Illinois Journal of Mathematics
Volume 46, Number 3, Fall 2002, Pages 797–818
S 0019-2082

GENERALIZED QUADRANGLES WITH A SPREAD OF SYMMETRY AND NEAR POLYGONS

BART DE BRUYN AND KOEN THAS

ABSTRACT. Let $\mathcal{S}$ be a finite generalized quadrangle of order (s,t), $s \neq 1 \neq t$. A spread is a set of $st+1$ mutually non-concurrent lines of $\mathcal{S}$. A spread $\mathbf{T}$ of $\mathcal{S}$ is called a spread of symmetry if there is a group of automorphisms of $\mathcal{S}$ which fixes $\mathbf{T}$ elementwise and which acts transitively on the points of at least one (and hence every) line of $\mathbf{T}$. From spreads of symmetry of generalized quadrangles, there can be constructed near polygons, and new spreads of symmetry would yield new near polygons. In this paper, we focus on spreads of symmetry in generalized quadrangles of order (s,s^2). Many new characterizations of the classical generalized quadrangle $\mathcal{Q}(5,q)$ which arises from the orthogonal group $\mathbf{O}^-(6,q)$ will be obtained. In particular, we show that a generalized quadrangle $\mathcal{S}$ of order (s,t), $s \neq 1 \neq t$, containing a spread of symmetry $\mathbf{T}$ is isomorphic to $\mathcal{Q}(5,s)$, under any of the following conditions: (i) $\mathcal{S}$ contains a point which is incident with at least three axes of symmetry (Theorem 6.4); (ii) $t = s^2$ with s even and $\mathcal{S}$ has a center of transitivity (Theorem 6.6); (iii) there exists a line $L \notin \mathbf{T}$ such that $\mathcal{S}$ is an EGQ with base-line L (Theorem 6.8).

1. Introduction

1.1. Finite generalized quadrangles. A (finite) *generalized quadrangle (GQ) of order* (s,t) is an incidence structure $\mathcal{S} = (P,B,I)$ in which P and B are disjoint (nonempty) sets of objects called *points* and *lines*, respectively, and for which I is a symmetric point-line incidence relation satisfying the following axioms:

(1) Each point is incident with $t+1$ lines ($t \geq 1$) and two distinct points are incident with at most one line.
(2) Each line is incident with $s+1$ points ($s \geq 1$) and two distinct lines are incident with at most one point.
(3) If p is a point and L is a line not incident with p, then there is a unique point-line pair (q,M) such that $pIMIqIL$.

Received October 30, 2001; received in final form February 27, 2002.
2000 *Mathematics Subject Classification.* 51E12, 51E20, 51E23, 51E26.

If $s = t$, then $\mathcal{S}$ is also said to be *of order* s.

Generalized quadrangles were introduced by J. Tits [41] in his celebrated work on triality as a subclass of a larger class of incidence structures, namely the *generalized polygons*, in order to understand the Chevalley groups of rank 2. The main results, up to 1983, on finite generalized quadrangles are contained in the monograph *Finite Generalized Quadrangles* by S. E. Payne and J. A. Thas [18], denoted by FGQ in the sequel. For an extensive survey on recent results on automorphisms and characterizations of GQ's see K. Thas [32].

Let $\mathcal{S} = (P, B, I)$ be a (finite) generalized quadrangle of order (s,t), $s \neq 1 \neq t$. Then $|P| = (s+1)(st+1)$ and $|B| = (t+1)(st+1)$. Also, $s \leq t^2$ and, dually, $t \leq s^2$, and $s+t$ divides $st(s+1)(t+1)$.

There is a point-line duality for GQ's of order (s,t) under which in any definition or theorem the words "point" and "line", as well as the corresponding parameters, are interchanged. Normally, we assume without further notice that the dual of a given theorem or definition has also been given. Also, sometimes a line will be identified with the set of points incident with it without further notice.

A GQ is called *thick* if every point is incident with more than two lines and if every line is incident with more than two points. Otherwise, a GQ is called *thin*. If $\mathcal{S}$ is a thin GQ of order $(s,1)$, then $\mathcal{S}$ is also called a *grid*. *Dual grids* are defined dually.

Let p and q be (not necessarily distinct) points of the GQ $\mathcal{S}$; we write $p \sim q$ and say that p and q are *collinear*, provided that there is some line L such that $pILIq$ (so $p \not\sim q$ means that p and q are not collinear). Dually, for $L, M \in B$, we write $L \sim M$ or $L \not\sim M$ according as L and M are *concurrent* or *non-concurrent*. If $p \neq q \sim p$, the line incident with both is denoted by pq, and if $L \sim M \neq L$, the point which is incident with both is sometimes denoted by $L \cap M$.

For $p \in P$, put $p^\perp = \{q \in P \parallel q \sim p\}$ (and note that $p \in p^\perp$). For a pair of distinct points $\{p,q\}$, the *trace* of $\{p,q\}$ is defined as $p^\perp \cap q^\perp$, and we denote this set by $\{p,q\}^\perp$. Then $|\{p,q\}^\perp| = s+1$ or $t+1$, according as $p \sim q$ or $p \not\sim q$. More generally, if $A \subseteq P$, $A^\perp$ is defined by $A^\perp = \bigcap\{p^\perp \parallel p \in A\}$. For $p \neq q$, the *span* of the pair $\{p,q\}$ is $\mathrm{sp}(p,q) = \{p,q\}^{\perp\perp} = \{r \in P \parallel r \in s^\perp$ for all $s \in \{p,q\}^\perp\}$. When $p \not\sim q$, then $\{p,q\}^{\perp\perp}$ is also called *the hyperbolic line* defined by p and q, and $|\{p,q\}^{\perp\perp}| = s+1$ or $|\{p,q\}^{\perp\perp}| \leq t+1$ according as $p \sim q$ or $p \not\sim q$. If $p \sim q$, $p \neq q$, or if $p \not\sim q$ and $|\{p,q\}^{\perp\perp}| = t+1$, we say that the pair $\{p,q\}$ is *regular*. The point p is *regular* provided $\{p,q\}$ is regular for every $q \in P \setminus \{p\}$. Regularity for lines is defined dually. One easily proves that either $s = 1$ or $t \leq s$ if $\mathcal{S}$ has a regular pair of non-collinear points.

Finally, if $\mathcal{S}$ is a GQ, then by $\mathcal{S}^D$ we denote its point-line dual.

Consider a nonsingular quadric of Witt index 2, that is, of projective index 1, in $\mathbf{PG}(3,q)$, $\mathbf{PG}(4,q)$, $\mathbf{PG}(5,q)$, respectively. The points and lines of the quadric form a generalized quadrangle which is denoted by $\mathcal{Q}(3,q)$, $\mathcal{Q}(4,q)$, $\mathcal{Q}(5,q)$, respectively, and has order $(q,1)$, (q,q), (q,q^2), respectively. Next, let $\mathcal{H}$ be a nonsingular Hermitian variety in $\mathbf{PG}(3,q^2)$, respectively $\mathbf{PG}(4,q^2)$. The points and lines of $\mathcal{H}$ form a generalized quadrangle $H(3,q^2)$, respectively $H(4,q^2)$, which has order (q^2,q), respectively (q^2,q^3). The points of $\mathbf{PG}(3,q)$ together with the totally isotropic lines with respect to a symplectic polarity form a GQ $W(q)$ of order q. The generalized quadrangles defined in this paragraph are the so-called *classical generalized quadrangles*; see Chapter 3 of FGQ. It is important to mention that $W(q)^D \cong \mathcal{Q}(4,q)$ and that $H(3,q^2)^D \cong \mathcal{Q}(5,q)$; see 3.2.1 and 3.2.3 of FGQ. Also, $W(q) \cong W(q)^D$ if and only if q is even [18, 3.2.1].

1.2. EGQ's and TGQ's. Let $\mathcal{S} = (P,B,I)$ be a GQ of order (s,t), $s,t > 1$. An *elation about the point* p is a collineation of $\mathcal{S}$ that fixes p linewise and fixes no point of $P \setminus p^\perp$. By definition, the identical permutation is an elation (about every point). If p is a point of the GQ $\mathcal{S}$ for which there exists a group G of elations about p which acts regularly on the points of $P \setminus p^\perp$, then $\mathcal{S}$ is said to be an *elation generalized quadrangle (EGQ)* with *elation point* p and *elation group* (or *base-group*) G, and we sometimes write $(\mathcal{S}^{(p)},G)$ for $\mathcal{S}$. An *axis of symmetry* L of $\mathcal{S}$ is a line for which there is a full group of size s of collineations of $\mathcal{S}$ fixing $L^\perp$ elementwise. Dually, one defines a *center of symmetry.* If a GQ $(\mathcal{S}^{(p)},G)$ is an EGQ with elation point p, and if each line incident with p is an axis of symmetry, then we say that $\mathcal{S}$ is a *translation generalized quadrangle (TGQ)* with *translation point* p and *translation group* (or *base-group*) G. In such a case, G is uniquely defined; G is generated by all symmetries about every line incident with p, and G is the set of all elations about p; see FGQ.

TGQ's were introduced by J. A. Thas in [23] for the case $s = t$ and by S. E. Payne and J. A. Thas in FGQ for the general case.

