ON 2-GROUPS OPERATING ON PROJECTIVE PLANES[1]

BY

CHRISTOPH HERING

1. Introduction

Let $\mathbf{E}$ be a projective plane of finite order n and G a group of automorphisms of $\mathbf{E}$, whose order is a power of 2. If $n \equiv 3 \pmod 4$, then the structure of G can be determined completely. This has been done in [5, Satz 1]. Here we consider the cases $n \equiv 5 \pmod 8$ and $n \equiv 1 \pmod 8$. If $n \equiv 5 \pmod 8$, we can again give a complete list of all possible isomorphism types (see Theorem 5). They are precisely those which also occur in the automorphism groups of desarguesian planes of the corresponding order (provided, of course, that such planes exist). If $n \equiv 1 \pmod 8$, this ceases to be true, and apparently it becomes quite difficult to describe all possibilities. However in many situations one still can determine the structure of the Sylow-2-subgroups of all composition factors of a given group of automorphisms of $\mathbf{E}$ (see Theorems 2 and 4).

2. Notations and definitions

Our notation is standard except perhaps in the following abbreviations:

Z_n the cyclic group of order n

D_n the dihedral group of order n

Q_n the (generalized) quaternion group of order n

M_{2^m} the group with generators a and b and relations $a^{2^{m-1}} = b^2 = 1$, $bab = a^{1+2^{m-2}}$

$\mathbf{Z}G$ the center of the group G

$\mathbf{C}_G H$ the centralizer of the subgroup H in the group G

$\mathbf{N}_G H$ the normalizer of the subgroup H in the group G

$G_{P_1,\cdots,P_n}$ the stabilizer of the permutation group G on the points $P_1, \cdots, P_n$

PQ the line containing the points P and Q

$\Omega_1(G) = \langle g \mid g \in G \text{ and } g^2 = 1\rangle$

$\mho_1(G) = \langle g^2 \mid g \in G\rangle$

$x \circ y = x^{-1}y^{-1}xy$

$x^y = y^{-1}xy$

$G(P, g)$ the subgroup of the automorphism group G consisting of all perspectivities with center P and axis g

$n \mid m$ n divides m

$2^n \parallel m$ 2^n divides m, but 2^{n+1} does not divide m

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$\mathbf{M} - \mathbf{U}$ the set of all elements in the set $\mathbf{M}$ which are not contained in the subset $\mathbf{U}$

An involution always is an element of order 2. A dihedral group is a group with two generators a and b and the relations

$$a^n = b^2 = 1, \qquad bab = a^{-1}$$

where $n \geqq 2$. Hence in particular every elementary abelian group of order 4 is a dihedral group. A quasidihedral group is a group with two generators a and b and the relations

$$a^{2^n} = b^2 = 1, \qquad bab = a^{-1+2^{n-1}}$$

where $n \geq 3$. Both ordinary and generalized quaternion groups will simply be called quaternion groups.

A projective plane is a pair $(\mathbf{P}, \mathbf{G})$ consisting of a set $\mathbf{P}$ and a collection $\mathbf{G}$ of subsets of $\mathbf{P}$ with the following properties:

(I) If $g, h \in \mathbf{G}$ and $g \neq h$, then $|g \cap h| = 1$.

(II) If $P, Q \in \mathbf{P}$ and $P \neq Q$, then there exists exactly one element $g \in \mathbf{G}$ such that P and Q are contained in g.

(III) There exist four elements of $\mathbf{P}$, no three of which are contained in an element of $\mathbf{G}$.

3. 2-groups containing a homology group of maximal possible order

LEMMA 1. *Let $\mathbf{E} = (\mathbf{P}, \mathbf{G})$ be a projective plane of finite odd order n and G a 2-group of automorphisms of $\mathbf{E}$ all of whose involutions are perspectivities. Then one of the following two statements holds:*

(a) *G does not contain any elementary abelian subgroup of order 8, and for any point $X \in \mathbf{P}$ and line $y \in \mathbf{G}$ the group $G(X, y)$ contains at most one involution.*

(b) *There exists a line $g \in \mathbf{G}$ and a point $P \in \mathbf{P} - g$ such that $\Omega_1(G) \subseteq G(P, g)$. Furthermore*

$$|\Omega_1(G)| > 2, n \equiv 1 \pmod 4 \quad \textit{and} \quad |G| \,|\, 2(n-1).$$

Proof. Let $G \neq 1$ and z be an involution contained in $\mathbf{Z}G$. Since n is odd, z is then a homology with a center P and an axis g, where $P \notin g$, and obviously P and g are left invariant by G. If all involutions of G have center P, they must all have axis g, and we get

$$\Omega_1(G) \subseteq G(P, g).$$

If now furthermore $|\Omega_1(G)| \leq 2$, then we obviously have case (a). So if statement (a) is false, we must have $|\Omega_1(G)| > 2$, and hence $|G(P, g)| > 2$ and $n \equiv 1 \pmod 4$. Also the group G which leaves invariant P and g must then operate regularly on $\mathbf{P} - g \cup \{P\}$. But this implies

$$|G| \,\big|\, |\mathbf{P} - g \cup \{P\}| = n^2 - 1 = (n-1)(n+1)$$

and $|G| \,|\, 2(n-1)$, since $2 \,\|\, n + 1$. Hence we have case (b).

Assume now that there exists an involution i whose center A is different from P: Since i commutes with z, it leaves invariant P and g. Therefore $A \in g$, and $P \in a$, if a is the axis of i. Now let M be an elementary abelian group of largest possible order containing i. Then M leaves invariant A, P and $a \cap g$. Since every element $\neq 1$ of M is an involution and so a perspectivity we have

$$M = M(P, g) \cup M(A, a) \cup M(a \cap g, AP).$$

Here the three subgroups have pairwise intersection 1, and $M(P, g)$ contains the central element z because of the maximality of M. These facts together imply $|M| = 4$ and

$$M = \langle i \rangle \times \langle z \rangle. \tag{1}$$

If $\bar{z}$ is any involution of $G(P, g)$, we have $\bar{z} \circ z = 1$ and, since $\bar{z}^{-1}i^{-1}\bar{z}i$ leaves a and g pointwise invariant, also $\bar{z} \circ i = 1$. Hence $M \cdot \langle \bar{z} \rangle$ is elementary abelian, and $\bar{z} \in M$, which means $\bar{z} \in M(P, g) = \langle z \rangle$. So $G(P, g)$ contains only one involution. Therefore any elementary abelian subgroup of order ≥ 4 must contain an involution i whose center is different from P and so must have order 4 because of (1).

