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EISENSTEIN SERIES AND CARTAN GROUPS

BY

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Introduction

The principal congruence subgroup $\Gamma(N)$ acts discontinuously on the upper half plane $\mathscr{H}$, to give a non-compact fundamental domain of finite volume. Given such a group, one can associate to each cusp κ_i an Eisenstein series $E_i(z, s)$, where $z \in \mathscr{H}$ and $s \in \mathbf{C}$. This Eisenstein series admits a Fourier expansion at each cusp κ_j. The zero Fourier coefficient involves a meromorphic function $\phi_{ij}(s)$, so that one obtains a matrix $\Phi(s) = (\phi_{ij}(s))_{i,j}$ (see §1 for precise definitions).

The determinant $\phi(s) = \det \Phi(s)$ plays a key role in the theory, mostly due to its appearance in the Selberg trace formula for the group in question. Of particular importance are the poles of $\phi(s)$, whose analysis is connected with the study of cusp forms for the group (see [11], [1]).

The problem of computing $\phi(s)$ for $\Gamma(N)$ was first addressed by Hejhal (see [4]), who treated the case of square free and odd N by some rather involved methods. Huxley [5] has recently solved the problem using other ingenious arguments, and gave an expression for $\phi(s)$ for any N. As for other groups, we mention the work in [2] where we compute these determinants for Hilbert modular groups, and in [1], where they are partially analyzed for congruence subgroups of Hilbert modular groups. Other relevant references are [3], [8], [9].

Our aim in this paper is to introduce the Cartan group $C(N)$ into the study of the Eisenstein series for $\Gamma(N)$, and to use it in order to give a short and simple proof of the precise formula for $\phi(s)$, for any N. Our main theorem (§3) shows that $\phi(s)$ is naturally expressed in terms of the L-functions on $C(N)$. These L-functions also come up in the work of Kubert and Lang on modular units [7].

1. The Eisenstein series

Let

$$\Gamma = \Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv I (\mathrm{mod}\, N) \right\}$$

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be the principal congruence subgroup of level $N > 2$, and choose a set of representatives for the cusps

$$\kappa_i = -\frac{\delta_i}{\gamma_i}, \quad i = 1, \ldots, h,$$

with $(\gamma_i, \delta_i) = 1$. Thus, if $1 \leq \gamma_i', \delta_i' \leq N$ with $\gamma_i' \equiv \gamma_i(N)$, $\delta_i' \equiv \delta_i(N)$, then (γ_i', δ_i'), $i = 1, \ldots, h$, are the primitive pairs mod N (i.e., $(\gamma_i', \delta_i', N) = 1$), identified mod ± 1. Also,

$$h = \frac{N^2}{2} \prod_{p|N} \left(1 - \frac{1}{p^2}\right).$$

For these standard facts, see [10] for example.

Let Γ_i be the stabilizer of κ_i in Γ, and choose $\alpha_i, \beta_i \in \mathbf{Z}$ with $\alpha_i\delta_i - \beta_i\gamma_i = 1$. Then

$$\rho_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix} \in SL_2(\mathbf{Z})$$

sends κ_i to ∞. Let

$$z^{(i)} = \rho_i z = \left(x^{(i)}, y^{(i)}\right).$$

Then the Eisenstein series at κ_i is defined in general as

$$E_i(z, s) = \sum_{\tau \in \Gamma_i \backslash \Gamma} y^{(i)}(\tau z)^s, \quad z \in \mathscr{H}, \operatorname{Re}(s) > 1,$$