THEOREM 1.1 (FGQ, 8.3.1). *Let $\mathcal{S} = (P,B,I)$ be a GQ of order (s,t), $s,t > 1$. Suppose each line through some point p is an axis of symmetry, and let G be the group generated by the symmetries about the lines through p. Then G is elementary abelian and $(\mathcal{S}^{(p)},G)$ is a TGQ.*

Suppose $H = \mathbf{PG}(2n+m-1,q)$ is the finite projective $(2n+m-1)$-space over $\mathbf{GF}(q)$, and let H be embedded in a $\mathbf{PG}(2n+m,q)$, say H'. Now define a set $\mathcal{O} = \mathcal{O}(n,m,q)$ of subspaces as follows: $\mathcal{O}$ is a set of q^m+1 $(n-1)$-dimensional subspaces of H every three of which generate a $\mathbf{PG}(3n-1,q)$, denoted by $\mathbf{PG}(n-1,q)^{(i)}$, and such that the following condition is satisfied: for every $i = 0,1,\ldots,q^m$ there is a subspace $\mathbf{PG}(n+m-1,q)^{(i)}$ of H of dimension $n+m-1$, which contains $\mathbf{PG}(n-1,q)^{(i)}$ and which is disjoint from any $\mathbf{PG}(n-1,q)^{(j)}$ if $j \neq i$. If $\mathcal{O}$ satisfies all these conditions for $n = m$, then $\mathcal{O}$ is called a *pseudo-oval* or a *generalized oval* or an $[n-1]$-*oval* of $\mathbf{PG}(3n-1,q)$; a generalized oval of $\mathbf{PG}(2,q)$ is just an oval of $\mathbf{PG}(2,q)$. For $n \neq m$, $\mathcal{O}(n,m,q)$ is called a *pseudo-ovoid* or a *generalized ovoid* or an $[n-1]$-*ovoid* or an *egg* of $\mathbf{PG}(2n+m-1,q)$; a $[0]$-ovoid of $\mathbf{PG}(3,q)$ is just an ovoid of $\mathbf{PG}(3,q)$. The spaces $\mathbf{PG}(n+m-1,q)^{(i)}$ are the *tangent spaces* of $\mathcal{O}(n,m,q)$, or just the *tangents*. Generalized ovoids were introduced for the case $n=m$ by J. A. Thas in [22] for some particular cases. S. E. Payne and J. A. Thas ([23], [18]) proved that from any egg $\mathcal{O}(n,m,q)$ there arises a GQ $T(n,m,q)=T(\mathcal{O})$ which is a TGQ of order (q^n,q^m), for some special point (∞), as follows.

- The Points are of three types.
  (1) A symbol (∞).
  (2) The subspaces $\mathbf{PG}(n+m,q)$ of H' which intersect H in a $\mathbf{PG}(n+m-1,q)^{(i)}$.
  (3) The points of $H' \setminus H$.
- The Lines are of two types.
  (1) The elements of the egg $\mathcal{O}(n,m,q)$.
  (2) The subspaces $\mathbf{PG}(n,q)$ of $\mathbf{PG}(2n+m,q)$ which intersect H in an element of the egg.
- Incidence is defined as follows: the point (∞) is incident with all the lines of type (1) and with no other lines; the points of type (2) are incident with the unique line of type (1) contained in it and with all the lines of type (2) which they contain (as subspaces), and finally, a point of type (3) is incident with the lines of type (2) that contain it.

Conversely, any TGQ can be seen in this way (that is, as a $T(n,m,q)$) [18], and thus the study of translation generalized quadrangles is equivalent to the study of the generalized ovals and generalized ovoids.

By Chapter 8 of FGQ, each TGQ is either of order s or $(s,s^{(k+1)/k})$, where k is odd and s a prime power. Each TGQ $\mathcal{S}$ of order $(s,s^{(k+1)/k})$, with translation point (∞), where k is odd and $s \neq 1$, has a *kernel* $\mathbb{K}$, which is a field with a multiplicative group isomorphic to the group of all collineations of $\mathcal{S}$ fixing the point (∞), and any given point not collinear with (∞), linewise. We have $|\mathbb{K}| \leq s$; see FGQ. The field $\mathbf{GF}(q)$ is a subfield of $\mathbb{K}$ if and only if $\mathcal{S}$ is of type $T(n,m,q)$ [18]. The TGQ $\mathcal{S}$ is isomorphic to a $T_3(\mathcal{O})$ of Tits [18] with $\mathcal{O}$ an ovoid of $\mathbf{PG}(3,s)$ if and only if $|\mathbb{K}|=s$.

Completely similar remarks can be made for the case $s=t$, and in that case, the TGQ is isomorphic to a $T_2(\mathcal{O})$ of Tits [18] with $\mathcal{O}$ an oval of $\mathbf{PG}(2,s)$ if and only if $|\mathbb{K}|=s$.

If $n \neq m$, then by 8.7.2 of [18] the q^m+1 tangent spaces of $\mathcal{O}(n,m,q)$ form an $\mathcal{O}^*(n,m,q)$ in the dual space of $\mathbf{PG}(2n+m-1,q)$. So in addition to $T(n,m,q)$ there arises a TGQ $T(\mathcal{O}^*)$, also denoted by $T^*(n,m,q)$, or $T^*(\mathcal{O})$. The TGQ $T^*(\mathcal{O})$ is called the *translation dual* of the TGQ $T(\mathcal{O})$. The GQ's $T_3(\mathcal{O})$ and $\mathcal{S}(\mathcal{F})^D$, where $\mathcal{F}$ is a Kantor flock (see below), are the only known

TGQ's of order (q^n, q^m), $n \neq m$, which are isomorphic to their translation dual. The TGQ $T(\mathcal{O})$ and its translation dual $T(\mathcal{O}^*)$ have isomorphic kernels.

A TGQ $T(\mathcal{O})$ of order (s,t), $s \neq 1 \neq t$, with $s \neq t$ is called *good* at an element $\pi \in \mathcal{O}$ if for any two distinct elements π' and π'' of $\mathcal{O} \setminus \{\pi\}$ the $(3n-1)$-space $\pi\pi'\pi''$ contains exactly q^n+1 elements of $\mathcal{O}$ and is skew to the other elements. If the egg $\mathcal{O}$ contains a good element, then the egg is called *good*, and a good egg $\mathcal{O}(n,m,q)$ satisfies $m = 2n$ (by [18, 8.7.2 (v)]).

2. The relation between spreads of symmetry, admissible triples and near polygons

2.1. Some definitions. A *near polygon* is a partial linear space with the property that for every point p and every line L there exists a unique point on L nearest to p (with respect to the distance in the point graph G). If d is the diameter of G, then the near polygon is called a *near* $2d$*-gon*. A near 0-gon is a point, a near 2-gon is a line, and the class of the near 4-gons coincides with the class of the generalized quadrangles. Also, generalized $2d$-gons and dual polar spaces are examples of near polygons. Near polygons were introduced by E. E. Shult and A. Yanushka in [21] because of the relationship with the so-called 'tetrahedrally closed systems of lines' in Euclidean spaces. For a survey on near polygons, see B. De Bruyn [7], and also F. De Clerck and H. Van Maldeghem [8].

Two lines L and M of a near polygon are called *parallel* if $\mathrm{d}(L,m)$ is independent of the chosen point $m\mathrm{I}M$. Clearly, any two disjoint lines of a generalized quadrangle are parallel. A *spread* of a near polygon is a set of lines which partition the point set. A *spread of symmetry* of a near polygon Γ is a spread $\mathbf{T}$ such that for every line $L \in \mathbf{T}$ and for any two points x and y of L, there exists an automorphism of Γ fixing each line of $\mathbf{T}$ and mapping x onto y. Clearly, any two lines of a spread of symmetry are parallel. If Γ is a GQ, then dually one defines *ovoids of symmetry*.

Suppose $\mathbf{T}$ is a spread of the GQ $\mathcal{S}$ of order (s,t), $s,t > 1$. Then $\mathbf{T}$ is *Hermitian* or *regular* or *normal* if for any two distinct lines L and M of $\mathbf{T}$, the pair $\{L,M\}$ is regular (so $|\{L,M\}^{\perp\perp}| = s+1$) and $\{L,M\}^{\perp\perp} \subseteq \mathbf{T}$. Let $\mathbf{T}$ be a spread of $\mathcal{S}$. Then $\mathbf{T}$ is *locally Hermitian* or *semiregular* or *seminormal* with respect to the line L if for every line $M \neq L$ of $\mathbf{T}$, the pair $\{L,M\}$ is regular and $\{L,M\}^{\perp\perp} \subseteq \mathbf{T}$.

2.2. Spreads of symmetry and admissible triples. By [4], every spread of symmetry of a GQ can be derived from a so-called *admissible triple* in the way we will describe now.

A triple $T = (\mathcal{D}, H, \Delta)$ is called *admissible* if the following conditions are satisfied:

(1) $\mathcal{D}$ is a linear space of order $(s, t-1)$ with s and t nonzero positive integers. Let $\mathcal{P}$ denote the point set of $\mathcal{D}$.

(2) H is a (multiplicative) group of order $s+1$. Let $\mathbf{1}$ denote its identity element.
(3) Δ is a map from $\mathcal{P} \times \mathcal{P}$ to H such that x, y and z are collinear if and only if $\Delta(x,y)\,\Delta(y,z) = \Delta(x,z)$.

Let Γ be the graph on the vertex set $H \times \mathcal{P}$; two vertices (h_1, x) and (h_2, y) are adjacent if and only if

(i) $x = y$ and $h_1 \neq h_2$, or
(ii) $x \neq y$ and $h_2 = h_1\,\Delta(x,y)$.

It is proved in B. De Bruyn [4] that Γ is the point graph of a generalized quadrangle $\mathcal{S}$ of order (s,t). The set $\mathbf{T} = \{L_x \parallel x \in \mathcal{P}\}$ with $L_x = \{(h,x) \parallel h \in H\}$ is a spread of $\mathcal{S}$ and we put $\Omega(T) := (\mathcal{S}, \mathbf{T})$. For every $h \in H$, the map $\theta_h : (g,x) \mapsto (h\,g, x)$, $g \in H$ and $x \in \mathcal{P}$, defines an automorphism of $\mathcal{S}$ that fixes each line of $\mathbf{T}$, proving that $\mathbf{T}$ is a spread of symmetry.

The following properties must be satisfied for an admissible triple:

(A) $\Delta(x,x) = \mathbf{1}$ for all $x \in \mathcal{P}$.
(B) $\Delta(y,x) = [\Delta(x,y)]^{-1}$ for all $x, y \in \mathcal{P}$.
(C) If $\mathcal{S}$ is not a grid, then $H = \{\Delta(a,x)\,\Delta(x,y)\,\Delta(y,a) \parallel x,y \in \mathcal{P}\}$ for every $a \in \mathcal{P}$.

Proof. Property (A) is obtained by putting $x = y = z$ in (3). Property (B) is obtained by putting $z = x$ in (3). Let h be an arbitrary element of $H \setminus \{\mathbf{1}\}$, and let $a, x \in \mathcal{P}$, $a \neq x$. Since $\mathcal{S}$ is not a grid, $(\Delta(a,x), x)$ and (h,a) have a common neighbour (g,y) with $a \neq y \neq x$. We have $g = \Delta(a,x)\,\Delta(x,y)$ and $h = g\,\Delta(y,a)$, proving Property (C). $\square$

Admissible triples (AT's for short) yield spreads of symmetry. It is possible, however, that two admissible triples, although different, yield equivalent spreads. In the following section, we examine why this happens and introduce the notion of *equivalent AT's*.