Finally let us assume that $G(A, a)$ contains more than one involution. Then $G(A, a)$ contains an elementary abelian subgroup E of order 4, and $E \times \langle z \rangle$ is an elementary abelain subgroup of order 8, which is impossible. This completes our proof.

THEOREM 1. *Let* $\mathbf{E} = (\mathbf{P}, \mathbf{G})$ *be a projective plane of finite odd order* n, $P \in \mathbf{P}$ *a point*, $g \in \mathbf{G}$ *a line and* G *a* 2*-group of automorphisms of* $\mathbf{E}$ *such that*

(a) $(n - 1)/|G(P, g)| \equiv 1 \pmod 2$ *and*

(b) $G(P, g) \subseteq \mathbf{Z}G$.

Then $G/G(P, g)$ *is either cyclic or a dihedral group.*

Proof. Because of (a) and the assumption $n \equiv 1 \pmod 2$ we have $G(P, g) \neq 1$. This implies that $P \notin g$. Furthermore P and g must be invariant under G, since $G(P, g) \subseteq \mathbf{Z}G$ by (b).

Let $i \in G$ be an involution which is no perspectivity. Then by Baer [1] the fixed elements of i determine a subplane $\mathbf{F}$ of $\mathbf{E}$ of order m, where $n = m^2$. Since P and g are invariant under i, P is a point and g a line of $\mathbf{F}$. Let a be a line of $\mathbf{F}$ incident with P. We denote $Z = G(P, g)$. Then Z commutes with i by (b) and therefore leaves invariant $\mathbf{F}$. But this implies that Z leaves invariant the $m + 1$ points of $\mathbf{F}$ on a. Hence $|Z| \mid m - 1$ and

$$2 \mid m + 1 = (m^2 - 1)/(m - 1) = (n - 1)/(m - 1) \mid (n - 1)/|Z|$$

which is a contradiction to (a). So we have:

PROPOSITION 1.1. *P and g are left invariant by G and $P \notin g$. Every involution of G is a homology.*

Assume now that Z is not cyclic. Then the group Z, which is abelian by (b), must contain more than one involution. By Lemma 1 this implies that all involutions of G are contained in Z and that the number of these involutions is larger than 1. Also $|G| \mid 2(n-1)$, and by (b) we get $[G:Z] \leq 2$. We have proved:

PROPOSITION 1.2. *If $G(P, g)$ is not cyclic, then $G(P, g)$ contains all involutions of G and $[G:G(P, g)] \mid 2$. Also $G(P, g)$ contains more than one involution.*

Assume now that G leaves invariant a point $A_1 \in g$. Since n is odd, G must then have a second fixed point $A_2 \neq A_1$ on g. We denote $a_i = A_iP$ for $i = 1, 2$. Since $Z \subseteq G$, each orbit of G in $a_i - \{A_i, P\}$ must have at least length 2^r, if $2^r = |Z|$. But on the other hand $2^r \parallel n-1$ by (b). So there must exist orbits $\mathbf{O}_i$ of G in $a_i - \{A_1, P\}$, which have precisely length 2^r. Let $P_i \in \mathbf{O}_i$ and $G_i = G_{P_i}$. Then

$$[G:G_i] = 2^r = |Z|,$$

and since obviously $Z \cap G_i = 1$, this implies $G = G_iZ$. Furthermore $Z \subseteq \mathbf{Z}G$, so $G_i \lhd G$ and therefore

$$G = Z \times G_1 = Z \times G_2 \tag{2}$$

Now by Proposition 1.1 every involution of G_i is a homology with axis a_i. Since $a_1 \neq a_2$ this implies

$$|G_i| \mid n-1, \qquad G_1 \cap G_2 = 1$$

and

$$G_1 \cong G_1/G_1 \cap G_2 \cong G_1G_2/G_2 \subseteq G/G_2 = G_2Z/G_2 \cong Z/Z \cap G_2 \cong Z.$$

If Z is not cyclic we have $G = Z$ by Proposition 1.2 and (2). If Z is cyclic, by the above equation G_1 is cyclic too, and so

$$G = Z \times G_1$$

is abelian. We have—for the case considered here—actually proved more than we claimed in the theorem:

PROPOSITION 1.3. *If G (as in Theorem 1) leaves invariant a point $A \in g$, then there exists a point $X \in AP - \{A, P\}$ such that*

$$G = G(P, g) \times G_X.$$

Furthermore G_X is cyclic and $|G_X| \mid n-1$.

If $n \equiv 3 \pmod 4$ then by [5, Satz 1] G is a cyclic group, a quaternion group, a dihedral group or a quasidihedral group. Since $Z \neq 1$ this implies in any case that G/Z is either cyclic or dihedral.

The proof of Theorem 1 now is completed by the following:

PROPOSITION 1.4. *If G (as in Theorem 1) does not leave invariant any point on g and $n \equiv 1 \pmod 4$ then one of the following statements holds:*

(a) $G = \langle z, a, b\rangle$, *where* $z^{2^r} = a^{2^s} = b^2 = 1$, $r \geq s$, $az = za$, $bz = zb$ *and*

$bab = a^{-1}z^{2^{r-s}}$. *Here* $\langle z\rangle = G(P, g)$ *and* $\langle a\rangle = G_X$ *for some point* $X \in \mathbf{P}$ *which is different from* P *and not incident with* g.

(b) $G = \langle a, b\rangle$, *where* $b^{2^{r+1}} = a^{2^s} = 1, r \geq s$ *and* $b^{-1}ab = a^{-1}b^{2^{r-s+1}}$. *Here* $\langle b^2\rangle = G(P, g)$ *and* $\langle a\rangle = G_X$ *for some point* $X \in \mathbf{P}$ *which is different from* P *and not incident with* g. *Also* $\Omega_1(G) = \langle a^{2^{s-1}}\rangle \times \langle b^{2^r}\rangle$, *if* $s > 0$.

