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A BEURLING-RUDIN THEOREM FOR H^{∞}

BY

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0. Introduction

Let H^{∞} be the Banach algebra of bounded analytic functions on the open unit disc $\mathbf{D} = \{z \in \mathbf{C}\colon |z| < 1\}$, supplied with the uniform norm. It is well known that we may regard H^{∞} as a closed subalgebra of L^{∞}, the uniform algebra of (equivalence classes of) essentially bounded Lebesgue measurable functions on the unit circle $\mathbf{T} = \partial\mathbf{D}$. Consider the linear space

$$H^{\infty} + C = \{f + g\colon f \in H^{\infty}, g \in C = C(\mathbf{T})\};$$

Donald Sarason [8, p. 377] has shown that this is a closed subalgebra of L^{∞}. In connection with $H^{\infty} + C$, Sarason introduced QC, the closed subalgebra of $H^{\infty} + C$ consisting of all functions whose complex conjugates also lie in $H^{\infty} + C$, and its analytic subalgebra $QA = QC \cap H^{\infty}$. Expressed differently, QC is the largest C^* algebra contained in $H^{\infty} + C$.

In the second section of this paper, we give a complete description of the closed ideals in QA. This result is hardly surprising, for Arne Beurling's (unpublished) and Walter Rudin's [15] independently obtained description of the closed ideals in the disc algebra $A = C \cap H^{\infty}$ and several results of Thomas Wolff [20], [21] suggest that this is possible. In fact, Srinivasan and Wang's proof [16] of Beurling's and Rudin's result can be extended to QA. We use this result in the third section of this paper to obtain some rather surprising results about closed ideals of H^{∞}. We will show that an arbitrary closed nonzero ideal in H^{∞} has the form $u(J \cap H^{\infty})$, where u is an inner function, and J is a closed ideal in $H^{\infty} + C$; this is what we mean by a Beurling-Rudin theorem for H^{∞}. Moreover, we shall see that the quotient algebras $H^{\infty}/J \cap H^{\infty}$ and $(H^{\infty} + C)/J$ are canonically isomorphic. And in $H^{\infty} + C$, as opposed to H^{∞}, a theorem of Šilov [9, §45] and a later refinement of it due to Errett Bishop and Irving Glicksberg [7, p. 61] give us quite a lot of information about the closed ideals. The first result in this direction was obtained by Håkan Hedenmalm [11], and later generalized by Raymond

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Mortini [14]. For a closed H^∞-ideal I whose hull $h(I)$ is contained in the Šilov boundary $\mathscr{M}(L^\infty)$ (for these concepts, see Section 1 below), we can say a great deal more: there exists a closed L^∞-ideal whose intersection with H^∞ equals I, and since a theorem of Šilov [9, §36] tells us that all closed ideals in $L^\infty = C(\mathscr{M}(L^\infty))$ are intersections of maximal ones, the same can be said about I. We will also show that

$$H^\infty/I \cong \hat{H}^\infty|_{h(I)} = C(h(I)).$$

Most of the results that appear in this paper extend easily to $H^\infty(\Omega)$ on finitely connected domains with the techniques of [11].

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1. Basic concepts

The bilinear form linking any Banach space A with its dual space A^* will always be denoted by $\langle \cdot, \cdot \rangle$.

All Banach algebras are assumed complex, commutative, and unital. For a Banach algebra B, we denote by $\mathscr{M}(B)$ its maximal ideal space; the elements of $\mathscr{M}(B)$ are the nonzero complex homomorphisms on B. With the Gelfand topology, $\mathscr{M}(B)$ is a compact Hausdorff space. The Gelfand transform, always denoted by $\hat{}$, defines a continuous homomorphism $B \to C(\mathscr{M}(B))$. The algebra B is said to be semisimple if the Gelfand transform is injective. A uniform algebra on a compact Hausdorff space X is a closed unital subalgebra of $C(X)$. A uniform algebra is a semisimple Banach algebra whose image under the Gelfand transform is a uniform algebra on the maximal ideal space. A C^* algebra is a uniform algebra which is closed under complex conjugation. The Stone-Weierstrass theorem allows us to conclude that a C^* algebra B is isomorphic to $C(\mathscr{M}(B))$. Let B be a uniform algebra on a compact Hausdorff space X. A closed subset E of X is a peak set if there is a function $p \in B$ such that $p|_E = 1$ and $|p| < 1$ on $X \setminus E$; we call p a peaking function. A weak peak set is an intersection of peak sets. An interpolation set E is a closed subset of X such that $B|_E = C(E)$. Unless X is specified, X is tacitly assumed to be $\mathscr{M}(B)$.