(see [11]). It has a Fourier expansion at κ_j of the form

$$\delta_{ij} y^{(j)^s} + \phi_{ij}(s) y^{(j)^{1-s}} + \textit{non-zero coefficients},$$

for some meromorphic function $\phi_{ij}(s)$. Let $\Phi(s) = (\phi_{ij}(s))_{i, j=1, \ldots, h}$. Our goal is to compute the determinant

$$\phi(s) = \det \Phi(s).$$

To this end, we begin by observing that for

$$\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

we have

$$y^{(i)}(\tau z) = y(\rho_i \tau z) = \frac{y^s}{|(\gamma_i a + \delta_i c)z + (\gamma_i b + \delta_i d)|^{2s}} = \frac{y^s}{|c'z + d'|^{2s}}$$

and $c' \equiv \gamma_i(N)$, $d' \equiv \delta_i(N)$. Conversely, for such c', d' we have $\alpha_i d' - \beta_i c' \equiv 1\ (N)$, so that (see [10, p. 74]) there exist $a', b' \in \mathbf{Z}$, $a' \equiv \alpha_i(N)$, $b' \equiv \beta_i(N)$ with $a'd' - b'c' = 1$. Let

$$\tau = \begin{pmatrix} \delta_i & -\beta_i \\ -\gamma_i & \alpha_i \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \Gamma(N).$$

Then

$$\rho_i \tau = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix},$$

and any other τ with this property is in the same coset of $\Gamma_i \setminus \Gamma$. We conclude that

$$E_i(z,s) = E_{\gamma_i,\delta_i}(z,s) = \sum_{\substack{(c,d)=1 \\ c\equiv\gamma_i,\, d\equiv\delta_i \ (\mathrm{mod}\ N)}} \frac{y^s}{|cz+d|^{2s}}.$$

To simplify further, we let

$$F_i(z,s) = \sum_{c\equiv\gamma_i,\, d\equiv\delta_i \ (\mathrm{mod}\ N)} \frac{y^s}{|cz+d|^{2s}}.$$

Then

$$\begin{aligned} F_i(z,s) &= \sum_{\substack{k=1 \\ (k,N)=1}}^{\infty} \sum_{\substack{(c,d)=k \\ c\equiv\gamma_i,\, d\equiv\delta_i(N)}} \frac{y^s}{|cz+d|^{2s}} \\ &= \sum_{\substack{k=1 \\ (k,N)=1}}^{\infty} \frac{1}{k^{2s}} \sum_{\substack{(c,d)=1 \\ c\equiv k^{-1}\gamma_i,\, d\equiv k^{-1}\delta_i(N)}} \frac{y^s}{|cz+d|^{2s}}. \end{aligned}$$

Here k^{-1} is the inverse of k mod N. Let $k_1, \ldots, k_r$ be representatives of

$$\mathbf{Z}(N)^{\pm} = (\mathbf{Z}/N\mathbf{Z})^{\times}/\pm 1, \quad r = \frac{1}{2}\phi(N).$$

Then the above becomes

$$\sum_{\nu=1}^{r} \zeta(2s, \pm k_\nu) E_{k_\nu^{-1}\gamma_i,\, k_\nu^{-1}\delta_i}(z,s),$$

where

$$\zeta(2s, \pm k_\nu) = \sum_{\substack{k=1 \\ k\equiv k_\nu(N)}}^{\infty} \frac{1}{k^{2s}} + \sum_{\substack{k=1 \\ k\equiv -k_\nu(N)}}^{\infty} \frac{1}{k^{2s}}.$$

We rewrite these relations as

$$\begin{bmatrix} F_1(z,s) \\ \vdots \\ F_h(z,s) \end{bmatrix} = \begin{bmatrix} \boxed{B} & & & \\ & \boxed{B} & & \\ & & \ddots & \\ & & & \boxed{B} \end{bmatrix} \begin{bmatrix} E_1(z,s) \\ \vdots \\ E_h(z,s) \end{bmatrix}$$

where each block B is the matrix

$$B = \left[\zeta\left(2s, \pm\, k_\nu^{-1}k_\mu\right)\right]_{\nu,\mu=1,\dots,r}.$$

This essentially reduces the study of $\Phi(s)$ to that of the corresponding matrix for the F_i's, so we now turn to the computation of the zero Fourier coefficient of F_i at κ_j. We have

$$\begin{aligned} F_i(z,s) &= F_i\left(\rho_j^{-1}z^{(j)}, s\right) \\ &= \sum_{c\equiv\gamma_i,\, d\equiv\delta_i(N)} \frac{y^{(j)^s}}{|(c\delta_i - d\gamma_j)z^{(j)} + (-c\beta_j + d\alpha_j)|^{2s}} \\ &= \sum_{c\equiv\lambda,\, d\equiv\mu(N)} \frac{y^{(j)^s}}{|cz^{(j)}+d|^{2s}}, \quad \lambda = \gamma_i\delta_j - \delta_i\gamma_j \quad \mu = -\gamma_i\beta_j + \delta_i\alpha_j. \end{aligned}$$

