generally

$$\sigma\circ\varphi^{[n]} = \sigma + nbi \qquad (n = 1, 2, \ldots). \tag{2.8}$$

Now the Blaschke condition for a sequence (z_n) in $\mathbb{U}$ is equivalent, via the map $w = (1+z)/(1-z)$ to the condition $\sum_n \operatorname{Re} w_n/(|1+w_n|)^2 < \infty$ for sequences (w_n) in the right half-plane; this is easily seen to be satisfied by the sequence $(\sigma(0)+nbi)_1^\infty$, which is therefore the zero-sequence of a bounded holomorphic function F on the right half-plane (see [13, Theorem 11.3, page 191]). The function $f = F\circ\sigma$ is thus a non-constant bounded holomorphic function on $\mathbb{U}$, and for each n:

$$f(\varphi^{[n]}(0)) = F(\sigma(\varphi^{[n]}(0))) = F(\sigma(0) + nbi) = 0,$$

where the second equality comes from evaluating (2.8) at the origin. Thus some nonconstant bounded holomorphic function on $\mathbb{U}$ vanishes at each point of $(\varphi^{[n]}(0))$, so that sequence satisfies the Blaschke condition. □

We remark that in part (a) of Lemma 2.6, the hypothesis can be restated as saying that $(\varphi^{[n]}(0))$ converges to the Denjoy-Wolff point along the radius from the origin to that point. In fact, it is enough to assume only that this orbit stays within an angular sector with vertex at the Denjoy-Wolff point (see [9, Theorem 3.5] and [8, Theorem 6.1, page 97]). We will see in §5.1 that the converse of part (b) need not hold, i.e., it is possible for φ to be of parabolic non-automorphic type yet have orbits that are Blaschke sequences. Indeed in [10, Proposition 4.9] Cowen shows that for maps of parabolic type the ones of automorphic type are characterized by orbits being *interpolating* sequences—in general a strictly smaller class of sequences than the Blaschke sequences. Our proof of part (b) of Lemma 2.6 follows Cowen's.

2.7. Composition operators. Each holomorphic selfmap φ of $\mathbb{U}$ induces on $H(\mathbb{U})$, the space of all functions holomorphic on $\mathbb{U}$, a linear *composition operator* C_φ by means of the definition

$$C_\varphi f = f \circ \varphi \qquad (f \in H(\mathbb{U})).$$

The interest in composition operators arises from the remarkable fact that they restrict to bounded operators on Hardy spaces. This results from *Littlewood's Subordination Theorem* which asserts that if $\varphi(0) = 0$ then $\|C_\varphi f\|_2 \leq \|f\|_2$ for each $f \in H^2$, i.e., that C_φ is a contraction on H^2. In fact the same is true for all the spaces H^p, but the case $p = 2$ will suffice for what we do here. If $\varphi(0) \neq 0$ then C_φ is no longer a contraction, but it is still bounded on H^2; this follows from Littlewood's Theorem and the easily proven fact that conformal automorphisms of $\mathbb{U}$ induce bounded composition operators on H^2. Littlewood's Theorem and the estimate that leads to the boundedness of automorphism-induced composition operators show that for any holomorphic selfmap φ of $\mathbb{U}$, the norm of C_φ on H^2 has this upper bound:

$$\|C_\varphi\| \leq \left(\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}\right)^{1/2} \tag{2.9}$$

(see, e.g., [25, §1.3] for the details).

If $\varphi(0) = 0$ then the above norm estimate gives $\|C_\varphi\| \leq 1$, in accordance with Littlewood's Theorem. In fact, there is equality here, since clearly $C_\varphi 1 = 1$ (where here "1" denotes the constant function). If we rule out constant functions by restricting C_φ to the invariant subspace of functions in H^2 that vanish at the origin, then *the norm of this restriction is* < 1*, except when* φ *is an inner function.* The first part of this statement comes from [26, Theorem 5.1], while the fact that inner functions that vanish at the origin induce composition operators that have norm-one restrictions is a consequence of a much stronger statement: such inner functions induce isometric composition operators on H^2 (see [19]).

2.8. Poisson Integrals. For $f \in L^1(\partial\mathbb{U})$ and $z = re^{i\theta} \in \mathbb{U}$, the *Poisson integral of* f *at* z is:

$$P[f](z) := \int_{\partial\mathbb{U}} \mathrm{Re}\left\{\frac{\zeta + z}{\zeta - z}\right\} f(\zeta)\, dm(\zeta) = \sum_{n=-\infty}^{\infty} \hat{f}(n) r^{|n|} e^{in\theta},$$

where m denotes normalized Lebesgue measure on $\partial\mathbb{U}$ and $\hat{f}(n)$ is the n-th Fourier coefficient of f:

$$\hat{f}(n) = \int_{\partial\mathbb{U}} f(\zeta)\zeta^{-n}\,dm(\zeta) \qquad (n \in \mathbb{Z}).$$

$P[f]$ is harmonic on $\mathbb{U}$ and has radial limit equal to f at almost every point of $\partial\mathbb{U}$. Note that $P[f](0) = \int f\,dm$.

Here is the key to all our applications of Poisson integrals:

2.9. Lemma ([19, Lemma 1]). *If φ is an inner function then for every $f \in L^1(\partial\mathbb{U})$,*

$$P[f \circ \varphi](z) = P[f](\varphi(z)) \qquad (z \in \mathbb{U}). \tag{2.10}$$

Proof. The proof consists of noting that if $f(\zeta) = \zeta^n$ with $\zeta \in \partial\mathbb{U}$ and n a non-negative integer, then for each $z \in \mathbb{U}$ the left-hand side of (2.10) is $P[\varphi^n](z) = \varphi^n(z)$, which is also the right-hand side. Upon taking complex conjugates and using the fact that $|\varphi| = 1$ a.e. on $\partial\mathbb{U}$ we get the same result for $n < 0$, thus the desired equation holds for all trigonometric monomials, hence by linearity for all trigonometric polynomials and therefore, by taking L^1-limits, for each $f \in L^1(\partial\mathbb{U})$. □

This Lemma can be interpreted as a statement about how inner functions transform the “Poisson measures” on $\partial\mathbb{U}$:

$$dm_z(\zeta) := \mathrm{Re}\left\{\frac{\zeta + z}{\zeta - z}\right\} dm(\zeta)$$

for $z \in \mathbb{U}$. With this notation Lemma 2.9 asserts that:

$$dm_z\varphi^{-1} = dm_{\varphi(z)} \qquad (\varphi \text{ inner},\ z \in \mathbb{U}). \tag{2.11}$$

In particular, if an inner function φ has its Denjoy-Wolff point ω in $\mathbb{U}$, i.e., $\varphi(\omega) = \omega \in \mathbb{U}$, then dm_ω is *invariant* for φ on $\partial\mathbb{U}$ in the sense that $dm_\omega\varphi^{-1} = dm_\omega$. As an important special case we have: *if φ is inner and fixes the origin, then Lebesgue measure on $\partial\mathbb{U}$ is invariant for φ.*

For general inner functions φ the case $z = 0$ of (2.11) is also of special interest. It asserts that $dm\varphi^{-1} = dm_{\varphi(0)}$; in particular: *$dm\varphi^{-1}$ and Lebesgue measure are mutually absolutely continuous on $\partial\mathbb{U}$.*

3. Norm convergence to the Denjoy-Wolff point

It is well known, and easy to prove, that if a sequence of functions in H^p is bounded in norm and converges uniformly on compact subsets of $\mathbb{U}$, then it converges in the weak-star topology. This applies in particular to holomorphic selfmaps of $\mathbb{U}$, so the Denjoy-Wolff Theorem can be rephrased:

> *Suppose φ is a holomorphic selfmap of $\mathbb{U}$ with Denjoy-Wolff point ω. Then for $1 \le p < \infty$ the iterate sequence $(\varphi^{[n]})$ converges to ω in the weak-star topology of H^p.*

This re-interpretation of the Denjoy-Wolff Theorem suggests that one should inquire about *norm* convergence of the iterate sequence to the Denjoy-Wolff point. Clearly norm convergence does not always happen: if φ is inner and has its Denjoy-Wolff point in $\mathbb{U}$ (e.g., $\varphi(z) = z^2$), then each iterate is inner (see the comment at the end of §2.2), and so has norm one in every H^p space. Thus, in this case, the iterate sequence cannot converge to the Denjoy-Wolff point of φ. As the following result shows, this is the only exceptional case.