2.3. Equivalence of admissible triples. Let $\mathbf{T}_i$, $i \in \{1,2\}$, be a spread of a generalized quadrangle $\mathcal{S}_i$. Then we say that $(\mathcal{S}_1, \mathbf{T}_1)$ and $(\mathcal{S}_2, \mathbf{T}_2)$ are *equivalent* if and only if there exists an isomorphism from $\mathcal{S}_1$ to $\mathcal{S}_2$ mapping $\mathbf{T}_1$ onto $\mathbf{T}_2$.

Let $T_1 = (\mathcal{D}_1, H_1, \Delta_1)$ and $T_2 = (\mathcal{D}_2, H_2, \Delta_2)$ be two admissible triples. If $\mathcal{D}_1$ and $\mathcal{D}_2$ are two lines of the same length, then T_1 and T_2 are said to be *equivalent*. Otherwise, T_1 and T_2 are called *equivalent* if there exist:

(1) an isomorphism from $\mathcal{D}_1$ to $\mathcal{D}_2$ determined by $\alpha : \mathcal{P}_1 \to \mathcal{P}_2$;
(2) an isomorphism β from H_1 to H_2;
(3) a map γ from $\mathcal{P}_1$ to H_1 such that, for all $x, y \in \mathcal{P}_1$, $\Delta_2(\alpha(x), \alpha(y)) = \beta(\gamma^{-1}(x)\,\Delta_1(x,y)\,\gamma(y))$ holds.

THEOREM 2.1. *Two admissible triples T_1 and T_2 are equivalent if and only if $\Omega(T_1)$ and $\Omega(T_2)$ are equivalent.*

Proof. Put $T_i = (\mathcal{D}_i, H_i, \Delta_i)$ and $\Omega(T_i) = (\mathcal{S}_i, \mathbf{T}_i)$ for every $i \in \{1,2\}$. We may suppose that $\mathcal{S}_1$ and $\mathcal{S}_2$ are not grids, or equivalently, that $\mathcal{D}_1$ and $\mathcal{D}_2$ are not lines. If T_1 and T_2 are equivalent, let α, β and γ be as above. Then the map $(h,x) \mapsto (\beta(h\,\gamma(x)), \alpha(x))$ defines an isomorphism from $\mathcal{S}_1$ to $\mathcal{S}_2$ mapping $\mathbf{T}_1$ onto $\mathbf{T}_2$.

Conversely, suppose that $(\mathcal{S}_1, \mathbf{T}_1)$ and $(\mathcal{S}_2, \mathbf{T}_2)$ are equivalent. We will freely use Properties (A), (B) and (C) of the previous section without further notice. Let θ be an isomorphism from $\mathcal{S}_1$ to $\mathcal{S}_2$ mapping $\mathbf{T}_1$ to $\mathbf{T}_2$. For a point x_i of $\mathcal{P}_i$, $i \in \{1,2\}$, let $L^{(i)}_{x_i} = \{(h, x_i) \parallel h \in H_i\} \in \mathbf{T}_i$. There exists a bijection $\alpha : \mathcal{P}_1 \to \mathcal{P}_2$ such that $\theta(L^{(1)}_{x_1}) = L^{(2)}_{\alpha(x_1)}$ for all $x_1 \in \mathcal{P}_1$. Since sets of the form $\{L, M\}^{\perp\perp}$ with $L, M \in \mathbf{T}_1$ are mapped by θ onto sets of the form $\{L', M'\}^{\perp\perp}$ with $L', M' \in \mathbf{T}_2$, α induces an isomorphism from $\mathcal{D}_1$ to $\mathcal{D}_2$. For every $x \in \mathcal{P}_1$, let β_x be the bijection from H_1 to H_2 such that $\theta[(h,x)] = (\beta_x(h), \alpha(x))$ for all $x \in \mathcal{P}_1$ and all $h \in H_1$. Consider now the adjacent points (h,x) and $(h\Delta(x,y), y)$ of $\mathcal{S}_1$. Then $(\beta_x(h), \alpha(x))$ and $(\beta_y(h\Delta_1(x,y)), \alpha(y))$ are two adjacent points of $\mathcal{S}_2$. Hence $\beta_y(h\Delta_1(x,y)) = \beta_x(h)\Delta_2(\alpha(x), \alpha(y))$. Let $a \in \mathcal{P}_1$ be fixed and put $\bar\beta := \beta_a$. Then $\beta_y(k) = \bar\beta(k\Delta_1(y,a))\Delta_2(\alpha(a), \alpha(y))$. Hence

$$\begin{aligned}&\bar\beta(h\Delta_1(x,y)\Delta_1(y,a))\Delta_2(\alpha(a),\alpha(y))\\ &\qquad\qquad = \bar\beta(h\Delta_1(x,a))\Delta_2(\alpha(a),\alpha(x))\Delta_2(\alpha(x),\alpha(y))\end{aligned}$$

for all $h \in H$ and all $x, y \in \mathcal{P}_1$. Putting $h = \Delta_1(a,x)$, we find

$$\begin{aligned}&\bar\beta(\Delta_1(a,x)\Delta_1(x,y)\Delta_1(y,a))\\ &\qquad\qquad = \bar\beta(\mathbf{1})\Delta_2(\alpha(a),\alpha(x))\Delta_2(\alpha(x),\alpha(y))\Delta_2(\alpha(y),\alpha(a)).\end{aligned}$$

Hence

$$\begin{aligned}&\bar\beta(h\Delta_1(x,a)\Delta_1(a,x)\Delta_1(x,y)\Delta_1(y,a))\\ &\qquad\qquad = \bar\beta(h\Delta_1(x,a))[\bar\beta(\mathbf{1})]^{-1}\bar\beta(\Delta_1(a,x)\Delta_1(x,y)\Delta_1(y,a)).\end{aligned}$$

Now, let h_1, h_1' be arbitrary elements of H_1. Since $\mathcal{S}_1$ is not a grid, we can choose the points x and y such that $\Delta_1(a,x)\Delta_1(x,y)\Delta_1(y,a) = h_1'$. Choose now h such that $h\Delta_1(x,a) = h_1$. Hence $\bar\beta(h_1h_1') = \bar\beta(h_1)[\bar\beta(\mathbf{1})]^{-1}\bar\beta(h_1')$ for all $h_1, h_1' \in H_1$. As a consequence the map $\beta : H_1 \to H_2, h \mapsto \bar\beta(h_1)[\bar\beta(\mathbf{1})]^{-1}$ is an isomorphism from H_1 to H_2. Hence $\Delta_2(\alpha(x), \alpha(y))$ is equal to

$$\begin{aligned}&\big(\Delta_2(\alpha(x),\alpha(a))[\bar\beta(\mathbf{1})]^{-1}\beta(\Delta_1(a,x))\big)\cdot\\ &\qquad\qquad \cdot\,\beta(\Delta_1(x,y))\,\big(\beta(\Delta_1(y,a)\bar\beta(\mathbf{1})\Delta_2(\alpha(a),\alpha(y))\big)\,.\end{aligned}$$

The theorem follows now if we put

$$\gamma : \mathcal{P}_1 \to H_1, \quad x \mapsto \Delta_1(x,a)\beta^{-1}(\bar\beta(\mathbf{1})\Delta_2(\alpha(a),\alpha(x))). \qquad \square$$

2.4. The known admissible triples. Every known admissible triple is equivalent to one of the following examples:

(1) Let $\mathcal{D}$ be the line of length $s+1$ and let H be the cyclic group of order $s+1$. Put $\Delta(x,y)$ equal to $\mathbf{1}$ for all points x and y of $\mathcal{D}$.
(2) Let $\mathcal{D}$ be the complete graph on $t+1$ vertices and let H be the group of order 2. Put $\Delta(x,y)$ equal to $\mathbf{1}$ if and only if $x=y$.
(3) Consider a nonsingular nondegenerate Hermitian form $(\cdot,\cdot)$ in the vector space $V(3,q^2)$ and let $\mathcal{U}$ be the corresponding Hermitian unital in $\mathbf{PG}(2,q^2)$. With this unital there is associated the following linear space $\mathcal{D}$:
 - The POINTS of $\mathcal{D}$ are the points of $\mathcal{U}$.
 - The LINES of $\mathcal{D}$ are all the sets of order $q+1$ arising as an intersection of $\mathcal{U}$ with lines of the projective plane.

 Put $H=\{x\in\mathbf{GF}(q^2)\parallel x^{q+1}=1\}$. Let $\alpha=\langle\bar{a}\rangle$ be a fixed point of $\mathcal{U}$. For any two points $\beta=\langle\bar{b}\rangle$ and $\gamma=\langle\bar{c}\rangle$ of $\mathcal{U}$, we define $\Delta(\beta,\gamma)=-(\bar{a},\bar{b})^{q-1}(\bar{b},\bar{c})^{q-1}(\bar{c},\bar{a})^{q-1}$ if $\alpha\neq\beta\neq\gamma\neq\alpha$; $\Delta(\beta,\gamma)=\mathbf{1}$ otherwise.

In Examples (4), (5) and (6), the linear space $\mathcal{D}$ is the Desarguesian affine plane $\mathbf{AG}(2,q)$ and H is the additive group of $\mathbf{GF}(q)$. In Examples (5) and (6) a function $f:\mathbf{GF}(q)\to\mathbf{GF}(q)$ occurs which satisfies one of the following two equivalent properties:

(I) The set $\mathcal{H}:=\{(1,0,0),(0,1,0)\}\cup\{(f(\lambda),\lambda,1)\parallel\lambda\in\mathbf{GF}(q)\}$ is a *hyperoval* [12] in $\mathbf{PG}(2,q)$ (and hence q is even; see [12]).
(II) We have $\begin{vmatrix} f(\lambda_1) & \lambda_1 & 1\\ f(\lambda_2) & \lambda_2 & 1\\ f(\lambda_3) & \lambda_3 & 1\end{vmatrix}\neq 0\Leftrightarrow\lambda_1\neq\lambda_2\neq\lambda_3\neq\lambda_1$.