(c) $G(P, g)$ *is not cyclic*, $\Omega_1(G) \subseteq G(P, g)$, $[G:G(P, g)] = 2$ *and* $|\Omega_1(G)| > 2$.

Proof. Since $n \equiv 1 \pmod 4$ we have $r > 1$ and $2 \parallel n + 1$. Hence the number of subsets of cardinality 2 of g is odd, and so G must leave invariant one of these subsets, say $\{A, B\}$. If we denote $H = G_{A,B}$, we have

$$Z \lhd H \lhd G \quad \text{and} \quad [G:H] = 2,$$

since G has no fixed point on g. Also by Proposition 1.3

$$H = Z \times H_X, \tag{3}$$

where X is a suitable point of $AP - \{A, P\}$, and H_X is cyclic, say $H_X = \langle a\rangle$.

If Z is not cyclic we must have case (c) by Proposition 1.2. Assume Z is cyclic. We denote $Z = \langle z\rangle$, $o(a) = 2^s$ and choose an arbitrary element $b \in G - H$. Then b interchanges A and B and therefore maps X on some point X^b on $BP - \{B, P\}$. Since every element of $\langle a\rangle \cap \langle a^b\rangle$ leaves invariant A, B, X and X^b, we have

$$\langle a\rangle \cap \langle a^b\rangle = 1 \tag{4}$$

by Proposition 1.1. So we get $2^{2s} = |\langle a\rangle|\,|\langle a^b\rangle| \leq |H| = 2^{r+s}$ and $s \leq r$. If $s = 0$ then $H = Z$, which implies that G is abelian, and we have one of the cases (a) and (b). So we can assume $s \geq 1$. Then (3) together with (4) implies that the involution in $\langle a^b\rangle$ is the product of the involutions of the cyclic groups $\langle z\rangle$ and $\langle a\rangle$. If we denote $a^b = a^t z^u$, we get from (3),

$$a^{2^{s-1}}z^{2^{r-1}} = (a^b)^{2^{s-1}} = a^{t2^{s-1}}z^{u2^{s-1}} \quad \text{and} \quad a^{2^{s-1}-t2^{s-1}} = z^{u2^{s-1}-2^{r-1}} = 1.$$

Hence $(u2^{s-1} - 2^{r-1})2^{-r}$ is an integer. Since

$$u = \left(2\,\frac{u2^{s-1} - 2^{r-1}}{2^r} + 1\right)2^{r-s}$$

and $z^{(2(u2^{s-1}-2^{r-1})2^{-r}+1)}$ is also a generator of Z we may assume $u = 2^{r-s}$. Now $b^2 \in H$, and H is abelian. So we have

$$a = a^{b^2} = a^{t^2}z^{t2^{r-s}}z^{2^{r-s}}, \qquad a^{1-t^2} = z^{(2^{r-s})(t+1)},$$

$$2^r \mid 2^{r-s}(t+1) \qquad \text{and} \quad 2^s \mid t+1.$$

Therefore $a^t = a^{-1}$ and $a^b = a^{-1}z^{2^{r-s}}$. Denote $b^2 = a^v z^w$. Then

$$a^v z^w = b^2 = (b^2)^b = (a^v z^w)^b = a^{-v}z^{v2^{r-s}}z^w,$$

$$a^{2v} = z^{v2^{r-s}} = 1 \quad \text{and} \quad 2^r \mid v2^{r-s}.$$

But this implies $2^s \mid v$ and $b^2 = z^w$. If now w is even, then we can find an involution in $G - H$, for example $bz^{-w/2}$. So in this case we may assume $b^2 = 1$ and have case (a). If w is odd we have $G = \langle b, a \rangle$ and case (b). (If we don't have case (a), then all involutions of G must lie in H. So $\Omega_1(G) = \Omega_1(H) = \langle a^{2^{s-1}} \rangle \times \langle b^{2^r} \rangle$.)

THEOREM 2. *Let* **A** *be a translation plane with kernel* **K** *and finite order* $n = |\mathbf{K}|^m$ *such that* $n \equiv m \equiv 1 \pmod 2$. *If G is a group of automorphisms*[2] *of* **A** *and C a composition factor of G, then either* $|C| = 2$, *or* $|C| \equiv 1 \pmod 2$, *or G and C have the following three properties:*

(a) *If S is a Sylow-2-subgroup of C, then S is a dihedral group and* $|S| \mid \frac{1}{2}(n^2 - 1)$.

(b) *G does not leave invariant any point on the line at infinity of* **A**.

(c) *C is the only composition factor of G whose Sylow-2-subgroups are not cyclic.*

Proof. Let T be the group of all translations and P an affine point of **A**. Then we have

$$G/G \cap T \cong GT/T = (GT)_P T/T \cong (GT)_P/(GT)_P \cap T \cong (GT)_P .$$

Since $G \cap T$ has odd order and we are looking for composition factors of even order, we may assume without loss of generality that G leaves invariant the point P. Let $|\mathbf{K}| = q$ and let H be the group of all automorphisms of **A**. Then by André [3, p. 132] the group $H(P, g_\infty)$ of all homologies with center P and axis g_∞ is a cyclic group of order $q - 1$. Since obviously $H(P, g_\infty) \lhd H_P$, the product $GH(P, g_\infty)$ is a group. Also we have

$$GH(P, g_\infty)/H(P, g_\infty) \cong G/G \cap H(P, g_\infty),$$

and since any composition factor of the cyclic group $G \cap H(P, g_\infty)$ is of prime order, we can assume $H(P, g_\infty) \subseteq G$. But then $G/\mathbf{C}H(P, g_\infty)$ is abelian, since $H(P, g_\infty)$ is cyclic. Therefore we may even require

$$H(P, g_\infty) \subseteq ZG. \tag{5}$$

Now let U be a Sylow-2-subgroup of G. Then

$$U(P, g_\infty) = U \cap H(P, g_\infty)$$

is a Sylow-2-subgroup of the normal subgroup $H(P, g_\infty)$. Therefore $|U(P, g_\infty)| = 2^r$ if $2^r \| q - 1$. Since m is odd we have $q^{m-1} + q^{m-2} + \cdots + q + 1 \equiv 1 \pmod 2$ and because of the equation

$$n - 1 = q^m - 1 = (q - 1)(q^{m-1} + q^{m-2} + \cdots + q + 1)$$

this implies $2^r \| n - 1$. This together with (5) allows us to apply Theorem 1. It follows that $U/U(P, g_\infty)$ is either cyclic or a dihedral group.