3.1. Theorem. *For all holomorphic selfmaps φ of $\mathbb{U}$ except the inner functions with Denjoy-Wolff point in $\mathbb{U}$, the iterate sequences converge in the H^p-norm ($1 \le p < \infty$) to the Denjoy-Wolff point.*

Proof. This proceeds in several steps. Let ω denote the Denjoy-Wolff point of φ.

STEP I. *It suffices to consider the case $p = 2$.*

Proof. What we mean here is that if $1 \le p < q < \infty$, then $(\varphi^{[n]})$ converges to ω in H^p if and only if it converges to ω in H^q. Since (on H^q) the H^q-norm is larger than the H^p norm, we need only show that H^p convergence of the iterate sequence implies H^q convergence. Since $|\varphi^{[n]} - \omega| < 2$ on $\mathbb{U}$ we have, after making a simple estimate of the integrals (2.1) that define the norms in question:

$$\|\varphi^{[n]} - \omega\|_q^q \le 2^{q-p} \|\varphi^{[n]} - \omega\|_p^p$$

which gives the desired result. □

From now on we drop the subscript "2" when referring to the H^2 norm.

STEP II. *The Theorem holds if φ has its Denjoy-Wolff point ω in $\mathbb{U}$.*

Proof. If $\omega = 0$ this follows quickly from the restriction theorem for composition operators mentioned at the end of §2.7: *If φ fixes the origin and is not inner, then the norm of $C_\varphi|_{H_0^2}$ (the restriction of C_φ to the subspace H_0^2 of H^2 consisting of functions that vanish at the origin) is* < 1. Now $C_\varphi^n z = \varphi^{[n]}$, and the monomial z is a unit vector in H_0^2, hence

$$\|\varphi^{[n]}\| = \|C_\varphi^n z\| \le \|(C_\varphi|_{H_0^2})^n\| \|z\| \le \|C_\varphi|_{H_0^2}\|^n \to 0 \tag{3.1}$$

as $n \to \infty$.

If $\omega \ne 0$ then we use the automorphism α_ω introduced in §2.2 above to get back to the previous case. Recall that α_ω interchanges the origin with ω and is self-inverse. Thus $\psi := \alpha_\omega \circ \varphi \circ \alpha_\omega$ is a holomorphic selfmap of $\mathbb{U}$ that has the origin as its Denjoy-Wolff point, so by the result of the last paragraph,

$$\|\psi^{[n]}\| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.2}$$

For each positive integer n we have $\psi^{[n]} = \alpha_\omega \circ \varphi^{[n]} \circ \alpha_\omega$, which for our purposes is best rewritten:

$$C_{\alpha_\omega}(\psi^{[n]}) := \psi^{[n]} \circ \alpha_\omega = \alpha_\omega \circ \varphi^{[n]}. \tag{3.3}$$

Since C_{α_ω} is a bounded operator on H^2, (3.2) and (3.3) combine to yield

$$\|\alpha_\omega \circ \varphi^{[n]}\| = \|C_{\alpha_\omega}(\psi^{[n]})\| \leq \|C_{\alpha_\omega}\| \, \|\psi^{[n]}\| \to 0 \qquad (n \to \infty). \tag{3.4}$$

Now at each point of $\mathbb{U}$:

$$|\alpha_\omega \circ \varphi^{[n]}| = \frac{|\omega - \varphi^{[n]}|}{|1 - \overline{\omega}\varphi^{[n]}|} \geq \frac{|\omega - \varphi^{[n]}|}{2} \ ,$$

so by (3.4) and the case $p = 2$ of the integral formula (2.1) for Hardy-space norms,

$$\|\omega - \varphi^{[n]}\| \leq 2\|\alpha_\omega \circ \varphi^{[n]}\| \to 0 \qquad (n \to \infty)$$

as desired. □

STEP III. *The Theorem holds if φ has its Denjoy-Wolff point on $\partial\mathbb{U}$.*

Proof. As noted (somewhat more generally) at the beginning of this section, the iterate sequence of φ converges to the Denjoy-Wolff point *weakly* in H^2. Thus the result we are asserting here is a special case of the following elementary Hilbert-space theorem:

> *Suppose (x_n) is a sequence in the unit ball of a Hilbert space $\mathcal{H}$ that converges weakly to a vector x of norm one. Then $x_n \to x$ in the norm of $\mathcal{H}$.*

For the proof note that for each n,

$$\|x_n - x\|^2 = \|x_n\|^2 - 2\,\mathrm{Re}\langle x_n, x\rangle + \|x\|^2 \leq 2(1 - \mathrm{Re}\langle x_n, x\rangle). \tag{3.5}$$

Moreover by weak convergence, $\langle x_n, x\rangle \to \|x\|^2 = 1$, hence $\|x_n - x\| \to 0$ as $n \to \infty$. □

Remark. Using (2.1) one may extend the definition of H^p to the range $0 < p < 1$, thus obtaining a family of F-spaces that are not locally convex [13, §7.5]. If now $0 < p \leq q < \infty$ then convergence in H^q still implies convergence in H^p, so Theorem 3.1 is true for the full range of p: $0 < p < \infty$.

4. Almost everywhere convergence to the Denjoy-Wolff point

In §2.1 we pointed out that for each $f \in H^2$ the H^2-norm of f coincides with its L^2-norm on the unit circle (as in §2.1 we continue to rely on context rather than notation in distinguishing between an H^2 function on $\mathbb{U}$ and its radial limit function on $\partial\mathbb{U}$). Thus the results of the last section imply that unless φ is an inner function with Denjoy-Wolff point in $\mathbb{U}$, some subsequence of the iterate sequence converges to the Denjoy-Wolff point a.e. (with respect to Lebesgue measure) on $\partial\mathbb{U}$. In this section we address the question of a.e. convergence for the *entire* iterate sequence—a mode of convergence which, thanks to the Dominated Convergence Theorem, is even stronger than norm convergence.

If φ has its Denjoy-Wolff point in $\mathbb{U}$ then, as the next result shows, the desired a.e. convergence is an easy consequence of the norm estimates of the previous section.

4.1. Theorem. *If φ is a non-inner holomorphic selfmap of $\mathbb{U}$ with Denjoy-Wolff point $\omega \in \mathbb{U}$ then $\varphi^{[n]} \to \omega$ a.e. on $\partial\mathbb{U}$.*

Proof. Suppose first that $\omega = 0$ and let M denote the norm of the restriction of C_φ to the subspace H_0^2. In the last section we exploited the fact that $M < 1$; now we do so more fully. Upon squaring both sides of the inequality that results from (3.1) of the last section we obtain

$$\int_{\partial\mathbb{U}} \left(\sum_{n=1}^{\infty} |\varphi^{[n]}|^2 \right) dm = \sum_{n=1}^{\infty} \int_{\partial\mathbb{U}} |\varphi^{[n]}|^2 \, dm = \sum_{n=1}^{\infty} \|\varphi^{[n]}\|^2 \leq \sum_{n=1}^{\infty} M^{2n} < \infty,$$

hence $\sum_{n=1}^{\infty} |\varphi^{[n]}|^2 < \infty$ a.e. on $\partial\mathbb{U}$, so in particular $\varphi^{[n]} \to 0$ a.e.