For an ideal I in a Banach algebra B, its hull is the closed set

$$h(I, B) = \{m \in \mathscr{M}(B) \colon m(x) = 0 \text{ for all } x \in I\}.$$

If I is closed, it is well known [7, p. 12] that we may identify $\mathscr{M}(B/I)$ with $h(I, B)$. Whenever possible, we will write $h(I)$ for $h(I, B)$.

A corollary of Beurling's famous invariant subspace theorem [8, p. 85] states that every weak $*$ closed ideal in H^∞ other than $\{0\}$ has the form uH^∞, where u is an inner function. The weak $*$ topology on H^∞ is the one inherited from $L^\infty = (L^1(\mathbf{T}))^*$. A weak $*$ closed subspace of H^∞ is an ideal if (and only if) it is invariant under multiplication by the coordinate function z; hence the weak $*$ closure in H^∞ of an ideal in H^∞ or QA is an H^∞-ideal. So, given an ideal I in H^∞ or QA other than $\{0\}$, we find an inner function u such that the weak $*$ closure of I equals uH^∞. We shall call the function u, which is unique except for a (unimodular) constant factor, the *inner factor* of I. It could also be described as the greatest common divisor of the inner factors of the functions in I [8, pp. 83–84].

Using the fact that $C \subset QC \subset H^\infty + C$, one can show that $QC = QA + C$. It follows from this that $\mathcal{M}(QA) = \mathcal{M}(QC) \cup \mathbf{D}$ [21]. It is standard to identify $\mathcal{M}(H^\infty + C)$ with $\mathcal{M}(H^\infty) \setminus \mathbf{D}$ and $\mathcal{M}(L^\infty)$ with the Šilov boundary of $\mathcal{M}(H^\infty)$. Let

$$\Gamma\colon \mathcal{M}(L^\infty) \to \mathcal{M}(QC) \quad \text{and} \quad \gamma\colon \mathcal{M}(H^\infty + C) \to \mathcal{M}(QC)$$

be the respective restriction mappings. Because QC is a C^* algebra, a theorem of Šilov [9, §12] tells us that Γ and γ are surjective. Let $D_m = \gamma^{-1}(\{m\})$ and $E_m = \Gamma^{-1}(\{m\})$ denote the *QC level sets* corresponding to the point $m \in \mathcal{M}(QC)$. The restriction of H^∞ to E_m (or D_m) is a uniform algebra with maximal ideal space D_m and Šilov boundary E_m.

2. Closed ideals in QA

Let ϕ be an element of the dual space of $QC = C(\mathcal{M}(QC))$. By the Riesz Representation Theorem, ϕ has the form

$$\langle f, \phi \rangle = \int_{\mathcal{M}(QC)} \hat{f}\, d\mu, \quad f \in QC,$$

where μ is a regular Borel measure on $\mathcal{M}(QC)$. In particular, the continuous linear functional ϕ_0 defined by

$$\langle f, \phi_0 \rangle = \int_{\mathbf{T}} f\, d\lambda, \quad f \in QC,$$

where λ is normalized Lebesgue measure on $\mathbf{T}$, defines a measure σ on $\mathcal{M}(QC)$ such that

$$\int_{\mathcal{M}(QC)} \hat{f}\, d\sigma = \int_{\mathbf{T}} f\, d\lambda, \quad f \in QC,$$

which we will call lifted Lebesgue measure. Lemmas 2.1 and 2.2 below are due to Thomas Wolff [20].