A term with $c = 0$ will come up iff $\lambda \equiv 0$ (N), in which case we get

$$y^{(j)^s} \sum_{d\equiv\mu(N)} \frac{1}{d^{2s}}.$$

Now, fixing $c \neq 0$, by the Poisson summation formula we have

$$\sum_{d\equiv\mu(N)} \frac{1}{|cz+d|^{2s}} = \sum_{t\in\mathbf{Z}} \frac{1}{|cz+\mu+tN|^{2s}} = \sum_{t\in\mathbf{Z}} \int_{-\infty}^{\infty} \frac{e^{2\pi iut}\,du}{|cz+\mu+uN|^{2s}},$$

and a change of variables gives

$$\frac{1}{N}\frac{1}{|c|^{2s-1}} \sum_{t\in\mathbf{Z}} \int_{-\infty}^{\infty} \frac{e^{2\pi icut/N}}{|z+u|^{2s}}\,du\, e^{-2\pi i\mu t/N}.$$

For the zero coefficient we put $t = 0$ and use

$$\int_{\infty}^{\infty} \frac{du}{|z+u|^{2s}} = \pi^{1/2}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}y^{1-2s}$$

to obtain:

Proposition 1. *The zero Fourier coefficient of F_i at κ_j is*

$$\iota \cdot \zeta\big(2s, \pm(-\gamma_i\beta_j + \delta_i\alpha_j)\big) y^{(j)^s} \\ + \pi^{1/2} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \frac{1}{N} \zeta\big(2s-1, \pm(\gamma_i\delta_j - \delta_i\gamma_j)\big) y^{(j)^{1-s}}$$

By comparing the zero coefficients of the E_i's and the F_i's we get:

Corollary.

$$\begin{bmatrix} \boxed{B} & & & \\ & \boxed{B} & & \\ & & \ddots & \\ & & & \boxed{B} \end{bmatrix} \big[\phi_{ij}(s)\big] \\ = \pi^{1/2} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \frac{1}{N} \big[\zeta\big(2s-1, \pm(\gamma_i\delta_j - \delta_i\gamma_j)\big)\big]$$

We shall identify the matrix on the right as essentially a group matrix for the Cartan group.

2. The Cartan groups

In this section we describe the basic aspects of these groups, essentially following [6]. We let

$$G(N) = GL_2(\mathbf{Z}/N\mathbf{Z}).$$

Then a primitive pair mod N (c, d) can be extended to an element $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of $G(N)$, and two such elements will differ on the left by an element of the subgroup

$$G_\infty(N) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in G(N) \right\}.$$

It follows that the cusps can be represented by the cosets in $G_\infty(N) \setminus G(N)$. We wish to establish a unique decomposition

$$G(N) = G_\infty(N) \cdot C(N),$$

where $C(N)$ is an abelian group, called the *Cartan group* of level N. This will imply that the cusps correspond naturally to the elements of $C(N)$, in that $C(N)$ acts on them simply and transitively.

Write $N = \prod_{p|N} p^{n(p)}$ and fix p. Let $R = [1, u]$ be the ring of integers of the unramified quadratic extension of $\mathbf{Q}_p$. Let $C_p = R^\times$ be the group of units of R. Then C_p consists of the primitive elements of R, i.e., those $d + cu \in R$ for which c and d are not both divisible by p. Since C_p is a group, it acts simply transitively on the primitive elements.