In case $\omega \neq 0$ we recall from the last section that $\varphi^{[n]} = \alpha_\omega \circ \psi^{[n]} \circ \alpha_\omega$, where $\psi := \alpha_\omega \circ \varphi \circ \alpha_\omega$ has its Denjoy-Wolff point at the origin. Then $\psi^{[n]} \to 0$ a.e., and it follows readily (since $\alpha_\omega(0) = \omega$) that $\varphi^{[n]} \to \omega$ a.e. □

Here is the main result of this section; part (a) extends a result of Doering and Mañe [12] who considered only inner functions.

4.2. Theorem. *Suppose φ is a holomorphic selfmap of $\mathbb{U}$ with Denjoy-Wolff point $\omega \in \partial\mathbb{U}$.*

(a) *If $\sum_n (1 - |\varphi^{[n]}(0)|) < \infty$ then $\varphi^{[n]} \to \omega$ a.e. on $\partial\mathbb{U}$.*
(b) *If φ is* inner *and $\varphi^{[n]} \to \omega$ a.e. on $\partial\mathbb{U}$ then $\sum_n (1 - |\varphi^{[n]}(0)|) < \infty$.*

Proof. (a) Without loss of generality we may take $\omega = 1$. By Julia's Lemma (see §2.3), φ takes the closed disc of radius 1/2 centered at 1/2 into itself, thus

$$|z|^2 \leq \operatorname{Re} z \implies |\varphi(z)|^2 \leq \operatorname{Re} \varphi(z) \qquad (z \in \mathbb{U}).$$

Upon applying this inequality successively to $0, \varphi(0), \varphi^{[2]}(0) \dots$, we see that for each positive integer n, $|\varphi^{[n]}(0)|^2 \leq \operatorname{Re}\varphi^{[n]}(0)$, thus

$$1 - \operatorname{Re}\langle \varphi^{[n]}, 1\rangle = 1 - \operatorname{Re}\varphi^{[n]}(0) \leq 1 - |\varphi^{[n]}(0)|^2 \leq 2(1 - |\varphi^{[n]}(0)|). \tag{4.1}$$

This estimate along with inequality (3.5) yields

$$\sum_{n=1}^{\infty} \|\varphi^{[n]} - 1\|^2 \leq 2\sum_{n=1}^{\infty}[1 - \operatorname{Re}\varphi^{[n]}(0)] \leq 4\sum_{n=1}^{\infty}[1 - |\varphi^{[n]}(0)|] < \infty,$$

so $\varphi^{[n]} \to 1$ a.e. on $\partial\mathbb{U}$ by the argument used to finish the proof of Theorem 4.1.

(b) We begin with an argument from ergodic theory. Let Δ be the open disc of radius $1/2$ centered at ω. By Egorov's Theorem there is a compact set $K \subset \partial\mathbb{U}\backslash\Delta$ of positive measure such that $\varphi^{[n]} \to \omega$ uniformly on K. Thus $\varphi^{[n]}(K) \cap K = \emptyset$ for all sufficiently large n, and so the same is true for $K \cap \varphi^{[-n]}(K)$ (since for any sets A and B and any map T we have $T(A \cap T^{-1}(B)) = T(A) \cap B)$).

Thus there is a least non-negative integer N for which

$$m\{K \cap \varphi^{[-n]}(K)\} = 0 \text{ for each } n > N. \tag{4.2}$$

Let $W = K \cap \varphi^{[-N]}(K)$. Then $m(W) > 0$ and for each positive integer k:

$$W \cap \varphi^{[-k]}(W) \subset K \cap \varphi^{-[N+k]}(K),$$

hence $m\{W \cap \varphi^{[-k]}(W)\} = 0$.

Recall from the discussion following the proof of Lemma 2.9 that the measures $dm\varphi^{-1}$ and dm are mutually absolutely continuous, hence (4.2) yields for each positive integer k and non-negative integer n

$$0 = m\{\varphi^{[-n]}(W \cap \varphi^{[-k]}(W))\} = m\{\varphi^{[-n]}(W) \cap \varphi^{[-(n+k)]}(W)\},$$

i.e., the sequence of sets $(\varphi^{[-k]}(W) : k \geq 0)$ is "essentially pairwise disjoint," and each has positive measure. (In the parlance of ergodic theory, W is called a "wandering set" for φ; and φ, being the possessor of a wandering set, is called "dissipative".)

For convenience write $W_n = \varphi^{[-n]}(W)$, and set $E = \bigcup_0^\infty W_n$. Some calculations with Poisson integrals now finish the proof; here I_A will denote the indicator function of the set A ($\equiv 1$ on A and $\equiv 0$ off A).

$$\begin{aligned}
P[I_E](0) &= P\left[\sum_{n=0}^{\infty} I_{W_n}\right](0) \qquad \text{(ess. disjointness of } W_n\text{'s)}\\
&= \sum_{n=0}^{\infty} P[I_{W_n}](0)\\
&= \sum_{n=0}^{\infty} P[I_W](\varphi^{[n]}(0)) \qquad \text{(eqn. 2.10 above)}\\
&\geq \sum_{n=0}^{\infty} m(W)\frac{1-|\varphi^{[n]}(0)|}{1+|\varphi^{[n]}(0)|} \qquad \text{(Harnack's inequality)}\\
&\geq \frac{1}{2}m(W)\sum_{n=0}^{\infty}(1-|\varphi^{[n]}(0)|),
\end{aligned}$$

so, because $m(W) > 0$ and $P[I_E](0) = m(E) \leq 1$, we see that

$$\sum_{n=0}^{\infty}(1-|\varphi^{[n]}(0)|) < \infty,$$

as desired. □

4.3. Remark. The conclusion of part (b) can fail for non-inner maps φ. Perhaps the simplest such examples arise from translation mappings on half-planes. Fix a complex number t with $\mathrm{Re}\, t \geq 0$ and let φ_t be the linear-fractional map that corresponds to the translation $w \to w + t$ via the transformation $w = (1 + z)/(1 - z)$ of $\mathbb{U}$ onto the right half-plane. A calculation shows that

$$\varphi_t(z) = \frac{t + (2 - t)z}{(2 + t) - tz},$$

from which it is clear that $\varphi_t(z) \to 1$ for each $z \in \partial\mathbb{U}$, as $t \to \infty$. From its origins as a translation it's clear that the n-th iterate of φ_t is just φ_{nt}, hence 1 is the Denjoy-Wolff point of each φ_t, and $\varphi_t^{[n]} \to 1$ pointwise on $\partial\mathbb{U}$ as $n \to \infty$. However $\varphi_t^{[n]}(0) = nt/(2 + nt)$, from which it is easy to check that $(\varphi_t^{[n]}(0))$ is a Blaschke sequence if and only if $\mathrm{Re}\, t = 0$. In this case φ_t is an inner function that illustrates part (b) of our theorem. When $\mathrm{Re}\, t > 0$, however, $\varphi = \varphi_t$ is not inner, and the orbit of 0 is not a Blaschke sequence, yet nevertheless $\varphi^{[n]} \to 1$ at each point of $\partial\mathbb{U}$. □

Recall from our discussion in §2.5 of linear-fractional models that the holomorphic selfmaps of $\mathbb{U}$ with Denjoy-Wolff point ω split into three disjoint classes: hyperbolic if $\varphi'(\omega) < 1$, parabolic automorphic if $\varphi'(\omega) = 1$ and orbits are separated, and parabolic non-automorphic if $\varphi'(\omega) = 1$ and orbits are *not* separated. The next result shows that Blaschke orbits (and hence a.e. convergence of iterates to the Denjoy-Wolff point) are guaranteed for two of these three classes.