[2] As in [3, p. 131] we define her translation planes to be affine planes. So even if **A** is desarguesian, G is a group which leaves invariant the line at infinity of **A**.

Since $G(P, g_\infty)$ is cyclic, we are only interested in the factor group $G/G(P, g_\infty)$. This group contains the Sylow-2-subgroup

$$UG(P, g_\infty)/G(P, g_\infty) \cong U/U \cap G(P, g_\infty) = U/U(P, g_\infty),$$

which is, as we have just seen, either cyclic or a dihedral group.

If the Sylow-2-subgroups of C are cyclic, then $|C| = 2$ or $|C| \equiv 1 \pmod 2$ by the Theorem of Burnside [6, p. 419, Satz 2.6]. Assume now that the Sylow-2-subgroups of C are not cyclic. Then $U/U(P, g_\infty)$ must be a dihedral group and hence we have the first part of (a). Furthermore (b) follows from Proposition 1.3, and (c) we get at once from the fact that every normal series of a dihedral group contains at most one noncyclic element.

We still have to prove the second part of (a). To do this, we assume in addition that G is a group of smallest possible order among all subgroups of H which have a composition factor isomorphic to C. Then G does not contain any normal subgroup of index 2. If $n \equiv 3 \pmod 4$, this implies by [5, Satz 2] that U is either cyclic or a quaternion group. But then $U(P, g_\infty)$ contains the only involution of U, and U operates regularly on the $n^2 - 1$ affine points $\neq P$. Hence $|U| \mid n^2 - 1$ and $U/U(P, g_\infty)$ is a dihedral group of order dividing $\frac{1}{2}(n^2 - 1)$. This implies (a).

Let $n \equiv 1 \pmod 4$ and assume $|U/U(P, g_\infty)| = 2^{s+1}$. Then by Proposition 1.3 and Proposition 1.4, U contains an element a of order 2^s leaving invariant a point $X \in \mathbf{P} - g_\infty \cup \{P\}$. By Proposition 1.1 the involution in $\langle a \rangle$ is a homology with axis XP and some center $A \in g_\infty - XP \cap g_\infty$. So $\langle a \rangle$ operates regularly on $g_\infty - \{XP \cap g_\infty, A\}$. Hence $2^s \mid n - 1$, and a has $2 + (n - 1)/2^s$ cycles on g_∞. But the group of permutations of g_∞ induced by G can contain only even permutations, because G does not contain any normal subgroup of index 2. In particular a must induce an even permutation on g_∞. This implies that $2^{s+1} \mid n - 1$ and completes (a).

By Feit and Thompson [4], every group of finite odd order is solvable. Therefore the group G in Theorem 2 must be solvable if it leaves invariant a point on g_∞. Hence the stabilizer $H_{P,A,B}$ of the full automorphism group H of $\mathbf{A}$ on an affine point P and two points A and B on g_∞ is always solvable, and we get the following:

COROLLARY 1 (Burmester and Hughes [2]). *If Q is a quasifield of finite odd order and odd dimension over its kernel, then the autotopism group of Q is solvable.*

THEOREM 3. *Let G be a 2-group which contains no elementary abelian group of order 8, and Q a subgroup of G isomorphic to a (generalized) quaternion group. Then one of the following statements holds for the centralizer $C = \mathbf{C}_G Q$ of Q in G:*

(I) *C does not contain any elementary abelian normal subgroup of order 4.*

(II) *C is a dihedral group of order 8.*

(III) $C = \mathbf{Z}Q \times A$, *where* A *is cyclic.*

(IV) $C/\mathbf{Z}Q$ *is a (generalized) quaternion group,* $|\Omega_1(C)| = 4$ *and* $\Omega_1(C) \subseteq \mathbf{Z}C$.

Proof. We define $Z = \mathbf{Z}Q$ and $H = QC$. Then obviously $|Z| = 2$ and $Z = Q \cap C$. Assume that C contains a normal subgroup $T \cong Z_2 \times Z_2$. Then $Q \subseteq \mathbf{C}T \subseteq \mathbf{N}T$ and $C \subseteq \mathbf{N}T$, so $H = QC \subseteq \mathbf{N}T$, i.e. $T \lhd H$. Now obviously $Z = \mathbf{Z}Q \subseteq \mathbf{Z}H$, and therefore $Z \subseteq T$, since G contains no elementary abelian subgroup of order 8. If $\bar{H} = \mathbf{C}_H T$ and $\bar{C} = \mathbf{C}_C T$, we have

$$\bar{H} = Q\bar{C} \quad \text{and} \quad [H:\bar{H}] = [C:\bar{C}] \mid 2.$$

Again because of our assumption that G contains no elementary abelian subgroup of order 8 and because of the fact $T \subseteq \mathbf{Z}\bar{H}$ we have

$$\Omega_1(\bar{H}) = \Omega_1(\bar{C}) = T. \tag{6}$$

Let $Z = \langle z \rangle$. We prove that z *does not have any root in* $\bar{C}$: Assume there exists an element $c \in \bar{C}$ such that $c^2 = z$. Take $q \in Q$ such that $q^2 = z$. Then $(qc)^2 = q^2c^2 = zz = 1$. So $qc \in \Omega_1(\bar{H}) = T \subseteq \bar{C}$ and $q \in Q \cap C = Z$, which is a contradiction.