LEMMA 2.1. *The Gelfand transform on QC has a unique extension to an isometric isomorphism $L^1(\mathbf{T}, \lambda) \to L^1(\mathcal{M}(QC), \sigma)$ (still denoted by $\hat{\ }$) such that*

$$\int_{\mathbf{T}} fg\, d\lambda = \int_{\mathcal{M}(QC)} \hat{f}\hat{g}\, d\sigma, \quad f \in L^1(\mathbf{T}, \lambda),\ g \in QC.$$

The following is the QA analogue of the classical F & M Riesz theorem.

LEMMA 2.2. *If $\mu \in QC^*$ annihilates QA, then μ is absolutely continuous with respect to σ, that is, $d\mu = f\, d\sigma$ for some $f \in L^1(\mathcal{M}(QC), \sigma)$.*

The easiest way to see that the above statement is true is to realize that the quotient spaces C/A and $QC/QA = (QA + C)/QA$ are canonically isomorphic, and therefore must have the same dual space $H_0^1 = \{f \in H^1\colon f(0) = 0\} \subset L^1(\mathbf{T}, \lambda)$.

LEMMA 2.3. *Let $f \in QA$ have the factorization $f = ug$, where u is an inner function and $g \in H^\infty$. Then $g \in QA$, and if $m(f) = 0$ for some $m \in \mathcal{M}(QC)$, $m(g) = 0$.*

Proof. Since we are assuming $g \in H^\infty$, and $\bar{g} = u\bar{f} \in H^\infty + C$ because $\bar{f} \in QC$, we see that $g \in QA$. To prove the remaining part of the assertion, find $\phi \in \mathcal{M}(L^\infty)$ such that $\phi|_{QC} = m$; this is possible because the restriction mapping $\Gamma\colon \mathcal{M}(L^\infty) \to \mathcal{M}(QC)$ is onto. Since u is inner, $|\phi(u)| = 1$, and hence $|\phi(g)| = |\phi(u)\phi(g)| = |\phi(ug)| = |\phi(f)| = |m(f)| = 0$. Since $g \in QA$, $m(g) = 0$.

Remark 2.4. Lemma 2.3 states that QA has the f-property in the sense of Havin, answering one of the two questions raised by Milne Anderson in [3]. The fact that QA has this property has been shown previously by Pamela Gorkin [10].

In [15] Walter Rudin described all closed ideals in the disc algebra. Srinivasan and Wang [16] later gave a proof that relied less on function theory. It is this proof that we shall modify to prove a similar result for QA. For a closed set $E \subset \mathcal{M}(QC)$, introduce the notation

$$I(E, QA) = \left\{f \in QA\colon \hat{f}|_E = 0\right\}.$$

Clearly, $I(E, QA)$ is a closed ideal in QA. Thomas Wolff [20] has shown that a closed subset of $\mathcal{M}(QC)$ is contained in the zero set of a nonidentically vanishing QA function if and only if it has lifted Lebesgue measure 0. Hence

$I(E, QA) \neq \{0\}$ if and only if $\sigma(E) = 0$. Assume for the moment that $\sigma(E) = 0$. Clearly, the inner factor of $I(E, QA)$ must be 1, in view of Lemma 2.3. If u is an inner function such that the set of all $m \in \mathcal{M}(QC)$ for which $\hat{u}|_{E_m}$ is nonconstant is contained in E, $uI(E, QA)$ will be contained in QA, because the functions in $uI(E, QA)$ are constant on all QC level sets; here we use a theorem of Šilov [9, §44]. Because u is inner, this is a closed ideal in QA. Our theorem below states that all (nontrivial) closed ideals in QA arise in this fashion.