4.4. Theorem. *Suppose φ is a holomorphic selfmap of $\mathbb{U}$ with Denjoy-Wolff point $\omega \in \partial\mathbb{U}$. If φ is of either hyperbolic type or of parabolic-automorphic type then $\sum_n (1 - |\varphi^{[n]}(0)|) < \infty$, hence $\varphi^{[n]} \to \omega$ a.e. on $\partial\mathbb{U}$.*

Proof. The result about parabolic-automorphic type has already been proved as part (b) of Lemma 2.6.

Suppose φ is of hyperbolic type. Recall that Julia's Lemma (§2.3) tells us that for each $\lambda > 0$

$$\varphi(H(\omega, \lambda)) \subset H(\omega, \varphi'(\omega)\lambda),$$

thus for each $\lambda > 0$ and $z \in \mathbb{U}$:

$$|\omega - z|^2 \le \lambda(1 - |z|^2) \quad \Longrightarrow \quad |\omega - \varphi(z)|^2 \le \varphi'(\omega)\lambda(1 - |\varphi(z)|^2)\ .$$

Now the origin lies in the closure of $H(\omega, 1)$, so an induction shows that for each positive integer n, the point $\varphi^{[n]}(0)$ lies in the closure of $H(\omega, \varphi'(\omega)^n)$, i.e.,

$$|\omega - \varphi^{[n]}(0)|^2 \le \varphi'(\omega)^n (1 - |\varphi^{[n]}(0)|^2) \le \varphi'(\omega)^n.$$

Thus for $n = 1, 2, \dots$:

$$1 - |\varphi^{[n]}(0)| \le |\omega - \varphi^{[n]}(0)| \le \varphi'(\omega)^{n/2},$$

hence (because $\varphi'(\omega) < 1$) $\sum_n (1 - |\varphi^{[n]}(0)|) < \infty$, as desired. $\square$

4.5. Remark. One might hope that if φ is of parabolic type and is in some sense twice differentiable at its Denjoy-Wolff point $\omega \in \partial\mathbb{U}$, then $\varphi''(\omega)$ would determine whether or not φ is of automorphic or non-automorphic type. If enough extra smoothness is assumed at ω then this does indeed happen, as shown in [8, Theorem 4.4, page 52]. This result is most easily stated if φ has Denjoy-Wolff point at 1, in which case *if $\varphi \in C^{3+\varepsilon}(1)$ with $\varphi'(1) = 1$ and $\varphi''(1)$ non-zero and pure imaginary, then φ is of parabolic automorphic type.* The $C^{3+\varepsilon}$ hypothesis at 1 means that, for some disc Δ centered at 1, φ is approximated in $\mathbb{U} \cap \Delta$ by a polynomial of degree three, with error term bounded uniformly in magnitude there by a constant multiple of $|z - 1|^{3+\varepsilon}$ (see [8, page 50]). Surprisingly, the smoothness hypothesis cannot be reduced to $C^{3-\varepsilon}$ (see [8, page 98]).

We close this section by applying the preceding ideas to an interesting class of inner functions, generalizations of which we will meet in Section 5.

4.6. Example. Recall the unit singular function S defined in §2.2(b). For $a > 0$ let

$$\varphi_a(z) = S(-z)^a = \exp\left\{-a\frac{1-z}{1+z}\right\} \qquad (z \in \mathbb{U});$$

this is the singular inner function induced by the atomic measure of mass a concentrated at the point -1. Clearly φ_a is holomorphic on $\mathbb{C}\backslash\{-1\}$ with $\varphi_a(1) = 1$ and $\varphi_a'(1) = a/2$. Since φ_a maps the interval $(-1, 1)$ into $(0, 1)$, its Denjoy-Wolff point ω, being the limit of each orbit, must lie in $(0, 1]$. If $a \le 2$

then $\varphi_a'(1) \le 1$ so, by Proposition 2.4, $\omega = 1$; moreover φ_a is of parabolic type when $a = 2$, and is of hyperbolic type when $a < 2$. If $a > 2$ then $\varphi_a'(1) > 1$, so the Denjoy-Wolff point cannot be at 1, and must therefore lie in $(0, 1)$. Thus the iterate sequence $(\varphi_a^{[n]})$ converges a.e. to the Denjoy-Wolff point of φ_a if $a < 2$, but not if $a > 2$.

The remaining case, $a = 2$, is more delicate:

4.7. Proposition. *φ_2 is of non-automorphic type and has non-Blaschke orbits.*

Thus φ_2 is an inner function of parabolic non-automorphic type whose iterate sequence does not converge a.e. on $\partial\mathbb{U}$ to the Denjoy-Wolff point. In the next section we will show how to produce singular inner functions of parabolic non-automorphic type whose iterate sequences *do* converge a.e. to the Denjoy-Wolff point.

Proof. For simplicity write $\varphi_2 = \varphi$. Since φ maps the unit interval into itself it is of non-automorphic type by Lemma 2.6(a). It remains to show that the orbit $(\varphi^{[n]}(0))$ does not satisfy the Blaschke condition.

A computation shows that $\varphi''(1) = 0$ and $\varphi'''(1) = -1/2$. Thus the Taylor expansion of φ with center at 1 looks like this:

$$\varphi(z) = 1 + (z-1) - \frac{1}{12}(z-1)^3 + (O(z-1)^4) \quad \text{as } z \to 1,$$

which we rewrite as

$$1 - \frac{1-\varphi(z)}{1-z} = \frac{1}{12}(1-z)^2 + O((1-z)^3) \quad \text{as } z \to 1. \tag{4.3}$$

Let's write $r_n := \varphi^{[n]}(0)$ and $y_n := 1/(1-r_n)$. As noted in the proof of Lemma 2.6(a), $r_n \nearrow 1$, hence $y_n \nearrow \infty$. Upon setting $z = r_n$ in (4.3) we obtain:

$$1 - \frac{y_n}{y_{n+1}} = \frac{1}{12}y_n^{-2} + O(y_n^{-3}) \quad \text{as } n \to \infty. \tag{4.4}$$

This will insure that (r_n) does not satisfy the Blaschke condition. To see why, rewrite (4.4) as

$$y_{n+1} - y_n = \frac{1}{12}y_n^{-1} + O(y_n^{-2}) \quad \text{as } n \to \infty,$$

where we have used the fact that $y_n/y_{n+1} \to 1$ (because φ has angular derivative equal to 1 at the point 1). Upon adding y_n to both sides of the last equation, squaring both sides of the result, and subtracting y_n^2 from both sides of the final expression we see that

$$y_{n+1}^2 - y_n^2 \to \frac{1}{6} \quad \text{as } n \to \infty.$$

A little argument with telescoping sums shows that $y_n^2/n \to 1/6$ as $n \to \infty$, which is equivalent to saying that $\sqrt{n}(1-r_n) \to \sqrt{6}$, hence $\sum_n (1-r_n) = \infty$, as promised. $\square$

4.8. Remarks. (a) A theorem of Burns and Krantz [6] (see also [7, Theorem 2.4(4)]) implies that if a holomorphic selfmap of $\mathbb{U}$ is of class C^3 at 1, in the sense that it has a Taylor expansion

$$\varphi(z) = 1 + (z-1) + a(z-1)^2 + b(z-1)^3 + o(z-1)^3 \quad \text{as } z \to 1 \text{ in } \mathbb{U},$$

and if $a = b = 0$, then $\varphi(z) \equiv z$ in $\mathbb{U}$. Suppose φ is real on the real axis with Denjoy-Wolff point at 1 and is of class C^3 at 1. If $a = 0$ in the above expansion then the fact that $\varphi(x) > x$ for each $x \in [0,1)$ implies that $b \leq 0$, so, by Burns-Krantz, $b < 0$. The argument we just applied to φ_2 shows that $\sqrt{n}(1-\varphi^{[n]}(0)) \to 1/\sqrt{-2b}$. If $a \neq 0$ then the argument is even easier (see below), and shows that $n(1-\varphi^{[n]}(0)) \to 1/a$. Thus in either case the orbit $(\varphi^{[n]}(0))$ is not Blaschke. In summary:

Theorem. *If φ is a holomorphic selfmap of $\mathbb{U}$ of parabolic type that is real on the real axis and of class C^3 at its Denjoy-Wolff point then φ is of non-automorphic type and its orbits do not satisfy the Blaschke condition.*

(b) In the arguments above the restriction that φ be real on the real axis can be relaxed. Suppose, for example, that φ is of class C^2 at 1, so that

$$\varphi(z) = 1 + (z-1) + a(z-1)^2 + o(z-1)^2 \quad \text{as } z \to 1 \text{ in } \mathbb{U},$$

and suppose that $\operatorname{Re} a \neq 0$. Then, repeating the previous arguments:

$$y_{n+1} - y_n = (a + o(1)) \text{ so } y_n/n \to a, \text{ i.e., } n(1-\varphi^{[n]}(0)) \to 1/a \text{ as } n \to \infty.$$

By Julia's Lemma each of the points $z_n := \varphi^{[n]}(0)$ lies in the closed disc Δ of radius $1/2$, centered at the point $1/2$, so a second application of inequality (4.1) provides:

$$n(1-|z_n|^2) \geq n \operatorname{Re}(1-z_n) \to \operatorname{Re}(1/a) \quad \text{as } n \to \infty.$$

Since Δ lies in the right half-plane, $\operatorname{Re}(1-z_n) > 0$ for each $n = 1, 2, \ldots$, hence $\operatorname{Re}(1/a) \geq 0$; actually > 0 since we are assuming $\operatorname{Re} a \neq 0$. Thus

$$\liminf_{n\to\infty} n(1-|z_n|^2) \geq \operatorname{Re}(1/a) > 0,$$

so once again the orbits of φ are not Blaschke sequences.

Although we have not shown it in this case, φ is again of non-automorphic type (see [8, Theorem 4.4, page 52]). In summary:

Theorem (see also [8, Theorem 4.4 and Lemma 4.5])**.** *Suppose φ is of parabolic type and of class C^2 at its Denjoy-Wolff point $\omega \in \partial\mathbb{U}$. If $\operatorname{Re}\varphi''(\omega) \neq 0$ (thus necessarily > 0) then φ is of non-automorphic type and has non-Blaschke orbits.*

(c) The ideas used in the proof of Theorem 4.7 to estimate the rate at which orbits approach the Denjoy-Wolff point are modifications of ones used by Aaronson in [1]. They will appear again in the next section.

5. Connections with ergodic theory

The setting for ergodic theory is a quadruple $(X, \mathcal{B}, \mu, T)$, where $(X, \mathcal{B}, \mu)$ is a sigma-finite measure space and $T : X \to X$ is $\mathcal{B}$-measurable and *nonsingular* in the sense that if $E \in \mathcal{B}$ and $\mu(E) = 0$ then $\mu(T^{-1}(E)) = 0$ (i.e., μT^{-1} is absolutely continuous with respect to μ). In our setting the measure space will be $\partial\mathbb{U}$ with Lebesgue measure m on the Borel sets of $\partial\mathbb{U}$. The transformation T will be the (radial limit) restriction of an inner function φ to $\partial\mathbb{U}$. It was noted in §2.8 that φ on $\partial\mathbb{U}$ is nonsingular for m, and even "quasi-invariant" in the sense that $m\varphi^{-1}$ and m are mutually absolutely continuous.

Since in this section we will be studying the dynamics of inner functions on the unit circle, we note once again that, as discussed at the end of §2.2, if φ is an inner function then $\varphi^{[n]}$ is also inner and, in fact, the n-th iterate of the radial limit function of φ is equal a.e. on $\partial\mathbb{U}$ to the radial limit function of $\varphi^{[n]}$.

5.1. Ergodicity. Returning to our general setting, we say that two $\mathcal{B}$-measurable sets are "equal mod μ" if their symmetric difference has μ-measure zero. Thus, for example, the definition of "nonsingular" for T can be rephrased: "If E is empty mod μ then so is $T^{-1}(E)$." We call $E \in \mathcal{B}$ *invariant* for T (and μ) if $T^{-1}(E) = E$ mod μ. If the only T-invariant sets are (mod μ) X and $\emptyset$ then T is said to be *ergodic*. Upon replacing sets by their indicator functions and using standard arguments there emerges this function-space characterization of ergodicity: *T is ergodic if and only if the only T-invariant measurable functions are the (μ a.e.) constants.* Here "T-invariant" for a function f means that $f \circ T = f$ (μ a.e.). In this statement "measurable" can just as well be replaced by "integrable" or "bounded measurable."

The ergodic theory of inner functions has been much studied, and exhibits beautiful connections with the theory of linear-fractional models. Work of Aaronson [1], [2] and Neuwirth [18], building on foundations laid by Valiron [27] (see also [22]) combine to show that:

> *An inner function is ergodic if and only if its orbits are non-separated,*

or equivalently (see §2.5):

An inner function φ is ergodic if and only if either its Denjoy-Wolff point is in $\mathbb{U}$ or the Denjoy-Wolff point is on $\partial\mathbb{U}$ and the function is of parabolic non-automorphic type.

For a complete exposition of this see [3, Chapter 6].

To get a feeling for this result, suppose φ has its Denjoy-Wolff point ω in $\mathbb{U}$, so each orbit $(\varphi^{[n]}(z))$ (for $z \in \mathbb{U}$) converges to ω; clearly these orbits are not separated. To see that φ acts ergodically on $\partial\mathbb{U}$ suppose $f \in L^1(\partial\mathbb{U})$ is φ-invariant. Then for $z \in \mathbb{U}$:

$$P[f](z) = P[f \circ \varphi^{[n]}](z) = P[f](\varphi^{[n]}(z)) \to P[f](\omega),$$

where the first equality follows from the φ-invariance of f, and the second from Lemma 2.9. Thus $P[f]$ is constant on $\mathbb{U}$, and hence f, which agrees almost everywhere with the radial limit function of $P[f]$, must be constant a.e. on $\partial\mathbb{U}$. Thus φ is ergodic.

The key to this little argument is the attracting nature of the interior Denjoy-Wolff point, which arises from the Schwarz Lemma. In case $\omega \in \partial\mathbb{U}$ a harmonic version of the Schwarz Lemma (see [4, Chapter 6]) plays a similar role in relating non-separation of orbits to ergodicity; for details see [3, Theorem 6.1.5, page 204].

5.2. Ergodicity and a.e. convergence. According to the Birkhoff Ergodic Theorem [29, §1.6, Theorem 1.14, page 34], if T is ergodic for a finite measure μ that is invariant for T (i.e., $\mu T^{-1} = \mu$), then for any $f \in L^1(\mu)$ the Cesaro means of the sequence $(T^n f)$ converge a.e. to the constant $\int f\, d\mu$. If φ is an inner function with Denjoy-Wolff point $\omega \in \mathbb{U}$, then φ preserves the Poisson probability measure m_ω, as discussed near the end of §2.8. Upon choosing f to be the identity map in the Birkhoff theorem we see that the Cesaro means of the sequence $(\varphi^{[n]})$ converge a.e. on $\partial\mathbb{U}$ to $\int_{\partial\mathbb{U}} \zeta\, dm_\omega(\zeta) = \omega$. Since the iterate sequence cannot converge a.e. to any point of $\mathbb{U}$, it therefore cannot converge a.e. to any function.