Next we embed C_p in $GL_2(\mathbf{Z}_p)$ by the regular representation over $\mathbf{Z}_p$:

$$d + cu \mapsto \begin{pmatrix} d & cu^2 \\ c & d \end{pmatrix}.$$

PROPOSITION 2. *Let*

$$G_{\infty, p} = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| b \in \mathbf{Z}_p, a \in \mathbf{Z}_p^* \right\} \subset GL_2(\mathbf{Z}_p).$$

Then we have a unique decomposition

$$GL_2(\mathbf{Z}_p) = G_{\infty, p} \cdot C_p.$$

Proof. We show that the multiplication map

$$G_{\infty, p} \times C_p \to GL_2(\mathbf{Z}_p)$$

is a bijection.

Since $G_{\infty, p} \cap C_p = \{1\}$ it is one-to-one. To prove that it is onto, let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbf{Z}_p).$$

For an element in C_p take

$$\begin{pmatrix} d & cu^2 \\ c & d \end{pmatrix}.$$

By the transitive action of C_p on the primitive pairs, there is a pair (a', b') so that

$$(a', b')\begin{pmatrix} d & cu^2 \\ c & d \end{pmatrix} = (a, b).$$

Hence

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a' & b' \\ 0 & 1 \end{pmatrix}\begin{pmatrix} d & cu^2 \\ c & d \end{pmatrix}.$$

Note that $G(p^n)$ is the restriction of $GL_2(\mathbf{Z}_p)$ mod p^n, and similarly for $G_\infty(p^n)$. Thus if we let $C(p^n)$ be the restriction of C_p mod p^n, we obtain a unique decomposition

$$G(p^n) = G_\infty(p^n) \cdot C(p^n).$$

Finally, let $C(N) = \prod_{p|N} C(p^{n(p)})$. Then we have a unique decomposition $G(N) = G_\infty(N) \cdot C(N)$, and since we identify primitive pairs mod ± 1, we actually need

$$G^{\pm}(N) = G_\infty(N) \cdot C^{\pm}(N)$$

where

$$G^{\pm}(N) = G(N)/\pm 1, C^{\pm}(N) = C(N)/\pm 1.$$

3. The main theorem

We recall the method of group determinants: If $A = \{a_1, \dots, a_n\}$ is an abelian group and f is a complex function on A, then the determinant of the "group matrix" $[f(a_i^{-1}a_j)]_{i,j=1,\dots,n}$ is given by

$$\prod_{\chi \in \hat{A}} \sum_{a \in A} \chi(a) f(a).$$

We wish to relate our

$$\left[\frac{1}{N}\zeta\left(2s-1, \pm(\gamma_i\delta_j - \delta_i\gamma_j)\right)\right]_{i,j=1,\dots,h}$$

to such a group matrix.

Now we saw in §2 that the cusp $\kappa = (\gamma, \delta) = -\delta/\gamma$ can be identified with the following element of $C(N)^{\pm}$:

$$\prod_{p|N} \begin{pmatrix} \delta(\mathrm{mod}\, p^n) & \gamma u^2(\mathrm{mod}\, p^n) \\ \gamma(\mathrm{mod}\, p^n) & \delta(\mathrm{mod}\, p^n) \end{pmatrix}$$

abbreviated by

$$\begin{pmatrix} \delta & \gamma u^2 \\ \gamma & \delta \end{pmatrix}.$$

For such a κ we let

$$N(\kappa) = \delta^2 - \gamma^2 u^2 \in \mathbf{Z}(N)^{\pm}$$

and define

$$\kappa' = (\gamma', \delta') = \left(N(\kappa)^{-1}\gamma, N(\kappa)^{-1}\delta\right) = N(\kappa)^{-1}\kappa.$$

(again we use the fact that $\mathbf{Z}(N)^{\pm}$ acts on the cusps). Then

$$\begin{pmatrix} \delta & -\gamma u^2 \\ -\gamma & \delta \end{pmatrix} = \frac{1}{N(\gamma', \delta')}\begin{pmatrix} \delta' & -\gamma' u^2 \\ -\gamma' & \delta' \end{pmatrix} = \kappa'^{-1},$$

and therefore

$$\kappa_i'^{-1} \cdot \kappa_j = \begin{pmatrix} \delta_i & -\gamma_i u^2 \\ -\gamma_i & \delta_i \end{pmatrix}\begin{pmatrix} \delta_j & \gamma_j u^2 \\ \gamma_j & \delta_j \end{pmatrix} = \begin{pmatrix} * & * \\ -\gamma_i\delta_j + \delta_i\gamma_j & * \end{pmatrix},$$