But this implies that $\bar{C}/Z$ contains at most one involution. Indeed, if X/Z is a subgroup of order 2 of $\bar{C}/Z$, then X must be elementary abelian, since z has no root in $\bar{C}$. Hence $X = \Omega_1(\bar{C}) = T$. So $\bar{C}/Z$ *is either cyclic or a quaternion group.*

Let N/Z be a largest cyclic subgroup of $\bar{C}/Z$ which is normal in C/Z. Then $N = \bar{C}$ if $\bar{C}/Z$ is cyclic, and $[C:N] = 2$ if $\bar{C}/Z$ is a quaternion group. This is also true if $\bar{C}/Z \cong Q_8$, because in this case the number of cyclic subgroups of order 4 of $\bar{C}/Z$ is odd. Let aZ be a generator of N/Z. Since z has no root, $Z \subseteq \langle a \rangle$ would then imply $|\langle a \rangle| = 2$, $|N| = 2$ and $|\bar{C}| \le 4$. But if $|\bar{C}| \le 4$, the factorgroup $\bar{C}/Z$ is cyclic, and hence $N = \bar{C}$ and $|\bar{C}| = 2$, which contradicts the fact $T \subseteq \bar{C}$. So $Z \cap \langle a \rangle = 1$, which implies

$$N = Z \times \langle a \rangle \quad \text{and} \quad \mho_1(N) = \langle a^2 \rangle.$$

Assume $\bar{C} \neq C$. Then the involution in $\langle a \rangle$ (obviously $|\langle a \rangle| \neq 1$) is not invariant in C. This together with the facts $N \lhd C$ and $\mho_1(N) = \langle a^2 \rangle$ implies now $\mho_1(N) = 1$, i.e. $N = T$. But then $|N/Z| = 2$, so $\bar{C}/Z$ cannot be a quaternion group. Hence $N = \bar{C} = T$, and C must be a dihedral group of order 8.

Assume $\bar{C} = C$. If C/Z is cyclic, we have $C = N = Z \times \langle a \rangle$, i.e. case (III). If C/Z is not cyclic, then C/Z is a quaternion group. Also obviously $T \subseteq \mathbf{Z}\bar{C} = \mathbf{Z}C$ and $\Omega_1(C) = T$ by (6).

THEOREM 4. *Let* $\mathbf{E} = (\mathbf{P}, \mathbf{G})$ *be a projective plane of finite odd order* n, $g \in \mathbf{G}$ *a line,* $P \in \mathbf{P} - g$ *a point not incident with* g *and* G *a group of automorphisms of* $\mathbf{E}$ *leaving invariant* P *and* g. *Furthermore let* H *be a Sylow-2-subgroup of the homology group* $G(P, g)$ *and* S *a Sylow-2-group of some composition factor of* $G/G(P, g)$.

(a) *If* $(n-1)/|H| \equiv 1 \pmod 2$ *and* $H \subseteq \mathbf{Z}G$, *then* S *is cyclic or a dihedral group.*

(b) *If* $(n-1)/|H| \equiv 1 \pmod 2$ *and every involution in* G *is a perspectivity, then* S *is cyclic, a centralized quaternion group or a dihedral group.*

(c) *If* H *is a quaternion group and every involution in* G *is a perspectivity, then* S *is cyclic, a centralized quaternion group or a dihedral group.*

Proof. If $n \equiv 3 \pmod 4$, then $|H| \mid 2$, and H is certainly cyclic. The theorem then follows immediately from [5, Satz 1]. So we may assume $n \equiv 1 \pmod 4$.

We consider the case $(n-1)/|H| \equiv 1 \pmod 2$: Let U be a Sylow-2-group of G containing H. Since $n \equiv 1 \pmod 4$, the number of subsets of g of cardinality 2 is odd. So U leaves invariant one of these sets, say $\{A, B\}$. Then $[U:U_{A,B}] \mid 2$. Assume now that all involutions of U lie in $U(P, g)$. Then $U_{A,B}$ operates regularly on $AP - \{A, P\}$, hence $|U_{A,B}| \mid n-1$, and our assumption $n - 1/|H| \equiv 1 \pmod 2$ implies $U_{A,B} = H$, $4 \nmid |G/G(P, g)|$ and $|S| \leq 2$.

So we can assume that U contains a homology with center on g and axis through P (in case (a) this follows from Proposition 1.1). But then by Lemma 1, H contains at most one involution. Hence in all remaining cases H is either cyclic or a quaternion group.

Since $G(P, g) \lhd G$, we have by the Frattini argument

$$G = \mathbf{N}H \cdot G(P, g)$$

and

$$G/G(P, g) = \mathbf{N}H \cdot G(P, g)/G(P, g) \cong \mathbf{N}H/\mathbf{N}H \cap G(P, g) = \mathbf{N}H/\mathbf{N}H(P, g).$$

Therefore we may assume without loss of generality that $H \lhd G$.

Now since H is cyclic or a quaternion group, the group of outer automorphisms of H is either a 2-group or isomorphic to the symmetric group S_3 (if $H \cong Q_8$). Therefore the composition factors of $\mathbf{N}H/\mathbf{C}H$ are all cyclic. Furthermore we get the Sylow-2-structure of $\mathbf{C}H$ with the help of Theorem 1 and Theorem 3. Indeed, if H is cyclic, then (a) and (b) follow from Theorem 1. Suppose H is a quaternion group, and let C be a Sylow-2-subgroup of $\mathbf{C}H$ and $K = HC$. Then $K(P, g) = H$, and therefore $|\Omega_1(K(P, g))| = 2$. This already implies by Lemma 1 that K does not contain any elementary abelian subgroup of order 8. Hence we can apply Theorem 3. Since $n \equiv 1 \pmod 4$, there exists a subset $\{A, B\}$ of g, which is left invariant by K and has cardinality 2. Then $K_{A,B} \lhd K$ and $[K:K_{A,B}] \mid 2$. If $K_{A,B}$ contains only one involution, then $K_{A,B}$ is a quaternion group containing H. But this implies $|C \cap K_{A,B}| = 2$ and $|C| \leq 4$. If $K_{A,B}$ contains more than one involution, then by Lemma 1, $\Omega_1(K_{A,B})$ is an elementary abelian group of order 4 consisting of the identity and three involutorial homologies. Now it is easy to see that $\Omega_1(K_{A,B}) \subseteq C$, and since obviously $\Omega_1(K_{A,B}) \lhd C$, we may assume that C contains an elementary abelian normal subgroup of order 4. Hence one of the statements (II)–(IV) in Theorem 3 holds. Obviously each of them implies the required properties of $\mathbf{C}H$.