Suppose now that $\omega \in \partial\mathbb{U}$. Can φ be ergodic, yet a.e. convergent on $\partial\mathbb{U}$? That this sort of thing can happen in general is illustrated by the simple example of the "translation-by-one" map $Tx = x + 1$ on the set of integers, endowed with the counting measure. If we adjoin ∞ to this set, give it measure zero, and make it fixed by T, then it is easy to see that T is ergodic, even though $T^n \to \infty$ a.e. For inner functions, Aaronson [2, page 241] proved that the same thing can happen: *there are ergodic inner functions whose iterate sequences converge a.e. on $\partial\mathbb{U}$ to the Denjoy-Wolff point.*

In view of Theorem 4.2 above, Aaronson's result can be stated in concrete function-theoretic terms:

There exists an inner function of parabolic non-automorphic type (i.e., Denjoy-Wolff point $\omega \in \partial\mathbb{U}$, $\varphi'(\omega) = 1$, and orbits non-separated) whose orbits satisfy the Blaschke condition.

Aaronson used exactly this kind of function-theoretic characterization to construct his examples as inner functions on the upper half-plane. However it is not clear, when those examples are transferred to the unit disc, just what kind of inner functions result. In the next result we modify Aaronson's method to produce the desired examples as singular inner functions. We then show how such examples can also arise as Blaschke products.

5.3. Theorem. *There exist singular inner functions on $\mathbb{U}$ that are ergodic, yet whose iterate sequences converge a.e. on $\partial\mathbb{U}$ to the Denjoy-Wolff point.*

Proof. We break the proof into several steps.

STEP I. *Overall strategy.*

We focus on positive, finite singular Borel measures on $\partial\mathbb{U}$ that are *symmetric about* 1, i.e., $\mu(E) = \mu(\overline{E})$ for every Borel set E, where $\overline{E}$ is the set of complex conjugates of points in E. For such a measure the associated inner function φ is positive on the interval $(-1, 1)$, indeed

$$\varphi(r) = \exp\{-P[\mu](r)\} \qquad (-1 < r < 1).$$

We will exhibit such measures μ for which φ is of parabolic type with Denjoy-Wolff point (necessarily) at 1, and for which φ has a first order Taylor expansion about 1 that looks like this:

$$\varphi(z) = 1 + (z-1) - (z-1)h(z) \qquad (z \in \mathbb{U}) \tag{5.1}$$

where, for some constants $c > 0$ and $0 < \gamma < 1$,

$$c(1-r)^\gamma \le h(r) \qquad (0 \le r < 1). \tag{5.2}$$

The fact that φ is of non-automorphic type follows immediately from part (a) of Lemma 2.6.

To show that the iterate sequence of φ converges a.e. to 1 we use (5.2) to conclude that $\sum_n (1 - r_n) < \infty$, where $r_n := \varphi^{[n]}(0)$. The argument here is a straightforward modification of the ones discussed in §4.8. In particular, since 1 is the Denjoy-Wolff point of φ, Julia's Theorem shows that $r_n \nearrow 1$ as $n \nearrow \infty$. From (5.1) we have

$$h(r) = 1 - \frac{1-\varphi(r)}{1-r} \qquad (0 \le r < 1),$$

so that

$$h(r_n) = 1 - \frac{1 - r_{n+1}}{1 - r_n} \qquad (n = 1, 2, \ldots).$$

Now set $y_n := \frac{1}{1-r_n}$, so that $y_n \nearrow \infty$ as $n \nearrow \infty$. The last equation then becomes, for each positive integer n:

$$1 - \frac{y_n}{y_{n+1}} = g(y_n) \quad \text{where} \quad g(y_n) := h\left(1 - \frac{1}{y_n}\right) \ge \frac{c}{y_n^\gamma}\,, \tag{5.3}$$

where c and γ are the constants occurring in (5.2).

Now we proceed as in §4.8, borrowing again from the work of Aaronson [1, page 250]. From (5.3) and the monotonicity of the sequence (y_n) we have for each positive integer n:

$$y_{n+1} - y_n = y_{n+1} g(y_n) \geq y_n g(y_n),$$

whereupon

$$y_{n+1} \geq y_n(1 + g(y_n)), \quad \text{so} \quad y_{n+1}^\gamma \geq y_n^\gamma (1 + g(y_n))^\gamma.$$

Since $g(y_n) \to 0$ the Binomial Theorem applies, and shows that, as $n \to \infty$:

$$(1 + g(y_n))^\gamma = 1 + \gamma\, g(y_n) + O(g(y_n)^2),$$

hence

$$y_{n+1}^\gamma - y_n^\gamma = y_n^\gamma\, (\gamma + o(1))\, g(y_n) \geq c\, y_n^\gamma \frac{1}{y_n^\gamma} = c.$$

As in §4.8, a telescoping series argument shows that (for a possibly different positive constant c) $y_n^\gamma \geq cn$, from which it follows that $0 < 1 - r_n \leq (cn)^{-1/\gamma}$, which implies, since $0 < \gamma < 1$, that $\sum_n (1 - r_n) < \infty$, as desired. This completes Step I.

Step II. *Insuring parabolic type.*

We claim that:

$$\int_{\partial\mathbb{U}} \frac{d\mu(\zeta)}{|1-\zeta|^2} = \frac{1}{2} \implies \lim_{r \to 1-} \frac{1 - \varphi(r)}{1 - r} = 1. \tag{5.4}$$

The result follows from a characterization, due to M. Riesz [23], of points at which singular inner functions have angular derivatives. Following Riesz, the key to the argument is the identity:

$$|r - \zeta|^2 = (1 - r)^2 + r|1 - \zeta|^2 \qquad (\zeta \in \partial\mathbb{U},\ 0 \leq r < 1), \tag{5.5}$$

which shows that

$$\frac{r}{|r-\zeta|^2} \leq \frac{1}{|1-\zeta|^2} \qquad (\zeta \in \partial\mathbb{U},\ 0 \leq r < 1),$$

whereupon the integrability hypothesis in (5.4) and Dominated Convergence yield

$$\frac{1}{2} = \lim_{r \to 1-} \int_{\partial\mathbb{U}} \frac{r\, d\mu(\zeta)}{|r-\zeta|^2} = \lim_{r \to 1-} \int_{\partial\mathbb{U}} \frac{d\mu(\zeta)}{|r-\zeta|^2}\ . \tag{5.6}$$

In particular this implies $P[\mu](r) = O(1 - r)$ as $r \to 1-$, so, upon recalling that $\varphi(r) = \exp\{-P[\mu](r)\}$ (due to the symmetry of μ) and doing some manipulations with the MacLaurin series of the exponential function, we obtain, as $r \to 1-$,

$$\frac{1 - \varphi(r)}{1 - r} = \frac{P[\mu](r)}{1 - r} + O(1 - r) = \frac{(1 - r^2)}{1 - r} \int_{\partial\mathbb{U}} \frac{d\mu(\zeta)}{|r-\zeta|^2} + O(1 - r), \tag{5.7}$$

from which (5.6) gives the desired result. Thus the integrability condition on the left-hand side of (5.4) insures that φ has its Denjoy-Wolff point at 1 and is of parabolic type.

STEP III. *Insuring the lower estimate (5.2).*

For this we make a "second order refinement" of the arguments of Step II. To review our hypotheses: we are assuming that μ is a positive finite Borel measure on $\partial\mathbb{U}$ singular with respect to Lebesgue measure and symmetric about 1, and that φ is the associated singular inner function. Now let's assume further that

$$\int_{\partial\mathbb{U}} |1-\zeta|^{-2}\, d\mu(\zeta) = 1/2, \tag{5.8}$$

so that by Step II, φ is of parabolic type with Denjoy-Wolff point at 1.