THEOREM 2.5. *Let I be a closed QA-ideal other than $\{0\}$. Then there exist an inner function u and a closed subset E of $\mathcal{M}(QC)$ with $\sigma(E) = 0$ such that $I = uI(E, QA)$.*

Proof. Let u be the inner factor of I (see Section 1), and let

$$J = \{ f \in QA \colon uf \in I \}.$$

First observe that because I is a closed QA-ideal, J is, too. In view of Lemma 2.3, we are done if we can show that $J = I(E, QA)$, where $E = \{m \in \mathcal{M}(QC)\colon m(f) = 0 \text{ for all } f \in I\}$. Clearly, J is contained in $I(E, QA)$, again by Lemma 2.3. To show the reverse inclusion, let μ be a regular Borel measure on $\mathcal{M}(QC)$ annihilating J. Since J is an ideal, for any $f \in J$ and $g \in QA$,

$$\int_{\mathcal{M}(QC)} \hat{g}\hat{f}\, d\mu = 0.$$

Thus $\hat{f}\, d\mu$ annihilates QA. By Lemma 2.2, $\hat{f}\, d\mu$ is absolutely continuous with respect to σ. Writing $\mu = \mu_a + \mu_s$ where μ_a is absolutely continuous with respect to σ and μ_s is singular, we see that $\hat{f}\, d\mu_s = 0$. Thus supp $\mu_s \subset \{m \in \mathcal{M}(QC)\colon m(f) = 0\}$ for any $f \in J$ and hence $\operatorname{supp} \mu_s \subset E$. Now it is clear that μ_s annihilates $I(E, QA)$. Since J is contained in $I(E, QA)$, both μ_s and μ annihilate J, so μ_a does, too. Since the inner factors of J and $I(E, QA)$ must be 1, we see that J and $I(E, QA)$ are both weak $*$ dense in H^∞. Recall that the weak $*$ topology of H^∞ is the weak $*$ topology of $L^\infty = (L^1(\mathbf{T}))^*$ restricted to H^∞. Since μ_a is absolutely continuous with respect to σ, Lemma 2.1 tells us that μ_a belongs to the predual of L^∞, and hence the fact that $\mu_a \perp J$ implies $\mu_a \perp I(E, QA)$. Now both μ_a and μ_s annihilate $I(E, QA)$, so μ does, too. Thus $J^\perp \subset I(E, QA)^\perp$, so $I(E, QA) \subset J$, as desired.

In this context, the QA analogue of the Rudin-Carleson theorem, which is due to Thomas Wolff [20], [21], takes the following form. An epimorphism is a surjective homomorphism.

THEOREM 2.6. *Let $E \subset \mathcal{M}(QC)$ be a closed set of lifted Lebesgue measure* 0. *Then there exists a unique continuous epimorphism*

$$QC \to QA/I(E, QA)$$

which is canonical on QA; its kernel is the closed QC-ideal

$$I(E, QC) = \{f \in QC\colon f|_E = 0\}.$$

Proof. Wolff's theorem [20], [21] states that $QA|_E = C(E)$. Because QA is a logmodular algebra on its Šilov boundary $\mathcal{M}(QC)$, E is a weak QA peak set [17, p. 216]. Hence we may deduce that

$$h(I(E, QA), QA) = E.$$

It is standard to identify $QA/I(E, QA)$ with $QA|_E$ [17, p. 117]. So, by letting L be the restriction mapping

$$QC \to QC|_E = C(E) \cong QA/I(E, QA),$$

we obtain a continous epimorphism that is canonical on QA, and clearly, its kernel is $I(E, QC)$. Observe that as a consequence, $QC = QA + I(E, QC)$.

It remains to be shown that L is unique. For the algebra $QC = C(\mathcal{M}(QC))$, a theorem of Šilov [9, §36] specializes to show that the closure of the QC-ideal generated by $I(E, QA)$ equals $I(E, QC)$, because $h(I(E, QA), QA) = E$. Hence any other continuous epimorphism $L'\colon QC \to QA/I(E, QA)$ which is canonical on QA must vanish on $I(E, QC)$, but since $QC = QA + I(E, QC)$, L' must coincide with L, which is our desired conclusion.