4. The case $n \equiv 5 \pmod 8$

Let W_{32} be the group of order 32 with generators a, b, c and relations

$$a^4 = b^4 = c^2 = 1, \qquad ab = ba,\ cac = b.$$

Then we have:

LEMMA 2. (a) *The maximal subgroups of W_{32} are*

$$S = \langle a \rangle \times \langle b \rangle, \qquad T = \langle ca, a^2 \rangle \quad \text{and} \quad U = \langle ab, a^2, c \rangle.$$

If $p = ca$ and $q = a^2$, then $T = \langle p, q \rangle$, where $p^8 = 1$, $q^2 = 1$ and $qpq = p^5$. Hence $T \cong M_{16}$.

If $p = ab$, $q = a^2$ and $r = c$, then $U = \langle p, q, r \rangle$, where $p^4 = q^2 = r^2 = 1$, $rqr = qp^2$, $qpq = p$ and $rpr = p$. Among the maximal subgroups of U are

$$\langle ca^2, a^2 \rangle \cong D_8 \quad \text{and} \quad \langle ca^2, cab \rangle \cong Q_8 .$$

Proof. Since $(ca)(ca) = ba$, we have $ba \in \Phi(W_{32})$ and

$$\langle a^2 \rangle \times \langle ab \rangle \subseteq \Phi(W_{32}).$$

On the other hand obviously $G/\langle a^2 \rangle \times \langle ab \rangle$ is elementary abelian. Hence

$$\Phi(W_{32}) = \langle a^2 \rangle \times \langle ab \rangle.$$

Therefore W_{32} has 3 maximal subgroups. The remaining details are easily verified.

LEMMA 3. *Let G be a group of order* 8 *which contains exactly* 3 *involutions. Then $G \cong Z_2 \times Z_4$.*

The proof is trivial.

THEOREM 5. *Let* $\mathbf{E} = (\mathbf{P}, \mathbf{G})$ *be a projective plane of finite order n, where* $n \equiv 5 \pmod 8$, *and let G be a* 2-*group of automorphisms of* $\mathbf{E}$. *Then G is isomorphic to a subgroup of W_{32}.*

Proof. Obviously n is odd. If n were a square, we would have $n \equiv 1 \pmod 8$. Hence n is a non-square, and by Baer [1] we have:

PROPOSITION 5.1. *Every involution in G is a homology.*

If G contains an elementary abelian subgroup E of order 8, then by Lemma 1, E must lie in some homology group $G(P, g)$, which implies $n \equiv 1 \pmod 8$. So we get:

PROPOSITION 5.2. *G does not contain any elementary abelian subgroup of order* 8.

Let $G \neq 1$ and let z be an involution belonging to the center of G. Then by Proposition 5.1, z is a homology with some axis g and some center $P \in \mathbf{P} - g$. Here P and g must be invariant under G, since all elements of G commute with z. Now $n \equiv 5 \pmod 8$ implies $n + 1 \equiv 2 \pmod 4$, and therefore there exists a subset $\{A, B\} \subseteq g$ such that $A \neq B$ and $\{A, B\}^G = \{A, B\}$. Let $\bar{G} = G_{A,B}$.
Then

$$\bar{G} \lhd G \quad \text{and} \quad [G:\bar{G}] \mid 2.$$

Since $n \not\equiv 1 \pmod 8$, $\bar{G}$ has an orbit $\mathbf{O}$ of length $|\mathbf{O}| \leq 4$ in $g - \{A, B\}$. Let $Q \in \mathbf{O}$. Then $|\bar{G}_Q| \mid 4$, since every involution of $\bar{G}_Q$ must be a perspectivity with center P and axis g, which implies that $\bar{G}_Q$ acts regularly on $AP - \{A, P\}$. Now we get $[\bar{G}:\bar{G}_Q] = |\mathbf{O}| \mid 4$ and the following:

PROPOSITION 5.3. $|\bar{G}| \mid 16$, $|G| \mid 32$.

As our next step we show:

PROPOSITION 5.4. *G does not contain any subgroup isomorphic to Z_{16} or Q_{16}, and $\bar{G}$ does not contain any subgroup isomorphic to Z_8 or Q_8.*

Proof. Assume that $\bar{G}$ contains a subgroup X of order 8 which is isomorphic to Z_8 or Q_8. Then X contains exactly one involution x, and this involution is a homology by Proposition 5.1. Since x leaves invariant A, B and P, one of these points must be its center. Let us assume without loss of generality that this is A. Then $x \in G(A, BP)$, and X operates regularly on $AB - \{A, B\}$. Hence we get $|X| \mid n - 1$ and the contradiction $|X| \mid 4$.

If G contains a subgroup Y of order 16 isomorphic to Z_{16} or Q_{16}, then in any case $\bar{G} \cap Y$ contains a subgroup isomorphic to Z_8 or Q_8, which is impossible by the above argument.

PROPOSITION 5.5. *If $|\bar{G}| \geq 8$, then $\bar{G}(A, BP)$, $\bar{G}(B, AP)$ and $\bar{G}(P, g)$ each contain exactly one involution.* $\Omega_1(\bar{G}) \subseteq \mathbf{Z}\bar{G}$ *and* $|\Omega 1(\bar{G})| = 4$.

Proof. If $\Omega_1(\bar{G}) \subseteq \bar{G}(P, g)$, then $\bar{G}$ acts regularly on $AP - \{A, P\}$, and hence $|\bar{G}| \mid 4$. So we may assume that there exists an involution $q \in \bar{G} - \bar{G}(P, g)$. Let $q \in \bar{G}(A, BP)$. Then since $z \in \mathbf{Z}\bar{G}$, qz is again an involution and by Proposition 5.1, a homology. Because qz has no fixed points on $g - \{A, B\}$ and $BP - \{P, B\}$, we must have $qz \in \bar{G}(B, AP)$. Now by Lemma 1, $\bar{G}(P, g)$, $\bar{G}(A, BP)$ and $\bar{G}(B, AP)$ each contain exactly one involution, which then of course is central, since these three groups are normal in $\bar{G}$. Since

$$\Omega_1(\bar{G}) \subseteq \bar{G}(P, g) \cup \bar{G}(A, BP) \cup \bar{G}(B, AP)$$

by Proposition 5.1, we finally get $|\Omega_1(\bar{G})| = 4$.

PROPOSITION 5.6. *Assume $|\bar{G}| = 8$. Then $\bar{G} \cong Z_2 \times Z_4$ and one of the* *following statements holds*:

(a) $G = \bar{G}$.