We make one final assumption: *there exist constants $c > 0$ and $2 < \alpha < 3$ such that*

$$c\,\delta^{\alpha} \le \mu\{\zeta \in \partial\mathbb{U} : |1-\zeta| < \delta\} \qquad (0 \le \delta \le 2). \tag{5.9}$$

We claim that under these conditions φ *is of parabolic non-automorphic type with orbits that obey the Blaschke condition.*

To get started with the proof, rewrite (5.7) as

$$\begin{aligned} \frac{1-\varphi(r)}{1-r} &= [2-(1-r)]\int_{\partial\mathbb{U}} \frac{d\mu(\zeta)}{|r-\zeta|^2} + O(1-r) \\ &= 2\int_{\partial\mathbb{U}} \frac{d\mu(\zeta)}{|r-\zeta|^2} + O(1-r), \end{aligned}$$

where the last line uses (5.6). Use this and (5.8) to obtain

$$\begin{aligned} h(r) &:= 1 - \frac{1-\varphi(r)}{1-r} \\ &= 2\int_{\partial\mathbb{U}} \left[\frac{1}{|1-\zeta|^2} - \frac{1}{|r-\zeta|^2}\right] d\mu(\zeta) + O(1-r) \\ &= 2(1-r)^2\int_{\partial\mathbb{U}} \frac{d\mu(\zeta)}{|1-\zeta|^2|r-\zeta|^2} - \frac{2}{1+r}P[\mu](r) + O(1-r), \end{aligned}$$

where one obtains the last line from the previous one thanks to some arithmetic facilitated by (5.5). Since $P[\mu](r) = O(1-r)$ as $r \to 1-$ the last calculation simplifies to

$$h(r) = 2(1-r)^2\int_{\partial\mathbb{U}} \frac{d\mu(\zeta)}{|1-\zeta|^2|r-\zeta|^2} + O(1-r) \qquad \text{as } r \to 1- \ . \tag{5.10}$$

For $0 < r < 1$ let $I(r) = \{\zeta \in \partial\mathbb{U} : |1-\zeta| < (1-r)\}$. Then, calling once again on (5.5), we see that the integral on the right-hand side of (5.10) is bounded below by

$$\int_{I(r)} \frac{d\mu(\zeta)}{|1-\zeta|^2|r-\zeta|^2} \geq \frac{\mu(I(r))}{2(1-r)^4} \geq \frac{c}{2}(1-r)^{\alpha-4} \ ,$$

where c is the constant in the lower distribution hypothesis (5.9) on μ. Using this in (5.10) we arrive at the estimate

$$h(r) \geq c(1-r)^{\alpha-2} + O(1-r) \qquad \text{as } r \to 1- \, .$$

Since $2 < \alpha < 3$ we have $0 < \gamma := \alpha - 2 < 1$, so (after appropriate modifications to the constant c)

$$h(r) \geq c(1-r)^{\gamma} \qquad (0 \leq r < 1),$$

where $c > 0$ and $0 < \gamma < 1$, with neither depending on r. This completes Step III.

Step IV. *Producing the measure.*

We have just shown that any positive finite singular measure on $\partial\mathbb{U}$ that is symmetric about the point 1 satisfies the integral condition (5.8), and places "sufficiently much" mass around 1 will do the job. Purely atomic measures are the easiest ones to produce; here is one.

For n a non-zero integer let $\zeta_n = e^{i/n}$. Fix α with $2 < \alpha < 3$, and let ν be the atomic measure that places mass $|n|^{-(\alpha+1)}$ at ζ_n. Then μ is finite and symmetric about 1, and if $r_n := 1 - 1/n$ then, borrowing some notation from Step III:

$$\nu(I(r_n)) = 2\sum_{k=n}^{\infty} \frac{1}{k^{\alpha+1}} \sim \frac{1}{n^{\alpha}} = (1-r_n)^{\alpha},$$

where we are using "$\sim$" to mean that the quantity on the left is bounded above and below by positive constant multiples of the one on the right (constants independent of n). It's easy to see that in fact $\nu(I(r)) \sim (1-r)^{\alpha}$ for all $0 \leq r < 1$. These inequalities insure that $c := \int_{\partial\mathbb{U}} |1-\zeta|^{-2}\, d\nu(\zeta) < \infty$, so the measure $\mu := \nu/2c$ satisfies the hypotheses of Step III, hence the associated inner function is parabolic non-automorphic (hence ergodic) with Denjoy-Wolff point at 1, and its orbits satisfy the Blaschke condition (so its iterates converge a.e. on $\partial\mathbb{U}$ to 1). □

5.4. Corollary. *There exist Blaschke products that are ergodic on $\partial\mathbb{U}$, yet have iterate sequences converging a.e. on $\partial\mathbb{U}$ to the Denjoy-Wolff point.*

Proof. A theorem of Frostman ([14], see also [13, page 30]) asserts that if φ is an inner function then for "quasi-every" standard automorphism α_λ the composition $\alpha_\lambda \circ \varphi$ is a Blaschke product! Here "for quasi-every α_λ" means "for all α_λ with $\lambda \in \mathbb{U}$ lying outside a subset of $\mathbb{U}$ having capacity zero. For our purposes all we need to know is that α_λ is Blaschke for *some* $\lambda \in \mathbb{U}$.

Suppose now that φ is one of the inner functions promised by Theorem 5.3 (ergodic, but with iterate sequence converging a.e. to the Denjoy-Wolff point).

Then the same is true, by the self-inverse property of α_λ, of $\varphi_\lambda := \alpha_\lambda \circ \varphi \circ \alpha_\lambda$. Choose $\lambda \in \mathbb{U}$ so that $B := \alpha_\lambda \circ \varphi$ is a Blaschke product. Then so is $B \circ \alpha_\lambda$, which equals φ_λ. □

5.5. Atomic singular inner functions. We conclude this section by using our results to settle an issue raised in [15] concerning ergodicity for the simplest family of singular inner functions, those whose singular measures are point masses. Let $M_{\zeta,\alpha}$ denote the function induced by the measure that places mass α at the point $\zeta \in \partial\mathbb{U}$:

$$M_{\zeta,\alpha}(z) = \exp\left(-\alpha\,\frac{\zeta + z}{\zeta - z}\right) \quad \alpha > 0, \zeta \in \partial\mathbb{U}. \tag{5.11}$$

Let us write $\zeta = e^{i\theta}$. In Theorem 4.1 of [15], Kim and Kim prove that $M_{\zeta,\alpha}$ is:

- ergodic if either $\alpha > 2$ or if $0 < \alpha \le 2$ and $|\theta| < \sqrt{\alpha(2-\alpha)} + 2\arcsin(\sqrt{\alpha/2})$;
- not ergodic if $\sqrt{\alpha(2-\alpha)} + 2\arcsin(\sqrt{\alpha/2}) < |\theta| \le \pi$.

Their characterization omits only the following "critical subfamily"

$$\theta = \pm\left(\sqrt{\alpha(2-\alpha)} + 2\arcsin\sqrt{\alpha/2}\right) \quad \text{and} \quad 0 < \alpha \le 2. \tag{5.12}$$

The critical subfamily includes the mapping φ_2 of Proposition 4.7 ($\alpha = 2$, $\zeta = -1$) which we have already observed to be ergodic (Proposition 4.7 and §5.2 above). The work we have done up to this point furnishes the last piece of the puzzle.