Now, let (α_1,β_1) and (α_2,β_2) be two arbitrary points of $\mathbf{AG}(2,q)$. Examples (4), (5) and (6) are then given as follows:

(4) We put $\Delta((\alpha_1,\beta_1),(\alpha_2,\beta_2))=\alpha_1\beta_2-\alpha_2\beta_1$.
(5) We put $\Delta((\alpha_1,\beta_1),(\alpha_2,\beta_2))=(\alpha_1-\alpha_2)f\left(\frac{\beta_1-\beta_2}{\alpha_1-\alpha_2}\right)$ if $\alpha_1\neq\alpha_2$ and 0 otherwise.
(6) We put $\Delta((\alpha_1,\beta_1),(\alpha_2,\beta_2))=(f(\alpha_1)-f(\alpha_2))\frac{\beta_1-\beta_2}{\alpha_1-\alpha_2}$ if $\alpha_1\neq\alpha_2$ and 0 otherwise.

2.5. Spreads of symmetry in the known GQ's. If a generalized quadrangle $\mathcal{S}$ of order (s,t) has a spread of symmetry, then we have the following restrictions on the parameters; see [4]:

(i) $s+1\mid t(t-1)$;
(ii) $s+t\mid s(s+1)(t+1)$;
(iii) $s+2\leq t\leq s^2$ if $s\neq 1\neq t$.

If $\mathcal{S}$ is one of the known GQ's, then (i), (ii) and (iii) imply that either $s = 1$, $t = 1$, $(s,t) = (q-1, q+1)$ or $(s,t) = (q, q^2)$. Here q denotes an arbitrary prime power. If $t = 1$, then $\mathcal{S}$ is a grid and the corresponding AT is given in (1). If $s = 1$, then $\mathcal{S}$ is a dual grid and the corresponding AT is given in (2). Every known GQ of order $(q-1, q+1)$ has a spread of symmetry and the corresponding admissible triples are given in (4), (5) and (6); they correspond, respectively, to the GQ $\mathcal{S} = \mathcal{P}(W(q), x)$ (where $\mathcal{S} \cong AS(q)$ if q is odd, and $\mathcal{S} \cong T_2^*(\mathcal{O})$, where $\mathcal{O}$ is a regular hyperoval of $\mathbf{PG}(2,q)$, if q is even), $T_2^*(\mathcal{O})$ with $\mathcal{O}$ an arbitrary hyperoval in $\mathbf{PG}(2,q)$, with q even, and the GQ $(S_{xy}^-)^D$ arising from a hyperoval in $\mathbf{PG}(2,q)$, q even. For more details, see B. De Bruyn [4]. Two AT's T_1 and T_2 of type (4), (5) or (6) are equivalent if and only if $\Omega(T_1)$ and $\Omega(T_2)$ are equivalent, and by a result of Payne [16] we know precisely when this happens. All spreads of symmetry in the classical GQ's of order (q, q^2) (i.e., the GQ's $\mathcal{Q}(5,q)$) were determined in [4]. The corresponding admissible triples are given in (3).

The problem that is still open today is whether there are known nonclassical GQ's of order (q, q^2) with a spread of symmetry. This is the main concern of this paper.

2.6. Glued near polygons and spreads of symmetry. Let k and s be nonzero integers and let X be a set of size $s+1$. For every $i \in \{1, \ldots, k\}$ consider the following objects:

(A) a near polygon Γ_i;
(B) a spread $\mathbf{T}_i = \{L_1^{(i)}, \ldots, L_{n_i}^{(i)}\}$ of Γ_i, consisting of lines which are two by two parallel;
(C) a bijection $\theta_i : X \mapsto L_1^{(i)}$.

Conditions (B) and (C) imply that all lines $L_j^{(i)}$, $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$, have the same length $s+1$. If x is a point of Γ_l and $L_m^{(l)} \in \mathbf{T}_l$, then $p_m^{(l)}(x)$ denotes the unique point of $L_m^{(l)}$ nearest to x. The following graph G can now be defined. The vertices of G are the elements of $X \times \mathbf{T}_1 \times \ldots \times \mathbf{T}_k$. Two vertices $(x, L_{i_1}^{(1)}, \ldots, L_{i_k}^{(k)})$ and $(y, L_{j_1}^{(1)}, \ldots, L_{j_k}^{(k)})$ are adjacent if and only if

(I) there exists an $l \in \{1, \ldots, k\}$ such that $i_m = j_m$ for all $m \in \{1, \ldots, k\} \setminus \{l\}$, and
(II) for every l like in (I), $p_{i_l}^{(l)} \circ \theta_l(x)$ and $p_{j_l}^{(l)} \circ \theta_l(y)$ are collinear points in Γ_l.

The following incidence structure Γ can then be defined:

- The POINTS of Γ are the vertices of G.
- The LINES of Γ are the maximal cliques of G.

THEOREM 2.2 (B. De Bruyn [7]). *The incidence structure* Γ *is a near polygon if and only if the permutations* $\theta_i^{-1} \circ p_1^{(i)} \circ p_\alpha^{(i)} \circ p_\beta^{(i)} \circ \theta_i$ *and* $\theta_j^{-1} \circ p_1^{(j)} \circ p_\gamma^{(j)} \circ p_\delta^{(j)} \circ \theta_j$ *commute for all possible* α, β, γ, δ, i *and* j *with* $i \neq j$.

The group $G_i := \langle p_1^{(i)} \circ p_\alpha^{(i)} \circ p_\beta^{(i)} \parallel \alpha, \beta \in \{1, \dots, n_i\}\rangle$, $i \in \{1, \dots, k\}$, is called the *group of projectivities of* $L_1^{(i)}$ *with respect to* $\mathbf{T}_i$. If Γ is a near polygon, then it is called a *glued near polygon*. In this case the following conditions necessarily are satisfied:

(i) $\mathbf{T}_i$ is a spread of symmetry of Γ_i.
(ii) There exists a group G such that G_i, $i \in \{1, \dots, k\}$, is either trivial or isomorphic to G.
(iii) G is abelian if there exist at least three elements $i \in \{1, \dots, k\}$ for which G_i is not trivial.

Conversely, if for fixed X, Γ_i, $\mathbf{T}_i$, $L_1^{(i)}$, $i \in \{1, \dots, k\}$, (i), (ii) and (iii) are satisfied, then there always exist maps θ_i, $i \in \{1, \dots, k\}$, such that Γ is a glued near polygon.

Hence, generalized quadrangles with a spread $\mathbf{T}$ of symmetry always yield glued near hexagons [5]. If the group of projectivities of a line $L \in \mathbf{T}$ with respect to $\mathbf{T}$ is commutative, then near $2d$-gons with $d \geq 4$ can also be derived. This is another motivation for our study of spreads of symmetry in generalized quadrangles.

3. Spreads of symmetry in generalized quadrangles: basic observations

Let $\mathbf{T} = \{L_1, \dots, L_{1+st}\}$ be a spread of a generalized quadrangle $\mathcal{S}$ of order (s, t) and let $H_\mathbf{T}$ be the group of automorphisms of $\mathcal{S}$ fixing each line of $\mathbf{T}$. If $\mathcal{S}$ is an $(s+1) \times (s+1)$-grid, then $|H_\mathbf{T}| = (s+1)!$ for both spreads of $\mathcal{S}$.

THEOREM 3.1. *If* $\mathcal{S}$ *is not a grid, then each nontrivial element of* $H_\mathbf{T}$ *has no fixed points; hence* $|H_\mathbf{T}| = (s+1)/n$ *with* n *some nonzero integer. In particular,* $\mathbf{T}$ *is a spread of symmetry if and only if* $|H_\mathbf{T}| = s+1$.

Proof. The fact that each nontrivial element of $H_\mathbf{T}$ has no fixed point readily follows from [18, 2.4.1]. This implies that $|H_\mathbf{T}| = (s+1)/n$ since $H_\mathbf{T}$ acts semiregularly on the set of points of any line of $\mathbf{T}$. □

THEOREM 3.2. *If a generalized quadrangle* $\mathcal{S}$ *of order* $(s, t) = (s, s^\alpha)$, $s \neq 1$ *and* $\alpha \in \mathbb{Q} \setminus \{0\}$, *contains a Hermitian spread, then* $\alpha = 2$.

Proof. Put $s = q^n$ and $t = q^m$, where q, n and m are strictly positive integers. Since $s, t > 1$ and $t \leq s^2$, we have that $m \leq 2n$. For every Hermitian spread $\mathbf{T}$ of $\mathcal{S}$, one can define the following linear space $\mathcal{L}(\mathbf{T})$. The points of $\mathcal{L}(\mathbf{T})$ are the elements of $\mathbf{T}$, the lines of $\mathcal{L}(\mathbf{T})$ are the sets $\{L, M\}^{\perp\perp}$ with

L and M distinct lines of $\mathbf{T}$, and incidence is containment. Counting the number of lines of $\mathcal{L}(\mathbf{T})$, one finds $s(s+1) \mid st(st+1)$ or $s+1 \mid t(t-1)$. From $s = q^n$ and $t = q^m$ with $0 < n, m \leq 2n$, it readily follows that $m = 2n$. □

4. Ovoids of symmetry and dual nets

4.1. Dual nets from GQ's with a regular point. A (finite) *net of order* $k(\geq 2)$ *and degree* $r(\geq 2)$ is an incidence structure $\mathcal{N} = (P, B, I)$ that satisfies the following properties:

(1) Each point is incident with r lines and two distinct points are incident with at most one line.
(2) Each line is incident with k points and two distinct lines are incident with at most one point.
(3) If p is a point and L a line not incident with p, then there is a unique line M incident with p and not concurrent with L.

A net of order k and degree r has k^2 points and kr lines. Also, $k \geq r-1$ with equality if and only if $\mathcal{N}$ is an affine plane. We refer to [1] for more information on nets. Nets are also related to the theory of GQ's in the following way.

Theorem 4.1 (FGQ, 1.3.1). *Let p be a regular point of a GQ $\mathcal{S} = (P, B, I)$ of order (s, t), $s \neq 1 \neq t$. Then the incidence structure with point set $p^\perp \setminus \{p\}$, with lineset the set of spans $\{q, r\}^{\perp\perp}$, where q and r are non-collinear points of $p^\perp \setminus \{p\}$, and with the natural incidence, is the dual $\mathcal{N}_p^*$ of a net $\mathcal{N}_p$ of order s and degree $t+1$.*

4.2. Ovoids of symmetry through a regular point. Let $\mathcal{S}$ be a GQ of order (s, t), $s \neq 1 \neq t$, with a regular point (∞). A *spread* $\mathbf{T}$ of $\mathcal{N}^*_{(\infty)}$ is a set of lines partitioning the point set of $\mathcal{N}^*_{(\infty)}$. Note that $|\mathbf{T}| = s$. The following observation was first made by J. A. Thas in [26], but we include a proof for the sake of completeness.