These are some of the many ways in which QA acts like the disc algebra. But there are some differences. Rudin [15] showed that every closed ideal in the disc algebra is the closure of a principal ideal. This is not true in QA. Take for example the maximal ideal $\{f \in QA\colon m(f) = 0\}$ for $m \in \mathcal{M}(QC)$; indeed, using the corona theorem for QA and Sundberg's and Wolff's description of QA interpolating sequences [19], it is not hard to check that any QA function vanishing at m must vanish on a set homeomorphic to $\beta\mathbf{N} \setminus \mathbf{N}$, where $\beta\mathbf{N}$ denotes the Stone-Čech compactification of the nonnegative integers $\mathbf{N}$. However, for countably generated closed ideals, we have a result analogous to that for the disc algebra [6, p. 73]. Namely, every countably generated closed ideal in QA is the principal ideal generated by a finite Blaschke product. This follows from a result of Dietrich [6, p. 72], after the observation that $\mathcal{M}(QC)$ is connected.

3. Closed ideals in H^∞

A Douglas algebra is a closed subalgebra of L^∞ containing H^∞. For certain Douglas algebras B and closed H^∞-ideals I we construct a continuous epimorphism (i.e., surjective homomorphism) $L_{I,B}\colon B \to H^\infty/I$ which is canonical on H^∞. Whenever possible we write L_I for $L_{I,B}$. This epimorphism will enable us to study closed ideals in H^∞ by replacing them with closed ideals in B.

The following lemma, which is probably known, will prove useful.

LEMMA 3.1. *For a uniform algebra B on a compact Hausdorff space X, let E be a peak set with peaking function p. Then the closure of the B-ideal generated by $1-p$ equals $\{f \in B\colon f|_E = 0\}$.*

Proof. Clearly, $\{f \in B\colon f|_E = 0\}$ is a closed ideal in B, and it contains the function $1-p$. Let $f \in B$ vanish on E. It is easy to check that

$$(1-p^n)f \to f \quad \text{as } n \to \infty,$$

and since $1 - p^n = (1-p)(1+p+\cdots+p^{n-1})$, the assertion follows.

The proposition below is what makes everything in this section work. It uses Wolff's remarkable result [21] that every L^∞ function can be multiplied into QC with an outer QA function.

For a closed set $E \subset \mathscr{M}(QC)$, $I(E, H^\infty)$ denotes the closed ideal

$$\left\{f \in H^\infty\colon \hat{f}|_{\gamma^{-1}(E)} = 0\right\}.$$

PROPOSITION 3.2. *Let $I \neq \{0\}$ be a closed ideal in H^∞ with inner factor 1. Then I contains $I(E, H^\infty)$ for some QA peak set $E \subset \mathscr{M}(QC)$ (which has lifted Lebesgue measure 0, of course).*

Proof. Thomas Wolff [21] has shown that to a given function $f \in L^\infty$, we can find an outer function $q \in QA$ such that $qf \in QC$. In particular, this result can be applied to the functions in I, showing that $I \cap QA \neq \{0\}$. Since q was outer, we can say even more, namely that the closed QA-ideal $I \cap QA$ has inner factor 1. By our characterization of the closed ideals in QA (Theorem 2.5), we find a closed set $E_0 \subset \mathscr{M}(QC)$ with $\sigma(E_0) = 0$ such that $I \cap QA = I(E_0, QA)$. We already noticed in the proof of Theorem 2.6 that E_0 is a weak QA peak set, meaning that it is an intersection of peak sets. So, we can find a QA peak set $E \subset \mathscr{M}(QC)$ containing E_0, and such sets have $\sigma(E) = 0$ according to Wolff [20]. Let p be a QA function that peaks on E. Then p is also a peaking function for the set

$$\gamma^{-1}(E) \subset \mathscr{M}(H^\infty + C) = \mathscr{M}(H^\infty) \setminus \mathbf{D}$$

in the algebra H^∞, and Lemma 3.1 tells us that the closure of the H^∞-ideal generated by $1-p$ coincides with $I(E, H^\infty)$. Since

$$1 - p \in I(E, QA) \subset I(E_0, QA) = I \cap QA,$$

the assertion follows.