5.6. Theorem. *Except for $\varphi_2 = M_{-1,2}$ no member of the critical subfamily is ergodic.*

Proof. Without loss of generality we focus on the those members of the critical subfamily for which $\theta = +\left(\sqrt{\alpha(2-\alpha)} + 2\arcsin\sqrt{\alpha/2}\right)$.

For $0 < \alpha \le 2$, define

$$\zeta_\alpha = \exp[i\left(\sqrt{\alpha(2-\alpha)} + 2\arcsin\sqrt{\alpha/2}\right)].$$

Kim and Kim show in [15] that the Denjoy-Wolff point of $M_{\zeta_\alpha,\alpha}$ is given by

$$\omega = \exp\left(i\sqrt{\alpha(2-\alpha)}\right).$$

The reader may check directly that $M_{\zeta_\alpha,\alpha}(\omega) = \omega$ and $M'_{\zeta_\alpha,\alpha}(\omega) = 1$ so that ω is indeed the Denjoy-Wolff point of $M_{\zeta_\alpha,\alpha}$ and we are in one of the parabolic cases. We will show that $M_{\zeta_\alpha,\alpha}$ is of parabolic-automorphic type when $\alpha < 2$ so that it is non-ergodic.

Let us simplify notation. Let $M_\alpha = M_{\zeta_\alpha,\alpha}$, let ω be the Denjoy-Wolff point of M_α, and let

$$\varphi(z) = \bar{\omega} M_\alpha(\omega z).$$

Observe that φ has Denjoy-Wolff point 1, $\varphi'(1) = 1$, and M_α has non-separated orbits if and only if the same is true for φ; in particular φ is of parabolic automorphic type if and only if the same is true of M_α. Clearly φ is analytic in a neighborhood of 1 and a calculation shows that

$$\varphi''(1) = \frac{-i\sqrt{2\alpha - \alpha^2}}{\alpha}.$$

Thus by [8, Theorem 4.15], φ is of parabolic-automorphic type when $0 < \alpha < 2$ and we conclude that M_α is not ergodic for this range of α values. □

Concluding Note. After submitting this paper for publication we received a message from Pietro Poggi-Corradini informing us of his paper [20], in which he obtained (independently) the same a.e. convergence results we did for the interior fixed-point case, and the boundary fixed-point hyperbolic and parabolic automorphism cases. His arguments are different from ours, utilizing potential-theoretic tools in the interior fixed-point case, and, in the boundary fixed-point cases, exploiting properties of the intertwining map σ of the applicable linear-fractional model. Our work in the parabolic non-automorphism case settles in the negative a conjecture, stated in [20], that in this case a.e. convergence takes place if and only if φ is not inner.

References

[1] J. Aaronson, *Ergodic theory for inner functions of the upper half plane*, Ann. Inst. H. Poincaré Sect. B (N.S.) **14** (1978), 233–253. MR 508928 (80b:28018)

[2] ———, *A remark on the exactness of inner functions*, J. London Math. Soc. (2) **23** (1981), 469–474. MR 616553 (82h:30018)

[3] ———, *An introduction to infinite ergodic theory*, Mathematical Surveys and Monographs, vol. 50, American Mathematical Society, Providence, RI, 1997. MR 1450400 (99d:28025)

[4] S. Axler, P. Bourdon, and W. Ramey, *Harmonic function theory*, Graduate Texts in Mathematics, vol. 137, Springer-Verlag, New York, 2001. MR 1805196 (2001j:31001)

[5] I. N. Baker and C. Pommerenke, *On the iteration of analytic functions in a halfplane. II*, J. London Math. Soc. (2) **20** (1979), 255–258. MR 551452 (83j:30024)

[6] D. M. Burns and S. G. Krantz, *Rigidity of holomorphic mappings and a new Schwarz lemma at the boundary*, J. Amer. Math. Soc. **7** (1994), 661–676. MR 1242454 (94j:32016)

[7] F. Bracci, R. Tauraso, and F. Vlacci, *Identity principles for commuting holomorphic self-maps of the unit disc*, J. Math. Anal. Appl. **270** (2002), 451–473. MR 1916591 (2003f:30030)

[8] P. S. Bourdon and J. H. Shapiro, *Cyclic phenomena for composition operators*, Mem. Amer. Math. Soc. **125** (1997). MR 1396955 (97h:47023)

[9] C. C. Cowen, *Iteration and the solution of functional equations for functions analytic in the unit disk*, Trans. Amer. Math. Soc. **265** (1981), 69–95. MR 607108 (82i:30036)

[10] ———, *Composition operators on H^2*, J. Operator Theory **9** (1983), 77–106. MR 695941 (84d:47038)

[11] C. C. Cowen and B. D. MacCluer, *Composition operators on spaces of analytic functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995. MR 1397026 (97i:47056)

[12] C. Doering and R. Mañe, *The dynamics of inner functions*, Ensaios Matemáticos **3** (1991), Soc. Brasiliera de Mat., 1–79.

[13] P. L. Duren, *Theory of H^p spaces*, Pure and Applied Mathematics, Vol. 38, Academic Press, New York, 1970; reprinted by Dover, 2000. MR 0268655 (42 #3552)

[14] O. Frostman, *Potential d'equilibre et capacité des ensembles avec quelques applications à la théorie des fonctions*, Meddel. Lunds Univ. Mat. Sem. **3** (1935), 1–118.

[15] H. O. Kim and H. S. Kim, *Iterations, Denjoy-Wolff points and ergodic properties of point-mass singular inner functions*, J. Math. Anal. Appl. **175** (1993), 105–117. MR 1216749 (94b:28005)

[16] G. Koenigs, *Recherches sur les intégrales de certaines équations fonctionnelles*, Annales Sci. École Norm. Sup. (3) **1** (1884), Supplément, 3–41.

[17] G. Letac, *Which functions preserve Cauchy laws?*, Proc. Amer. Math. Soc. **67** (1977), 277–286. MR 0584393 (58 #28433)

[18] J. H. Neuwirth, *Ergodicity of some mappings of the circle and the line*, Israel J. Math. **31** (1978), 359–367. MR 516157 (80e:28031)

[19] E. A. Nordgren, *Composition operators*, Canad. J. Math. **20** (1968), 442–449. MR 0223914 (36 #6961)

[20] P. Poggi-Corradini, *Pointwise convergence on the boundary in the Denjoy-Wolff Theorem*, Preprint; available at `http://front.math.ucdavis.edu/math.CV/0407133`.

[21] C. Pommerenke, *On the iteration of analytic functions in a half-plane*, J. London Math. Soc. (2) **19** (1979), 439–447. MR 540058 (83j:30023)

[22] ———, *On ergodic properties of inner functions*, Math. Ann. **256** (1981), 43–50. MR 620121 (82h:30019)

[23] M. Riesz, *Sur certaines inégalités dans la théorie des fonctions*, Fysiogr. Sällsk. Lund förh. **1** (1931).

[24] W. Rudin, *Real and complex analysis*, McGraw-Hill Book Co., New York, 1987. MR 924157 (88k:00002)

[25] J. H. Shapiro, *Composition operators and classical function theory*, Universitext: Tracts in Mathematics, Springer-Verlag, New York, 1993. MR 1237406 (94k:47049)

[26] ———, *What do composition operators know about inner functions?*, Monatsh. Math. **130** (2000), 57–70. MR 1762064 (2001a:47029)

[27] G. Valiron, *Sur l'itération des fonctions holomorphes dans un demi-plan*, Bull. Sci. Math. (2) **55** (1931), 105–128.

[28] ———, *Fonctions analytiques*, Presses Universitaires de France, Paris, 1954. MR 0061658 (15,861a)

[29] P. Walters, *An introduction to ergodic theory*, Graduate Texts in Mathematics, vol. 79, Springer-Verlag, New York, 1982. MR 648108 (84e:28017)

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