Theorem 4.2. *Let $\mathcal{S} = (P, B, I)$ be a GQ of order (s, t), $s, t > 1$, with a regular point (∞). Let $\mathbf{T} = \{L_1, \ldots, L_s\}$ be a spread of $\mathcal{N}^*_{(\infty)}$. Then $O_\mathbf{T} = \{(\infty)\} \cup \{x \parallel x \in P \setminus (\infty)^\perp \text{and } \{x, (\infty)\}^\perp \in \mathbf{T}\}$ is an ovoid of $\mathcal{S}$.*

Proof. Let x_1 and x_2 be two collinear points of $O_\mathbf{T}$. Then $\{x_1, (\infty)\}^\perp$ and $\{x_2, (\infty)\}^\perp$ have at least one point in common. Hence $\{x_1, (\infty)\}^\perp = \{x_2, (\infty)\}^\perp$ and $x_2 \in \{x_1, (\infty)\}^{\perp\perp}$. This implies that x_1 and x_2 are not collinear, contradicting our assumption. Since $O_\mathbf{T}$ is a set of $st+1$ two by two non-collinear points, the theorem follows. □

A spread $\mathbf{T}$ of $\mathcal{N}^*_{(\infty)}$ is called *regular* if for every line M_1 of $\mathcal{N}^*_{(\infty)}$ not belonging to $\mathbf{T}$—so there exist $t+1$ lines $L_1, \ldots, L_{t+1}$ of $\mathbf{T}$ intersecting M_1 —there are t other lines $M_2, \ldots, M_{t+1}$ such that M_i is disjoint from M_j for all $i,j \in \{1,\dots,t+1\}$ with $i \neq j$, and so that M_i meets L_j in a point for all $i,j \in \{1,\dots,t+1\}$.

Theorem 4.3. *Let $\mathcal{S}$ be a GQ of order (s,t), $s,t > 1$, with a regular point (∞) which is contained in an ovoid of symmetry* $\mathbf{O}$. *Then there exists a regular spread* $\mathbf{T}$ *of* $\mathcal{N}^*_{(\infty)}$ *such that* $\mathbf{O} = O_{\mathbf{T}}$.

Proof. We first prove that $\{o_1,(\infty)\}^\perp$ and $\{o_2,(\infty)\}^\perp$ are equal or disjoint for any two points o_1, o_2 of $\mathbf{O} \setminus \{(\infty)\}$. Suppose $x \in \{o_1,(\infty)\}^\perp \cap \{o_2,(\infty)\}^\perp$ and let θ be a nontrivial automorphism of $\mathcal{S}$ which fixes each point of O. Then $x, x^\theta \in \{o_1,(\infty)\}^\perp \cap \{o_2,(\infty)\}^\perp$ and hence $\{o_1,(\infty)\}^\perp = \{o_2,(\infty)\}^\perp = \{x,x^\theta\}^{\perp\perp}$. Hence $\mathbf{T} = \{\{o,(\infty)\}^\perp \parallel o \in \mathbf{O} \setminus \{(\infty)\}\}$ is a spread of $\mathcal{N}^*_{(\infty)}$. We now show that $\mathbf{T}$ is regular. Let M_1 be an arbitrary line of $\mathcal{N}^*_{(\infty)}$ not belonging to $\mathbf{T}$ and let $M_1 = \{(\infty),x_1\}^\perp$. Let $\{x_1,\dots,x_{t+1}\}$ denote the orbit of x_1 determined by the group of automorphisms fixing each point of $\mathbf{O}$ and put $M_i = \{(\infty),x_i\}^\perp$ for all $i \in \{1,\dots,t+1\}$. The above conditions are then satisfied. □

Every automorphism θ of $\mathcal{S}$ which fixes (∞) induces an automorphism $\bar{\theta}$ of $\mathcal{N}^*_{(\infty)}$. If θ fixes every point of $O_{\mathbf{T}}$, then $\bar{\theta}$ fixes every line of $\mathbf{T}$.

Theorem 4.4. *Let ϕ be an automorphism of $\mathcal{N}^*_{(\infty)}$ fixing each line of a spread* $\mathbf{T}$ *of $\mathcal{N}^*_{(\infty)}$. Then there exists at most one automorphism θ of $\mathcal{S}$, with $\bar{\theta} = \phi$, which fixes each point of* $O_{\mathbf{T}}$.

Proof. If θ and θ' are two such automorphisms, then by [18, 2.2.2], $\theta[\theta']^{-1}$ is the identity on $\mathcal{S}$. □

If θ exists, then it must be equal to the following map. Put $(\infty)^\theta = (\infty)$ and put $z^\theta = z^\phi$ for every $z \in (\infty)^\perp \setminus \{(\infty)\}$. For $z \notin (\infty)^\perp$, let L_1 and L_2 denote two lines through z. The line L_i, $i \in \{1,2\}$, intersects $(\infty)^\perp$ in a point z_i and the ovoid $O_{\mathbf{T}}$ in a point o_i. Now, z^θ is the intersection of the lines $z_1^\theta o_1$ and $z_2^\theta o_2$. This is another proof for Theorem 4.4.

To determine all ovoids of symmetry of $\mathcal{S}$ through the regular point (∞), one could proceed as follows.

(i) Determine all regular spreads of $\mathcal{N}^*_{(\infty)}$.
(ii) For each regular spread $\mathbf{T}$ of $\mathcal{N}^*_{(\infty)}$, calculate the ovoid $O_{\mathbf{T}}$ of $\mathcal{S}$.
(iii) Determine all the automorphisms of $\mathcal{N}^*_{(\infty)}$ fixing each line of $\mathbf{T}$. There have to be at least $t+1$ such automorphisms (including the identity); otherwise $O_{\mathbf{T}}$ cannot be an ovoid of symmetry.
(iv) Check for each of these automorphisms whether it corresponds to an automorphism of $\mathcal{S}$ which fixes each point of $O_{\mathbf{T}}$. If we find $t+1$ such automorphisms, then $O_{\mathbf{T}}$ is an ovoid of symmetry.

This method allows us to determine the group of automorphisms that fixes each point of an ovoid of symmetry $O_{\mathbf{T}}$ if the dual net $\mathcal{N}^*_{(\infty)}$ satisfies the *Axiom of Veblen* (i.e., if $L_1 I x I L_2, L_1 \neq L_2, M_1 \not\!I x \not\!I M_2$, and if the line L_i is concurrent with the line M_j for all $i, j \in \{1, 2\}$, then M_1 is concurrent with M_2).

Since $s \neq t$, it follows by J. A. Thas and F. De Clerck [28] that $\mathcal{N}^*_{(\infty)} \cong H^n_q$, $n > 2$, where H^n_q is the dual net obtained as follows:

- The POINTS of H^n_q are the points of $\mathbf{PG}(n, q)$ not in a given subspace $\mathbf{PG}(n-2, q) \subseteq \mathbf{PG}(n, q)$.
- The LINES of H^n_q are the lines of $\mathbf{PG}(n, q)$ which have no point in common with $\mathbf{PG}(n-2, q)$.
- INCIDENCE in H^n_q is the one derived from $\mathbf{PG}(n, q)$.

From $s = q^{n-1}$, $t = q$ and $s \leq t^2$, it then follows that $n = 3$. Adding $L = \mathbf{PG}(n-2, q)$ to the spread $\mathbf{T}$ of $\mathcal{N}^*_{(\infty)}$, we then obtain a spread $\tilde{\mathbf{T}}$ of $\mathbf{PG}(3, q)$.

THEOREM 4.5. *The spread $\tilde{\mathbf{T}}$ is a regular spread of $\mathbf{PG}(3, q)$ and the group of automorphisms of $\mathcal{S}$ fixing each point of $O_{\mathbf{T}}$ is isomorphic to the cyclic group C_{q+1}.*

Proof. Let θ_i, $i \in \{1, \ldots, q+1\}$, denote the $q+1$ automorphisms of $\mathcal{S}$ fixing each point of $O_{\mathbf{T}}$. The automorphism θ_i corresponds to an automorphism $\tilde{\theta}_i$ of $\mathcal{N}^*_{(\infty)}$ which can be extended to an automorphism $\tilde{\theta}_i$ of $\mathbf{PG}(3, q)$; see, e.g., Theorem 1.4.3 of [3]. The automorphism $\tilde{\theta}_i$ fixes each line of $\tilde{\mathbf{T}}$. Now, let M be any line of $\mathbf{PG}(3, q)$ not belonging to $\tilde{\mathbf{T}}$. Then M meets the lines $L_1, \ldots, L_{q+1}$ of $\tilde{\mathbf{T}}$. Put $\{M_1, \ldots, M_{q+1}\} = \{\tilde{\theta}_i(M) \parallel i \in \{1, \ldots, q+1\}\}$. Clearly $L_1, \ldots, L_{q+1}$ is a regulus of $\mathbf{PG}(3, q)$; hence $\tilde{\mathbf{T}}$ is regular. The group $\{\tilde{\theta}_i \parallel i \in \{1, \ldots, q+1\}\}$ is a subgroup of the full group of automorphisms of $\mathbf{PG}(3, q)$ which fix each element of $\tilde{\mathbf{T}}$ and this latter group is isomorphic to C_{q+1}; see, e.g., Section 1.4.3 of [3]. The theorem now follows from Theorem 3.1. $\square$

In this section, we restricted our search to those ovoids of symmetry through a fixed regular point (∞). These restrictions are justified if the GQ comes from a *flock* $\mathcal{F}$ of the quadratic cone $\mathcal{K}$ in $\mathbf{PG}(3, q)$; this is just a partition of $\mathcal{K}$ minus its vertex in q disjoint irreducible conics. From the work of W. M. Kantor [13], S. E. Payne [15] and J. A. Thas [24], we know that each flock $\mathcal{F}$ gives rise to an elation generalized quadrangle $\mathcal{S}(\mathcal{F})$ of order (q^2, q) for some special base-point (∞). If $\mathcal{S}(\mathcal{F})$ is not isomorphic to $H(3, q^2)$, then the regular point (∞) is fixed by each nontrivial automorphism of $\mathcal{S}(\mathcal{F})$; see [19].