THEOREM 3.3. *Let $I \neq \{0\}$ be a closed ideal in H^∞ with inner factor 1. Then there exists a unique continuous epimorphism $H^\infty + C \to H^\infty/I$ which is canonical when restricted to H^∞.*

Proof. Let us do the uniqueness part first. Observe that any such epimorphism $L: H^\infty + C \to H^\infty/I$ must have $L(z^{-n}f) = (z + I)^{-n}(f + I)$ for $f \in H^\infty$ and $n \geq 0$, and since such functions span a dense subspace of $H^\infty + C$, continuity shows that L must be unique whenever it exists.

By Proposition 3.2, we can find a QA peak set E in $\mathscr{M}(QC)$ such that $I \supset I(E, H^\infty)$. In the proof of Proposition 3.2, we mentioned that $\gamma^{-1}(E)$ is a peak set for H^∞, so by [17, p. 117], $H^\infty|_{\gamma^{-1}(E)}$, which is isomorphic to $H^\infty/I(E, H^\infty)$, is a closed subalgebra of $C(\gamma^{-1}(E))$. Our epimorphism will be the composition of the following maps:

$$H^\infty + C \to H^\infty + C|_{\gamma^{-1}(E)} = H^\infty|_{\gamma^{-1}(E)} \cong H^\infty/I(E, H^\infty) \to H^\infty/I.$$

The first map is the restriction of the Gelfand transform. The equality sign holds because of Wolff's interpolation theorem $QA|_E = C(E)$ [20], [21]. The last map is well defined because $I \supset I(E, H^\infty)$.

We now arrive at our main result.

THEOREM 3.4. *Let $I \neq \{0\}$ be a closed ideal in H^∞ with inner factor u. Then $I = u(J \cap H^\infty)$, where J is a closed ideal in $H^\infty + C$. Also, the quotient algebras $H^\infty/J \cap H^\infty$ and $(H^\infty + C)/J$ are canonically isomorphic.*

Proof. Clearly, $I_0 = \{f \in H^\infty : uf \in I\}$ is a closed H^∞-ideal because I is, and $I = uI_0$. By construction, I_0 has inner factor 1. Theorem 3.3 gives us a continuous epimorphism

$$L_{I_0}: H^\infty + C \to H^\infty/I_0$$

which is canonical on H^∞, so putting $J = \ker L_{I_0}$, we obtain a closed $(H^\infty + C)$-ideal whose intersection with H^∞ is I_0. The map L_{I_0} induces a topological isomorphism $(H^\infty + C)/J \to H^\infty/I_0$, which is the inverse of the canonical homomorphism $H^\infty/I_0 \to (H^\infty + C)/J$ because L_{I_0} is canonical on H^∞.

An antisymmetric set for $H^\infty + C$ is a set $S \subset \mathscr{M}(L^\infty)$ such that whenever $f \in H^\infty + C$ and $f|_S$ is real valued, then $f|_S$ is constant. Bishop's antisymmetric decomposition theorem for ideals, which is due to Glicksberg, tells us that Theorem 3.4 has the following corollary [7, p. 61], [17, p. 115].

COROLLARY 3.5. *Let $I \neq \{0\}$ be a closed H^∞-ideal with inner factor* 1. *Then an H^∞ function f is an element of I if (and only if) $f|_S \in I|_S$ for all maximal antisymmetric sets S for $H^\infty + C$.*

Since QC level sets are unions of maximal antisymmetric sets for $H^\infty + C$, Corollary 3.5 has the following consequence. Corollary 3.6 may also be deduced from Theorem 3.4 by applying Šilov's decomposition theorem for ideals [9, §45].

