INNER FUNCTIONS ANALYTIC AT A POINT[1]

BY
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1. Main result

We will prove the following theorem.

THEOREM 1. *Let U be an inner function in the upper half-plane and analytic at the origin. If $U'(0)a = 0$ for some vector a, then Ua is constant.*

This theorem generalizes a recent result of Helson [1] concerning operator-valued inner functions. Our proof is based on some calculations of Potapov [3] which were originally done for a finite Hilbert space but which carry over to the infinite dimensional case.

2. Preliminaries

All symbols keep the same meaning throughout the paper. Complex variables will be denoted by z and w, while x will denote a real variable. K is a Hilbert space of finite or infinite dimension. H_K^2 is the usual Hilbert space of boundary values of K-vector valued functions analytic on the upper half-plane. U is an inner function, that is, a function on the real line whose values are unitary operators in K such that if $F(x) \in H_K^2$, then $U(x)F(x) \in H_K^2$. The operator-valued function U has a bounded analytic extension to the upper half-plane.

For a more thorough development of these ideas see Helson's book [2].

3. Proof of Theorem 1

We can assume without loss of generality that $U(0) = I$, where I is the identity operator in K. Let a be a vector in K such that $U'(0)a = 0$. Rotate the half-plane by the map $w = -iz$ and define $V(w) = U(z)$. V is contractive in the right half-plane and analytic at zero. It has a series expansion in some neighborhood of zero:

$$V(w) = I + A_1 w + A_2 w^2 + \cdots.$$

From the definition of V we see that $A_1 = iU'(0)$. (The differentiation on U will always be with respect to z or x.) In [1] it is observed that $U'(0) = iA$ where A is a self-adjoint positive operator in K. Thus A_1 is self-adjoint.

Using the fact that A_1 is self-adjoint and that

$$([I - V(w)V^*(w)]a, a) \geq 0, \qquad \operatorname{Re} w \geq 0,$$

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Potapov proves [3, p. 154] that $A_1 a = 0, \cdots, A_{m-1} a = 0$ implies $A_m a = 0$ for every $m \geq 2$. Thus $U'(0)a = 0$ implies $V(w)a = a$ identically in w. So $U(z)a = a$ identically in z and the theorem is proven.

4. Application

With Theorem 1 we can simplify the proof of the following theorem of Helson [1]. We use only two general lemmas he establishes relating continuity on the boundary with analyticity.

THEOREM 2. *Suppose an inner function $U(x)$ satisfies the differential equation $U'(x) = ip(x)M(x)U(x)$. Assume $M(x)$ is norm-continuous and projection-valued. Assume furthermore that $p(x)$ is a continuous bounded real function on the line such that $1/p(z)$ is entire, and such that $-\log | p(z) |$ possesses a positive harmonic majorant in the upper half-plane. Then $M(x)$ is constant.*

Proof. We assume $U(0) = I$. Helson easily shows that the norm-continuity of $p(x)M(x)$ implies that $U(x)$ is analytic on the real axis.

Let $N(x)$ be the null space of $M(x)$. If a is in $N(0)$, Theorem 1 implies $U(x)a = a$, $(U(0) = I)$. Thus $U'(x)a = 0$ and $N(0) \subset N(x)$. It remains only to show $N(x) \subset N(0)$ for any x. Let b be any real number unequal to zero. Suppose c is a vector in $N(b)$. Let $V(x) = U(x + b)$. The vector $U^*(b)c$ is such that $V'(0)U^*(b)c = 0$. By Theorem 1, $V(x)U^*(b)c$ is identically constant. Evaluations at $x = 0$ and $x = -b$ show that $U(x + b)c = c$ identically in x. Differentiation with respect to x followed by evaluation at $-b$ give $U'(0)c = 0$. Thus c is in $N(0)$ and Theorem 2 is proven.

The only properties of $p(x)$ that we used were that it was continuous and never zero. If we only assume that $p(x)$ has isolated zeros, then a restriction of the calculations to the set of real numbers b such that $p(b)$ and $p(-b)$ are both nonzero and an appeal to the norm-continuity of $U(x)$ carry the proof through.

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REFERENCES

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