This implies that (∞) is contained in every ovoid of symmetry. Also, the case where the dual net is isomorphic to H_q^3 occurs, e.g., in the GQ's arising from the Kantor flocks. These GQ's will be treated in the following section.

5. The nonexistence of ovoids of symmetry in nonclassical Kantor flock quadrangles

In this section we will show that each *Kantor flock generalized quadrangle* of order (q^2, q) with an ovoid of symmetry is classical (i.e., isomorphic to $H(3, q^2)$). This will be a crucial observation for the next section.

5.1. Kantor generalized quadrangles. Let $\mathcal{K}$ be the quadratic cone of $\mathbf{PG}(3, q)$, where q is odd, with equation $X_0X_1 = X_2^2$. Then the q planes π_t with equation $tX_0 - mt^\sigma X_1 + X_3 = 0$, $t \in \mathbf{GF}(q)$, m a given non-square in $\mathbf{GF}(q)$ and σ a given automorphism of $\mathbf{GF}(q)$, define a flock $\mathcal{F}$ of $\mathcal{K}$; see [24]. All the planes π_t contain the exterior point $(0, 0, 1, 0)$ of $\mathcal{K}$. This flock is *linear*, that is, all the planes π_t contain a common line, if and only if $\sigma = \mathbf{1}$. Conversely, every nonlinear flock $\mathcal{F}$ of $\mathcal{K}$ for which the planes of the q conics share a common point, is of the type just described; see [24]. The corresponding GQ is called a *Kantor (flock) generalized quadrangle*. The described quadrangle is a TGQ for some baseline, and the following was shown by Payne in [17].

Theorem 5.1 (S. E. Payne [17]). *The point-line dual of a Kantor flock generalized quadrangle is a TGQ which is isomorphic to its translation dual.*

Remark 5.2. (i) It is well-known (see, e.g., [25] and [29]) that if $\mathcal{S}(\mathcal{F})$ is a Kantor flock GQ of order (q^2, q), $q > 1$, with special point (∞), then there are (precisely) $q^3 + q^2$ subGQ's of order q which contain the point (∞). These subGQ's are all isomorphic to $W(q)$. We will use this property (or its dual) in the sequel without further notice.

(ii) By the main result of K. Thas [37], each TGQ $\mathcal{S}$ of order (q, q^2), $q > 1$, of which the translation dual is the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$, has a line $[\infty]$ of translation points. In particular, if $\mathcal{F}$ is a Kantor flock, the property holds, and as $\mathcal{S} \cong \mathcal{S}^*$ by Theorem 5.1, $\mathcal{S}^*$ also has a line of translation points.

5.2. Nonexistence of ovoids of symmetry in nonclassical Kantor flock GQ's. We now come to the main result of this section; for the sake of convenience, we will work with the dual of $\mathcal{S}(\mathcal{F})$, $\mathcal{F}$ a Kantor flock.

Theorem 5.3. *Suppose $\mathcal{S} = \mathcal{S}(\mathcal{F})^D$ is the dual of the flock GQ $\mathcal{S}(\mathcal{F})$ of order (q^2, q), where $\mathcal{F}$ is a Kantor flock. If $\mathcal{S}$ admits a spread of symmetry $\mathbf{T}$, then $\mathcal{F}$ is linear, that is, $\mathcal{S} \cong \mathcal{Q}(5, q)$.*

Proof. Let $[\infty]$ be the line of $\mathcal{S}(\mathcal{F})^D$ which corresponds to the point (∞) in $\mathcal{S}(\mathcal{F})$. Consider a (classical) subGQ $\mathcal{S}'$ of order q through $[\infty]$. As $\mathcal{S}$ is the dual of a flock GQ, $\mathbf{T}$ must contain the line $[\infty]$ if $\mathcal{S}$ is not classical. Since $\mathcal{S}'$ is of order q, there are lines U and V of $\mathbf{T}$ so that $\{U,V\}^{\perp\perp} \subseteq \mathbf{T}\cap\mathcal{S}'$. If $H_{\mathbf{T}}$ is the group of automorphisms of $\mathcal{S}$ which fix $\mathbf{T}$ linewise, then $|[\mathcal{S}']^{H_{\mathbf{T}}}| = q+1$. This implies that the q^3+q^2 subGQ's of order q through $[\infty]$ are all in the same $\mathrm{Aut}(\mathcal{S})$-orbit as, since $[\infty]$ is a line of translation points by Remark 5.2, $\mathrm{Aut}(\mathcal{S})_{[\infty]}$ acts transitively on the pairs of non-concurrent lines in $[\infty]^{\perp}$. This yields a contradiction since non-classical Kantor GQ's have two such $\mathrm{Aut}(\mathcal{S})$-orbits of subGQ's of order q [27]. □

6. TGQ's and EGQ's with a spread of symmetry

6.1. Span-symmetric generalized quadrangles. Suppose $\mathcal{S}$ is a GQ of order (s,t), $s,t \neq 1$, and suppose L and M are distinct non-concurrent axes of symmetry; then it is easy to see that every line of $\{L,M\}^{\perp\perp}$ is an axis of symmetry, and $\mathcal{S}$ is called a *span-symmetric generalized quadrangle (SPGQ) with base-span* $\{L,M\}^{\perp\perp}$.

Throughout the rest of this paper, we will use the following notation and terminology.

First, the base-span will always be denoted by $\mathcal{L}$. The group which is generated by all the symmetries about the lines of $\mathcal{L}$ is G, and sometimes we will call this group the *base-group*. This group clearly acts 2-transitively on the lines of $\mathcal{L}$, and fixes every line of $\mathcal{L}^{\perp}$. The set of all the points which are on lines of $\{L,M\}^{\perp\perp}$ is denoted by Ω. We will refer to $\Gamma = (\Omega, \mathcal{L}\cup\mathcal{L}^{\perp}, I')$, with I' being the restriction of I to $(\Omega\times(\mathcal{L}\cup\mathcal{L}^{\perp}))\cup((\mathcal{L}\cup\mathcal{L}^{\perp})\times\Omega)$, as *the base-grid*.

THEOREM 6.1 (K. Thas [33], W. M. Kantor [14]). *A span-symmetric generalized quadrangle of order s, $s\neq 1$, is always isomorphic to $\mathcal{Q}(4,s)$.*

The following theorem is a consequence of the classification of the *finite split BN-pairs of rank* 1; see [20] and [11].

THEOREM 6.2 (K. Thas [35],[36]). *Every SPGQ of order (s,t), $s\neq 1\neq t$ and $s\neq t$, contains $s+1$ subquadrangles isomorphic to the classical GQ $\mathcal{Q}(4,s)$. Moreover, the base-group G acts semiregularly on $\mathcal{S}\setminus\Omega$ and $G\cong \mathbf{SL}_2(s)$.*

Finally, recall that $|\mathbf{PSL}_2(s)| = (s+1)s(s-1)$ or $(s+1)s(s-1)/2$, according as s is even or odd, respectively. Also, $|\mathbf{SL}_2(s)| = (s+1)s(s-1)$ for arbitrary s.

6.2. Generalized quadrangles with translation points or elation points with a spread of symmetry.

THEOREM 6.3. *Let $\mathcal{S} = T(\mathcal{O})$ be a TGQ of order (s,t), $s \neq 1 \neq t$, with base-point (∞), and which contains a spread of symmetry* $\mathbf{T}$. *Then $\mathcal{S}$ is isomorphic to $\mathcal{Q}(5,s)$.*

Proof. Suppose L is the line of $\mathbf{T}$ which is incident with the base-point (∞) of $\mathcal{S}$. Since there is a group $H_{\mathbf{T}}$ of $s+1$ automorphisms of $\mathcal{S}$ which fixes $\mathbf{T}$ elementwise and which acts transitively on the points of any line of $\mathbf{T}$, it follows that each point on L is a translation point (so every line of $L^{\perp}$ is an axis of symmetry, and hence $\mathcal{S}$ is an SPGQ for any two non-concurrent lines of $L^{\perp}$). By Theorem 3 of K. Thas [35] it follows that one of the following holds:

(i) s is even and $\mathcal{S} \cong \mathcal{Q}(5,s)$.
(ii) s is odd and $\mathcal{S}^{(\infty)}$ is the translation dual of the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$, that is, $\mathcal{O}$ is good at the element π which corresponds to L.

Recall that $t > s$ by the assumption. For the remainder of the proof, we assume we are in case (ii). Suppose $M \neq L$ is a line of $\mathbf{T}$, and put $\mathcal{L} = \{L, M\}^{\perp}$. Then every line of $\mathcal{L}$ is an axis of symmetry. Let G be the group which is generated by the symmetries about the lines of $\mathcal{L}$, and define H by $H = \langle G, H_{\mathbf{T}} \rangle$. First, note that any element of H fixes $\mathcal{L}^{\perp}$ linewise. Also, by Theorem 6.2, G acts semiregularly on $\mathcal{S} \setminus \Omega$ (where Ω is as in the previous section), and $G \cong \mathbf{SL}_2(s)$. Since s is odd, the kernel of the action of G on the lines of $\mathcal{L}$ has size 2. We now show that $H_{\mathbf{T}} \cap G = \{\mathbf{1}\}$.

By Theorem 6.2, G acts semiregularly on the points of $\mathcal{S} \setminus \Omega$ and G has order $(s+1)s(s-1)$. Let Λ be an arbitrary G-orbit in $\mathcal{S} \setminus \Omega$, and fix a line W of $\mathcal{L}^{\perp}$. By the semiregularity of G on the point set of $\mathcal{S} \setminus \Omega$, the fact that $|G| = (s+1)s(s-1)$ and that G acts transitively on the points of W, we have that any point on W is incident with exactly $s-1$ lines of $\mathcal{S}$ which are completely contained in Λ except for the point on W which is in Ω, and every point of Λ is incident with a line which meets W (recall that G is generated by groups of symmetries). Now define the following incidence structure $\mathcal{S}' = (P', B', I')$;

- LINES. The elements of B' are the lines of $\mathcal{S}'$ and they are essentially of two types:
  (1) the lines of Γ;
  (2) the lines of $\mathcal{S}$ which contain a point of Λ and a point of Ω.
- POINTS. The elements of P' are the points of the incidence structure and they are just the points of $\Omega \cup \Lambda$.
- INCIDENCE. Incidence I' is the 'induced incidence'.