COROLLARY 3.6. *Let $I \neq \{0\}$ be a closed H^∞-ideal with inner factor* 1. *Then an H^∞ function f is an element of I if (and only if) $f|_{E_m} \in I|_{E_m}$ for all QC level sets E_m.*

COROLLARY 3.7. *If $I \neq \{0\}$ is a closed ideal in H^∞ with inner factor* 1, *then $I \cap QA = I(E, QA)$, where $E = \gamma(h(I, H^\infty))$.*

Proof. By Proposition 3.2, the inner factor of $I \cap QA$ is 1, and by our description of the closed ideals in QA, $I \cap QA = I(E, QA)$ for some closed set $E \subset \mathcal{M}(QC)$ of lifted Lebesgue measure 0, which clearly must contain $\gamma(h(I, H^\infty))$. On the other hand, if $f \in QA$ vanishes on $\gamma(h(I, H^\infty))$, then $f|_{E_m} \in I|_{E_m}$ trivially for all QC level sets E_m, since the maximal ideal space of $H^\infty|_{E_m}$ is $D_m = \gamma^{-1}(\{m\})$. Corollary 3.6 now tells us that $f \in I$. That is, $I \cap QA = I(\gamma(h(I, H^\infty)), QA)$, as asserted.

One may wonder whether the ideal J in the formulation of Theorem 3.4 is uniquely determined by I. This turns out to be the case, indeed, J is the closure of the $(H^\infty + C)$-ideal generated by $J \cap H^\infty$. Here is our precise statement.

THEOREM 3.8. *The mapping $J \mapsto J \cap H^\infty$ is one-to-one from the set of all closed $(H^\infty + C)$-ideals with $\sigma(\gamma(h(J, H^\infty + C))) = 0$ onto the set of all closed H^∞-ideals with inner factor* 1. *Also, if $\sigma(\gamma(h(J, H^\infty + C))) > 0$, $J \cap H^\infty = \{0\}$.*

Proof. If $\sigma(\gamma(h(J, H^\infty + C))) > 0$, J cannot contain any nonidentically vanishing QA function [20], and hence $J \cap H^\infty$ must equal $\{0\}$, by Wolff's generalized Fatou theorem [21].

Let J be a closed $(H^\infty + C)$-ideal such that the lifted Lebesgue measure of $E = \gamma(h(J, H^\infty + C))$ is zero. We shall show that J contains the ideal

$$I(E, H^\infty + C) = \left\{ f \in H^\infty + C \colon f|_{\gamma^{-1}(E)} = 0 \right\}.$$

To this end, observe that if $m \in \mathcal{M}(QC) \setminus E$, then $J|_{D_m} = J|_{\gamma^{-1}(\{m\})}$ is not contained in any maximal ideal of $\mathcal{M}(H^\infty + C|_{D_m}) = D_m$, so $J|_{D_m} = H^\infty + C|_{D_m}$. An application of Šilov's decomposition theorem for ideals [9, §45] yields $J \supset I(E, H^\infty + C)$, as desired. Using this we have $J \supset I(E, QA)$. Since we already know that $I(E, QA)$ has inner factor 1, we see that $J \cap H^\infty$ is a closed H^∞-ideal with inner factor 1. If $I \neq \{0\}$ is a closed ideal in H^∞ with inner factor 1, taking the kernel of the epimorphism of Theorem 3.3 provides us with a closed $(H^\infty + C)$-ideal J whose intersection with H^∞ equals I. By the above remark, $\sigma(\gamma(h(J, H^\infty + C))) = 0$. Hence the mapping $J \to J \cap H^\infty$

is onto, as asserted. What remains to be shown is that it is one-to-one. To this end, let $I \neq \{0\}$ be an arbitrary closed ideal in H^∞ with inner factor 1, and let J and J' be two closed ideals in $H^\infty + C$ such that $J \cap H = J' \cap H = I$. We wish to show that $J = J'$. According to what we have done so far, the lifted Lebesgue measure of the sets

$$E = \gamma(h(J, H^\infty + C)) \quad \text{and} \quad E' = \gamma(h(J', H^\infty + C))$$