(b) $G \cong M_{16}$.

(c) $G = \langle p, q, r\rangle$, *where* $p^4 = q^2 = r^2 = 1, rqr = qp^2, qpq = p$ *and* $rpr = p$. *Here* $p^2 = z$ *and* p *has at least* 6 *fixed points on* g.

Proof. The first claim follows at once from Proposition 5.5 and Lemma 3. Assume $G \neq \bar{G}$, i.e., $|G| = 16$. Then by Proposition 5.5, $\bar{G}$ contains besides the involution z two further involutions $q \in \bar{G}(A, BP)$ and $qz \in \bar{G}(B, AP)$, and each element in $G - \bar{G}$ interchanges q and qz. So obviously G is not abelian. We treat two separate cases:

(I) Assume G contains an element x of order 8. Then by Proposition 5.4 we have

$$|\langle x\rangle \cap \bar{G}| = 4.$$

Hence the involution q centralizes a cyclic subgroup of index 2 in $\langle x\rangle$; q does not centralize x since x interchanges q and qz. So G is isomorphic to M_{16} (see [6, Satz 14.9, p. 90]).

(II) Assume G does not contain any element of order 8, and let r be any element of $G - \bar{G}$. Then either $r^2 = 1$, or r^2 is an involution in $\bar{G}$. But in the second case $r^2 = z$, since r does not commute with q or qz. This implies that $G^2 \subseteq \langle z\rangle$, i.e., $G/\langle z\rangle$ is elementary abelian. Hence the conjugacy class r^G is contained in the residue class $\langle z\rangle r$, and $|\mathbf{C}r| \geq 8$. Now $\mathbf{C}r \cap \bar{G}$ is a group of order ≥ 4 which has trivial intersection with $\langle q\rangle$. So $\mathbf{C}r \cap \bar{G}$ is a cyclic group of order 4, say $\langle p\rangle$, and $|\mathbf{C}r| = 8$. Furthermore $\mathbf{C}r = \langle r\rangle\langle p\rangle$ is obviously abelian, but by our assumption not cyclic. Hence there exists an involution in $\mathbf{C}r - \langle p\rangle$, and we may assume that r itself is an involution. Since r commutes with p^2, we have $p^2 = z$. Also $G - \bar{G}$ contains four invoutions, which by Proposition 5.1 are homologies whose centers lie on g. One easily sees that these centers are pairwise different and different from A and B. Hence we have six homology centers on g, and these are all invariant under p, since $p \in \mathbf{Z}G$. This completes our proof.

PROPOSITION 5.7. *If* $|\bar{G}| = 16$, *then* $\bar{G} \cong Z_4 \times Z_4$ *and either* $G = \bar{G}$ *or* $|G| = 32$ *and* $G = \langle a, b, c\rangle$, *where*

$$a^4 = 1, \quad b^4 = 1, \quad c^2 = 1, \quad ab = ba \quad \text{and} \quad cac = b.$$

Proof. As we have seen in the proof of Proposition 5.3, there exists a point $Q \in g - \{A, B\}$ such that $|\bar{G}_Q| \mid 4$ and $[\bar{G}:\bar{G}_Q] \mid 4$. If $|\bar{G}| = 16$, this implies $|\bar{G}_Q| = 4$. In the same way we get points $R \in AP - \{A, P\}$ and $S \in BP - \{B, P\}$ such that $|\bar{G}_R| = |\bar{G}_S| = 4$. By Proposition 5.5, these groups each contain exactly one involution, and different groups contain different involutions. So $\bar{G}_Q, \bar{G}_R, \bar{G}_S$ are cyclic and have pairwise intersection 1. Denote $\bar{G}_R = \langle a\rangle$ and $\bar{G}_S = \langle b\rangle$. Then $\bar{G} = \langle a\rangle\langle b\rangle$, and $\bar{G}_Q = \langle a^n b^m\rangle$ for suitable n, m. Since A and B are invariant under $\bar{G}$, the involutions a^2

and b^2 lie in the center $\mathbf{Z}\bar{G}$. Therefore n and m must be odd; otherwise we have $(a^n b^m)^2 = a^2$ or $(a^n b^m)^2 = b^2$, which is impossible, since the groups $\bar{G}_Q$, $\bar{G}_r$ and $\bar{G}_S$ have pairwise trivial intersection. So we may assume $n = m = 1$. But then $(ab)^2$ must be the third involution a^2b^2, and we have

$$a^2b^2 = (ab)^2 = abab$$

and $ab = ba$, i.e., $\bar{G}$ is abelian.

Assume now that $G \neq \bar{G}$ and $c \in G - \bar{G}$. Then c maps A onto B and Q onto some point on $PB - \{P, B\}$. Therefore we may assume that $Q^c = R$, and even that $c^{-1}ac = b$. Then

$$c^{-1}bc = c^{-1}(c^{-1}ac)c = a$$

since c^2 belongs to the abelian group $\bar{G}$. Denote $c^2 = a^n b^m$. Then c commutes with $a^n b^m$, and

$$a^n b^m = c^{-1} a^n b^m c = b^n a^m,$$

so that

$$a^{n-m} = b^{n-m}.$$

Since $\langle a \rangle \cap \langle b \rangle = 1$, we get $b^n = b^m$ and

$$(ca^{-n})^2 = c^2c^{-1}a^{-n}ca^{-n} = c^2b^{-n}a^{-n} = 1.$$

Hence in any case there exists some involution in $G - \bar{G}$, and we can assume $c^2 = 1$.

Using Lemma 2 we can prove now that G always is isomorphic to a subgroup of W_{32}: If $|G| \leq 4$ this is trivial. If $|G| = 8$ it follows from Proposition 5.2, since all groups of order 8 which are not elementary abelian occur as subgroups of W_{32}. Now by Proposition 5.3, there remain only the cases $|G| = 16$ and $|G| = 32$. But in these cases the theorem follows from Proposition 5.6 or Proposition 5.7.