Then $\mathcal{S}'$ is a generalized quadrangle of order s, and hence any line of $\mathcal{S}$ intersects $\mathcal{S}'$ in 1 or $s+1$ points. Now suppose that $\theta \in H_{\mathbf{T}} \cap G$, $\theta \neq \mathbf{1}$. Then by the semiregularity of G on $\mathcal{S} \setminus \Omega$ it immediately follows that each line of

$\mathbf{T} \setminus \mathcal{L}^{\perp}$ intersects $\mathcal{S}' \setminus \Gamma$ in at least two points, and hence in $s+1$ points, which is clearly a contradiction. Thus $H_{\mathbf{T}} \cap G = \{\mathbf{1}\}$. Since $H_{\mathbf{T}} \cap G = \{\mathbf{1}\}$, we have $|H| \geq (s^3 - s)(s+1)$. Actually, since it is now clear that H acts transitively on the points of $\mathcal{S} \setminus \Gamma$, we have that $(s+1)(s^3-s)$ divides $|H|$. Consider the dual net $\mathcal{N}^*_{[\infty]}$ which arises from the regular line $[\infty]$. Since $\mathcal{O}$ is good at π, there then follows by [29, Theorem 5.3] that $\mathcal{N}^*_{[\infty]}$ satisfies the Axiom of Veblen. Hence by J. A. Thas and F. De Clerck [28], $\mathcal{N}^*_{[\infty]} \cong H^3_s$. The points of H^3_s are the points of $\mathbf{PG}(3,s)$ not on a given line Z of $\mathbf{PG}(3,s)$; the lines are the lines of $\mathbf{PG}(3,s)$ which have no point in common with Z. With $\{L,M\}^{\perp}$ there corresponds a line $Z' \not\sim Z$ of $\mathbf{PG}(3,s)$ and with each line L_i, $i = 0,1,\ldots,s$, of $\{L,M\}^{\perp}$, there corresponds a point z'_i on Z'. Now we interpret the group $H = \langle G, H_{\mathbf{T}}\rangle$ as a group of collineations of H^3_s. First, the subgroup of G of symmetries about L_j, $j = 0,1,\ldots,s$, clearly induces the group of all elations of $\mathbf{PG}(3,s)$ with axis $\langle z'_j, Z\rangle$ and center z'_j. Hence if G induces G' on $\mathbf{PG}(3,s)$ then G' is a subgroup of $\mathbf{PGL}(4,s)$. Also, $H_{\mathbf{T}}$ induces the full group $H'_{\tilde{\mathbf{T}}}$ of automorphisms of $\mathbf{PG}(3,s)$ which fix the spread $\tilde{\mathbf{T}}$ of $\mathbf{PG}(3,s)$ (recall Section 4.2) linewise. As $H'_{\tilde{\mathbf{T}}}$ preserves the cross-ratio of $\mathbf{PG}(3,s)$, it follows that $H'_{\tilde{\mathbf{T}}}$ also is a subgroup of $\mathbf{PGL}(4,s)$ (see B. De Bruyn [3, p. 12] for an alternative proof of the latter observation). Hence H induces a subgroup H' of $\mathbf{PGL}(4,s)$ on $\mathbf{PG}(3,s)$ (which fixes Z and Z'). Hence the following property holds:

(E) If $\theta \in H$ fixes three lines of $\{L,M\}^{\perp}$, then θ fixes every line of $\{L,M\}^{\perp}$.

Now fix some point u in $\mathcal{S} \setminus \Gamma$, and consider u^N. There is at most one nontrivial $\theta \in N$ which fixes u, since the fixed elements structure of such an element is a subGQ $\mathcal{S}_\theta$ of $\mathcal{S}$ of order s (by [18, 2.2.2,2.4.1]), and such a subGQ can be fixed pointwise by at most one nontrivial collineation (which is then an involution), as it is well-known that in a GQ of order (s,s^2), $s>1$, for each two distinct non-collinear points u and v we have $\{u,v\}^{\perp\perp} = \{u,v\}$; see FGQ. Hence $|H| \in \{(s+1)(s^3-s), 2(s+1)(s^3-s)\}$, and so, as H acts transitively on the points of $\mathcal{S} \setminus \Gamma$, it follows in both cases that $|u^N| = s+1$, by Property (E). Now also suppose that $u \in (\infty)^{\perp}$. Recall that u is not a point of Γ. Suppose $\mathcal{S}^{(\infty)} = T(\mathcal{O})$ for the generalized ovoid $\mathcal{O}$ in $\mathbf{PG}(4n-1,q)$, where $q^n = s$. Then u^N is a set of q^n+1 points of type (2) which—as subspaces of $\mathbf{PG}(4n,q)$—all contain the same q^n points in $\mathbf{PG}(4n,q) \setminus \mathbf{PG}(4n-1,q)$. Note that in the GQ, (∞) is also a point of $(u^N)^{\perp}$. Interpreted in the translation dual $T(\mathcal{O}^*)$ of $T(\mathcal{O})$, the q^n+1 tangent spaces to $\mathcal{O}$ as defined by u^N become q^n+1 elements $\pi_0,\ldots,\pi_{q^n}$ of $\mathcal{O}^*$ which are contained in a $\mathbf{PG}(3n-1,q)$. Hence $\{\pi_0,\ldots,\pi_{q^n}\}$ is a generalized oval which lies on $\mathcal{O}^*$. Since $T(\mathcal{O}^*)$ is the point-line dual of a flock GQ and since q is odd, we can now conclude by [2] that $T(\mathcal{O}^*)$ is the point-line dual of a Kantor flock GQ $\mathcal{S}(\mathcal{F})$. Since the dual Kantor flock GQ's are isomorphic to their translation duals, we can conclude that $\mathcal{S}^{(\infty)}$ is the point-line dual of a Kantor flock GQ. Now Theorem 5.3 applies. $\square$

The following theorem is a strong generalization of Theorem 6.3 and relies on [36].

THEOREM 6.4. *Let $\mathcal{S}$ be a GQ of order (s,t), $s \neq 1 \neq t$, with a point x which is incident with at least three axes of symmetry. Moreover, suppose that $\mathcal{S}$ contains a spread of symmetry $\mathbf{T}$. Then $\mathcal{S}$ is isomorphic to $\mathcal{Q}(5,s)$.*

Proof. Suppose L is the line of $\mathbf{T}$ which is incident with x. Then, as before, each point on L is incident with three axes of symmetry. By K. Thas [36], it follows that each point on L is a translation point. The theorem now follows from the proof of Theorem 6.3. $\square$

REMARK 6.5. By [36], it is in fact sufficient to require that $\mathcal{S}$ be a GQ of order (s,t), $s \neq 1 \neq t$, with x a point incident with at least two axes of symmetry U and V, and $\mathcal{S}$ having a spread of symmetry $\mathbf{T}$ which does not contain U or V, in order to conclude that $\mathcal{S}$ is isomorphic to $\mathcal{Q}(5,s)$.

The following very general theorem eliminates almost all known classes of GQ's in the even characteristic case. First recall that a *center of transitivity* x of a GQ $\mathcal{S} = (P,B,I)$ of order (s,t), $s,t > 1$, is a point x so that there is a group of collineations of $\mathcal{S}$ fixing x linewise, which acts transitively on $P \setminus x^{\perp}$.

THEOREM 6.6. *Let $\mathcal{S} = (P,B,I)$ be a GQ of order (s,s^2), $s > 1$ and s even, with a center of transitivity (∞), and which contains a spread of symmetry $\mathbf{T}$. Then $\mathcal{S}$ is isomorphic to $\mathcal{Q}(5,s)$.*

Proof. As before, there is some line L incident with (∞) of which each point is a center of transitivity, and which is a line of the spread of symmetry $\mathbf{T}$. Suppose U and V are two distinct concurrent lines in $B \setminus L^{\perp}$, and let $u = U \cap V$. Suppose u' is the unique point on L which is collinear with u. Then, since u' is a center of transitivity, there is a collineation of $\mathcal{S}$ which fixes u' linewise and which maps U onto V. Using this observation, one easily derives that the group of automorphisms of $\mathcal{S}$ which fixes L, say Γ_L, acts transitively on the lines of $B \setminus L^{\perp}$. By Lemma 4.1, we know that the spread $\mathbf{T}$ is semiregular with respect to L. By the transitivity of Γ_L on $B \setminus L^{\perp}$, we can hence conclude that L is a regular line of $\mathcal{S}$. Also, by K. Thas [39] (see also Chapter 6 of [38]), the fact that $\mathcal{S}$ is of order (s,s^2) implies that $\mathcal{S}^{(\infty)}$ is an EGQ with base-point (∞) for some elation group G. By Theorem 5.1 of [31], we then conclude that, since s is even, $\mathcal{S}$ is a TGQ for every point incident with L. The theorem now follows from Theorem 6.3. $\square$

From the proof of Theorem 6.6, we immediately have:

Corollary 6.7. *Let $\mathcal{S}$ be an EGQ (for some elation point) of order (s,t), $s \neq 1 \neq t$ and s even, which contains a spread of symmetry* $\mathbf{T}$. *Then $\mathcal{S}$ is isomorphic to $\mathcal{Q}(5,s)$.*

6.3. Generalized quadrangles with translation lines or elation lines with a spread of symmetry.

Theorem 6.8. *Suppose $\mathcal{S}$ is an EGQ of order (s,t), $s \neq 1 \neq t$, with base-line L. Assume that* $\mathbf{T}$ *is a spread of symmetry of $\mathcal{S}$ which does not contain the line L. Then $\mathcal{S}$ is isomorphic to $\mathcal{Q}(5,s)$.*

Proof. Since L is not contained in $\mathbf{T}$, one easily observes that each line of $\mathcal{S}$ is an elation line, and hence, as $\mathrm{Aut}(\mathcal{S})$ clearly acts transitively on the pairs of non-concurrent lines of $\mathcal{S}$ and as $\mathbf{T}$ is a Hermitian spread, it follows that every line of $\mathcal{S}$ is regular. Now fix an arbitrary line M of $\mathcal{S}$. Then by K. Thas [34], we have two possibilities:

(i) M is an axis of symmetry, or

(ii) there is a subGQ $\mathcal{S}'$ of $\mathcal{S}$ of order s which contains M.