so that if we define a function on the cusps by

$$f(\kappa) = f(\gamma, \delta) = \frac{1}{N}\zeta(2s-1, \pm\gamma),$$

our matrix above becomes $[f(\kappa_i'^{-1} \cdot \kappa_j)]_{i,j=1,\dots,h}$. This is not quite a group matrix, but if we multiply it by the permutation matrix $P = (p_{ij})$ defined by

$$p_{ij} = \begin{cases} 1 & \text{if } \kappa_j' = \kappa_i \\ 0 & \text{otherwise} \end{cases}$$

then we get the group matrix $[f(\kappa_i^{-1} \cdot \kappa_j)]_{i,j=1,\dots,h}$.

We can finally compute the determinant $\phi(s)$. To the map $T(\gamma, \delta) = \gamma$ and the character χ of $C(N)^{\pm}$ we associate the L-function

$$L(s, \chi, T) = \frac{1}{N}\sum_{\kappa \in C(N)} \chi(\kappa)\zeta(s, T\kappa)$$

where

$$\zeta(s, \gamma) = \sum_{\substack{k=1 \\ k \equiv \gamma(N)}}^{\infty} \frac{1}{k^s}.$$

Then by the method of group determinants,

$$\det\left[f\left(\kappa_i^{-1}\kappa_j\right)\right] = \prod_{\chi \in \hat{C}(N)\pm} L(2s-1, \chi, T).$$

Similarly, if we go back to the corollary of §1, we see that the matrix B is a group matrix for $\mathbf{Z}(N)^{\pm}$, so that

$$\det(B) = \prod_{\chi \in \mathbf{Z}^{\wedge}(N)^{\pm}} L(2s, \chi)$$

where $L(2s, \chi)$ is a Dirichlet L-function.

Turning finally to the permutation matrix, we see that

$$\det(P) = (-1)^{(h-h_0)/2},$$

where h_0 is the number of cusps κ for which $N(\kappa) = 1$. Thus

$$h_0 = \frac{|C(N)^{\pm}|}{|\mathbf{Z}(N)^{\pm}|} = \frac{h}{r} = N \prod_{p|N} \left(1 + \frac{1}{p}\right).$$

Putting all these results together, we obtain our main theorem:

THEOREM.

$$\phi(s) = (-1)^{(h-h_0)/2} \left(\pi^{1/2} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \right)^h \frac{\prod_{\chi \in \hat{C(N)}^{\pm}} L(2s-1, \chi, T)}{\prod_{\chi \in \hat{\mathbf{Z}(N)}^{\pm}} L(2s, \chi)^{h_0}}$$

Remarks. The L-function $L(s, \chi, T)$ above are exactly the ones that appear in [7] where it is shown how they can be related to ordinary Dirichlet L-functions. Assume first that χ is primitive, and let

$$S(\chi, T) = \sum_{\kappa \in C(N)} \chi(\kappa) e^{2\pi i T\kappa/N}$$

be its Gauss sum of $C(N)$ with respect to T. Furthermore, let $\chi_{\mathbf{Z}}$ be the restriction of χ to $\mathbf{Z}(N)$ (of conductor c, say), and let $S_{\mathbf{Z}}(\chi_{\mathbf{Z}})$ be its standard Gauss sum. Then

$$L(s, \chi, T) = \frac{1}{N} \frac{S(\chi, T)}{S_{\mathbf{Z}}(\chi_{\mathbf{Z}})} \prod_{\substack{p|N \\ p\nmid c}} \left(1 - \frac{\bar{\chi}(p)}{p^{1-s}}\right) L(s, \chi_{\mathbf{Z}}).$$

Finally, if χ is not primitive, so that it factors through $C(M)$ for some $M|N$, then

$$L(s, \chi, T) = \prod_{\substack{p|N \\ p\nmid M}} \left(1 - \frac{\chi_M(p)}{p^{s+1}}\right) L(s, \chi_M, T_M).$$

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