If $\mathbf{E}$ is desarguesian, then the Sylow-2-subgroups of the full automorphism group of $\mathbf{E}$ have order 32 and therefore are isomorphic to W_{32}. Hence by Theorem 5, every 2-group which operates as a group of automorphisms on a projective plane of order $n \equiv 5 \pmod 8$, is isomorphic to a subgroup of the automorphism group of the desarguesian plane of the same order (if such a plane exists).

THEOREM 6. *Let* $\mathbf{A}$ *be an affine plane of finite order* n, *where* $n \equiv 5 \pmod 8$ *and* G *a group of automorphisms of* $\mathbf{A}$, *which does not contain any normal subgroup of index* 2. *Furthermore let* D *be the normal subgroup of* G *consisting of all elements which leave invariant all points on the line at infinity. Then the orders of the Sylow-2-subgroups of* D *and* G/D *are at most* 4.

Proof. Let S be a Sylow-2-subgroup of a counter-example G. Then S leaves invariant an affine point P and a subset $\{A, B\}$ of cardinality 2 of g_∞, since the number n^2 of affine points of $\mathbf{A}$ is odd and the number $n + 1$ of points on g_∞ is not divisible by 4. Let $\bar{\tilde{S}} = S_{A,B}$. Then again

$$\bar{\tilde{S}} \lhd S \quad \text{and} \quad [S:\bar{\tilde{S}}] \leq 2.$$

Since G is a counterexample, we have $\bar{\tilde{S}} \neq 1$, $\bar{\tilde{S}} \cap \mathbf{Z}S \neq 1$, and hence there exists an involution $z \in \bar{\tilde{S}} \cap \mathbf{Z}S$.

Assume now that $\bar{\tilde{S}}$ acts regularly on $g_\infty - \{A, B\}$. Then $|\bar{\tilde{S}}| \mid n - 1$ and therefore $|\bar{\tilde{S}}| \mid 4$. Also the center of z must then be one of the points A and B. So A and B are invariant under S, i.e. $S = \bar{\tilde{S}}$ and $|S| \leq 4$, contradicting our assumption that G is a counter-example. Hence there exists a point $X \in g_\infty - \{A, B\}$ such that $\bar{\tilde{S}}_X \neq 1$. Then $\bar{\tilde{S}}_X$ contains an involution, which must be a homology with center P and axis g_∞. So $S(P, g_\infty) \neq 1$. Since $S(P, g_\infty) \lhd S$, we have $S(P, g_\infty) \cap \mathbf{Z}S \neq 1$, and we can assume $z \in S(P, g_\infty) \cap \mathbf{Z}S$. This allows us to apply Propositions 5.1–5.7.

Now $S \cap D = S(P, g_\infty)$ is a Sylow-2-subgroup of D, and

$$\tilde{S} = SD/D \cong S/S \cap D = S/S(P, g_\infty)$$

is a Sylow-2-subgroup of $\tilde{G} = G/D$. Since obviously $|S(P, g_\infty)| \mid 4$, we must have

$$|S/S(P, g_\infty)| \geq 8$$

and $|S| \geq 16$. If $|\bar{\tilde{S}}| = 16$, then as we have seen in the proof of Proposition 5.7, $\bar{\tilde{S}}$ contains an element of order 4 whose square has on g_∞ only two fixed points, namely A and B. But because $8 \nmid n - 1$, this element is an odd permutation and G must contain a normal subgroup of index 2, contradicting our assumptions. So by Proposition 5.3, $|S| = 16$, $|\bar{\tilde{S}}| = 8$ and $S(P, g_\infty) = \langle z \rangle$. By Proposition 5.6, we have now only two possibilities for S, and $\tilde{S}$ *must be either isomorphic to* $Z_2 \times Z_4$ *or to an elementary abelian group of order* 8. If $\tilde{S} = Z_2 \times Z_4$, then the automorphism group of $\tilde{S}$ is a 2-group, hence $\mathbf{N}_{\tilde{G}}\tilde{S} = \mathbf{C}_{\tilde{G}}\tilde{S}$, and by a theorem of Burnside (see [6, p. 419, Satz 2.6]) $\tilde{G}$ has a normal 2-complement. So $\tilde{S}$ is an elementary abelian group of order 8. But then $\mathbf{A}\tilde{S} = \mathbf{N}_{\tilde{G}}\tilde{S}/\mathbf{C}_{\tilde{G}}\tilde{S}$ is isomorphic to a subgroup of the group $PSL(3, 2)$ of order $168 = 8 \cdot 3 \cdot 7$, and since by Proposition 5.5 and Proposition 5.6, $\tilde{S}$ contains involutions with two and six fixed points on g_∞, this subgroup cannot act transitively on $\tilde{S}$. Hence $7 \nmid |\mathbf{A}\tilde{S}|$, which implies that $|\mathbf{A}\tilde{S}| \mid 3$ and that some subgroup E of order 4 of $\tilde{S}$ must be normal in $\mathbf{N}_{\tilde{G}}\tilde{S}$. Furthermore by the Theorem of Zassenhaus (see [6, p. 126, Satz 18.1]) $\mathbf{N}_{\tilde{G}}\tilde{S}$ has a 2-complement C. Now the subgroup EC has index 2 in $\mathbf{N}_{\tilde{G}}\tilde{S}$ and hence is normal. Using the First Theorem of Grün (see [6, p. 423, Satz 3.4]) we get that G contains a normal subgroup of index 2, which is a contradiction.

REFERENCES

1. R. BAER, *Projectivities with fixed points on every line of the plane*, Bull. Amer. Math. Soc., vol. 52 (1946), pp. 273–286.
2. M. V. D. BURMESTER AND D. R. HUGHES, *On the solvability of autotopism groups*, Arch. Math., vol. 16 (1965), pp. 178–183.

3. P. DEMBOWSKI, *Finite geometries*, Springer, New York, 1968.
4. W. FEIT AND J. G. THOMPSON, *Solvability of groups of odd order*, Pacific J. Math., vol. 13 (1963), pp. 775–1029.
5. C. HERING, *Eine Bemerkung über Automorphismengruppen von endlichen projektiven Ebenen und Möbiusebenen*, Arch. Math., vol. 18 (1967), pp. 107–110.
6. HUPPERT, B., *Endliche Gruppen I*, Springer, New York, 1967.

UNIVERSITY OF ILLINOIS AT CHICAGO CIRCLE
CHICAGO, ILLINOIS