Suppose we are in case (i). Then each line is an axis of symmetry, and hence $\mathcal{S}$ is *half Moufang* as each line is *Moufang*; see, e.g., [30]. Thus, by the main theorem of J. A. Thas, H. Van Maldeghem and S. E. Payne [30], $\mathcal{S}$ is Moufang. Hence, by Fong and Seitz ([9], [10]) it follows that $\mathcal{S}$ is classical. Since $\mathcal{S}$ cannot be of order s and since $\mathcal{S}$ contains regular lines, we now obtain that $\mathcal{S} \cong \mathcal{Q}(5,s)$ [18, 3.3.1].

Next suppose we are in case (ii). As all lines of $\mathcal{S}$ are regular, it easily follows that $\mathcal{S}' \cong \mathcal{Q}(4,s)$, by [18, 5.2.1]. Consider two lines U and V of $\mathbf{T}$. Then $\{U,V\}^{\perp\perp} \subseteq \mathbf{T}$, and there is a subGQ $\mathcal{S}'' \cong \mathcal{Q}(4,s)$ which contains U and V, and hence also $\{U,V\}^{\perp\perp}$, since $\mathrm{Aut}(\mathcal{S})$ acts transitively on the pairs of non-concurrent lines. It is clear that $\mathbf{T} \cap \mathcal{S}'' = \{U,V\}^{\perp\perp}$ (both $\mathbf{T}$ and $\mathcal{S}''$ are viewed as line sets here). Hence, if $H_{\mathbf{T}}$ is the group of automorphisms of $\mathcal{S}$ which fix $\mathbf{T}$ linewise, then $|[\mathcal{S}'']^{H_{\mathbf{T}}}| = s+1$, and there are $s+1$ (classical) subGQ's of order s which mutually intersect in the induced subgeometry of $\mathcal{S}$ defined by $\{U,V\}^{\perp\perp}$. By the transitivity of $\mathrm{Aut}(\mathcal{S})$ on the pairs of non-concurrent lines of $\mathcal{S}$, the result now follows from [18, 5.3.5]. □

Remark 6.9. A generalized quadrangle $\mathcal{S}$ of order (s,t), $s,t > 1$, with a translation line L cannot have a spread of symmetry. For, suppose that this is the case. As L contains centers of symmetry, and hence regular points, we have $t \leq s$, a contradiction to (iii) of Section 2.5.

7. Remaining open cases

We end our paper with mentioning the most important remaining open cases.

Problem A. Classify all generalized quadrangles of order (s,t), where $s,t>1$ and s is odd, with an elation point p which have a spread of symmetry $\mathbf{T}$.

Problem B. Classify all generalized quadrangles of order (s,t), $s,t>1$, with an elation line L which admit a spread of symmetry $\mathbf{T}$ for which $L \in \mathbf{T}$.

We consider Problem B as the hardest but also the least interesting of both problems. By Corollary 6.6, the solution of Problem A would yield a complete classification of the elation generalized quadrangles (with respect to a point) which admit a spread of symmetry. Since almost all known GQ's are EGQ's for some base-point, this would be a very interesting result. For more on Problem A see [40].

Acknowledgements. The first author is a postdoctoral fellow supported by the Fund for Scientific Research — Flanders (Belgium). The second author is a research fellow supported by the Flemish Institute for the promotion of Scientific and Technological Research in Industry (IWT), grant no. IWT/SB/991254/Thas.

References

[1] T. Beth, D. Jungnickel, and H. Lenz, *Design theory, Vol. I. Second edition*, Encyclopedia of Math. and its Appl., vol. 69, Cambridge University Press, Cambridge, 1999.

[2] M. R. Brown and J. A. Thas, *Subquadrangles of order s of generalized quadrangles of order* (s,s^2), manuscript in preparation.

[3] B. De Bruyn, *Constructions and Characterizations of Near Polygons*, Ph.D. Thesis, Universiteit Gent (2000), iv + 203 pp..

[4] ______, *Generalized quadrangles with a spread of symmetry*, European J. Combin. **20** (1999), 759–771.

[5] ______, *On near hexagons and spreads of generalized quadrangles*, J. Algebraic Combin. **11** (2000), 211–226.

[6] ______, *On the number of nonisomorphic glued near polygons*, Bull. Belg. Math. Soc. Simon Stevin **7** (2000), 493–510.

[7] ______, *Recent results on near polygons: a survey*, Proceedings of the Academy Contact Forum 'Generalized Polygons', 20 October 2000, Palace of the Academies, Brussels, 2001, pp. 39–60.

[8] F. De Clerck and H. Van Maldeghem, *Some classes of rank* 2 *geometries*, Handbook of incidence geometry, North-Holland, Amsterdam, 1995, pp. 433–475.

[9] P. Fong and G. M. Seitz, *Groups with a (B,N)-pair of rank 2, I*, Invent. Math. **21** (1973), 1–57.

[10] ______, *Groups with a (B,N)-pair of rank 2, II*, Invent. Math. **24** (1974), 191–239.

[11] C. Hering, W. M. Kantor, and G. M. Seitz, *Finite groups with a split BN-pair of rank* 1, I, J. Algebra **20** (1972), 435–475.

[12] D. R. Hughes and F. C. Piper, *Projective planes*, Graduate Texts in Mathematics, vol. 6, Springer-Verlag, New York/Berlin, 1973.

[13] W. M. Kantor, *Generalized quadrangles associated with* $G_2(q)$, J. Combin. Th. Ser. A **29** (1980), 212–219.

[14] ———, *Note on span-symmetric generalized quadrangles*, Adv. Geom. **2** (2002), 197–200.

[15] S. E. Payne, *Generalized quadrangles as group coset geometries*, Proceedings of the Eleventh Southeastern Conference on Combinatorics, Graph Theory and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1980), Vol II, Congr. Numer. **29** (1980), 717–734.

[16] ———, *Hyperovals yield many GQ*, Simon Stevin **60** (1986), 211–225.

[17] ———, *An essay on skew translation generalized quadrangles*, Geom. Dedicata **32** (1989), 93–118.

[18] S. E. Payne and J. A. Thas, *Finite generalized quadrangles*, Research Notes in Mathematics, vol. 110, Pitman, Boston, 1984.

[19] ———, *Conical flocks, partial flocks, derivation, and generalized quadrangles*, Geom. Dedicata **38** (1991), 229–243.

[20] E. Shult, *On a class of doubly transitive groups*, Illinois J. Math. **16** (1972), 434–455.

[21] E. Shult and A. Yanushka, *Near n-gons and line systems*, Geom. Dedicata **9** (1980), 1–72.

[22] J. A. Thas, *The m-dimensional projective space $S_m(M_n(GF(q)))$ over the total matrix algebra $M_n(GF(q))$ of the $n \times n$-matrices with elements in the Galois field $GF(q)$*, Rend. Mat. (6) **4** (1971), 459–532.

[23] ———, *Translation* 4*-gonal configurations*, Rend. Accad. Naz. Lincei **56** (1974), 303–314.

[24] ———, *Generalized quadrangles and flocks of cones*, European J. Combin. **8** (1987), 441–452.

[25] ———, *Generalized quadrangles of order (s, s^2), I*, J. Combin. Th. Ser. A **67** (1994), 140–160.

[26] ———, 3*-Regularity in generalized quadrangles: a survey, recent results and the solution of a longstanding conjecture*, Rend. Circ. Mat. Palermo (2) Suppl. **53** (1998), 199–218.

[27] ———, *Orbits of subGQ's of the Kantor flock GQ's*, private communication (2001).

[28] J. A. Thas and F. De Clerck, *Partial geometries satisfying the axiom of Pasch*, Simon Stevin **51** (1977), 123–137.

[29] J. A. Thas and H. Van Maldeghem, *Generalized quadrangles and the axiom of Veblen*, Geometry, combinatorial designs and related structures (Spetses, 1996), London Math. Soc. Lecture Note Ser., vol. 245, Cambridge University Press, Cambridge, 1997, pp. 241–253.

[30] J. A. Thas, S. E. Payne, and H. Van Maldeghem, *Half Moufang implies Moufang for finite generalized quadrangles*, Invent. Math. **105** (1991), 153–156.

[31] K. Thas, *On symmetries and translation generalized quadrangles*, Finite geometries, Proceedings of the fourth Isle of Thorns conference (Brighton, UK, 2000), Kluwer Academic Publishers, Dordrecht, 2001, pp. 333–345.

[32] ———, *Automorphisms and characterizations of finite generalized quadrangles*, Proceedings of the Academy Contact Forum 'Generalized Polygons', 20 October 2000, Palace of the Academies, Brussels, 2001, pp. 111–172.

[33] ———, *Classification of span-symmetric generalized quadrangles of order s*, Adv. Geom. **2** (2002), 189–196.

[34] ———, *A theorem concerning nets arising from generalized quadrangles with a regular point*, Des. Codes Cryptogr. **25** (2002), 247–253.

[35] ———, *The classification of generalized quadrangles with two translation points*, Beiträge Algebra Geom. (Contributions to Algebra and Geometry) **43** (2002), 365–398.

[36] ———, *A Lenz-Barlotti classification for finite generalized quadrangles*, submitted, 110 pp.

[37] ———, *On translation generalized quadrangles of which the translation dual arises from a flock*, Glasgow Math. J., to appear, 26 pp.

[38] ———, *Automorphisms and Combinatorics of Finite Generalized Quadrangles*, Ph.D. Thesis, Universiteit Gent (2002), xxviii + 412 pp..

[39] ———, *A Note on the structure of certain groups which act on generalized quadrangles*, unpublished notes.

[40] ———, *A characterization of the generalized quadrangle $\mathcal{Q}(5,q)$ and the nonexistence of certain near polygons*, preprint.

[41] J. Tits, *Sur la trialité et certains groupes qui s'en déduisent*, Inst. Hautes Etudes Sci. Publ. Math. **2** (1959), 13–60.

Ghent University, Department of Pure Mathematics and Computer Algebra, Galglaan 2, B-9000 Ghent, Belgium

E-mail address, Bart De Bruyn: `bdb@cage.rug.ac.be`

E-mail address, Koen Thas: `kthas@cage.rug.ac.be`