must be zero. From our work above we know that $J \supset I(E, H^\infty + C)$ and $J' \supset I(E', H^\infty + C)$. By Wolff's interpolation result $QA|_E = C(E)$ [20], [21], we may conclude that $H^\infty + I(E, H^\infty + C) = H^\infty + C$; just take an arbitrary function $f = g + h \in H^\infty + C$, find a $q \in QA$ with $q|_E = h|_E$, and observe that

$$f = (g + q) + (h - q) \in H^\infty + I(E, H^\infty + C).$$

Hence $H^\infty + J = H^\infty + C$, and in the same fashion, $H^\infty + J' = H^\infty + C$. Elementary algebra now tells us that

$$H^\infty/I = H^\infty/J \cap H^\infty \cong (H^\infty + J)/J = (H^\infty + C)/J$$

and

$$H^\infty/I = H^\infty/J' \cap H^\infty \cong (H^\infty + J')/J' = (H^\infty + C)/J'$$

algebraically, and hence topologically, by the open mapping theorem. These isomorphisms induce two continuous epimorphisms $H^\infty + C \to H^\infty/I$ that are canonical on H^∞, with kernels J and J', respectively. Theorem 3.3 tells us that these two epimorphisms must be identical, and hence $J = J'$. The proof of the theorem is complete.

Whenever we can get a continuous epimorphism L_I: $L^\infty \to H^\infty/I$ which is canonical on H^∞, we can say much more about the closed H^∞-ideal I. If such a map exists, then $L^\infty/\ker L_I \cong H^\infty/I$. Thus we may identify the maximal ideal spaces of these algebras. But $\mathcal{M}(L^\infty/\ker L_I)$ may be identified with $h(\ker L_I, L^\infty)$, and $\mathcal{M}(H^\infty/I)$ may be identified with $h(I, H^\infty)$ [7, p. 12]. From this it follows that a necessary condition for the existence of such a map is that $h(I, H^\infty)$ be contained in $\mathcal{M}(L^\infty)$. Surprisingly enough, it is also sufficient. We obtain this result as a corollary of the following theorem, the proof of which is based on Sheldon Axler's neat result on factorization of L^∞ functions [4], and its proof, as in [18].

THEOREM 3.9. *Let I be an ideal in $H^\infty + C$ such that*

$$h(I, H^\infty + C) \subset \mathcal{M}(L^\infty).$$

Then there exists a unique epimorphism $\mathscr{L}_I$: $L^\infty \to (H^\infty + C)/I$ which is canonical on $H^\infty + C$. If I is closed, $\mathscr{L}_I$ is continuous.

Proof. By Axler's theorem, for each $f \in L^\infty$, there exists a Blaschke product b such that $bf \in H^\infty + C$. We define $\mathscr{L}_I$ as follows:

$$\mathscr{L}_I(f) = (b + I)^{-1}(bf + I).$$

Note that $(b + I)^{-1}$ exists since $h(I, H^\infty + C) \subset \mathscr{M}(L^\infty)$ and $|\hat{b}| = 1$ on $\mathscr{M}(L^\infty)$. We will show that the choice of the Blaschke product b does not affect the definition of $\mathscr{L}_I$. Suppose that we have $f \in L^\infty$ and Blaschke products b and c such that bf and cf are in $H^\infty + C$. Then

$$\begin{aligned}(b + I)^{-1}(bf + I) &= (b + I)^{-1}(c + I)^{-1}(bcf + I) \\ &= (b + I)^{-1}(c + I)^{-1}(b + I)(cf + I) \\ &= (c + I)^{-1}(cf + I).\end{aligned}$$

Thus $\mathscr{L}_I$ is well defined, and in the same way one checks that it is a homomorphism. That $\mathscr{L}_I$ is canonical on $H^\infty + C$ is obvious.

For the uniqueness, let L: $L^\infty \to (H^\infty + C)/I$ be an arbitrary epimorphism that is canonical on $H^\infty + C$. Then for $f \in L^\infty$ and a Blaschke product b with $bf \in H^\infty + C$,

$$(b + I)L(f) = L(b)L(f) = L(bf) = bf + I,$$

so $L = \mathscr{L}_I$.

From now on, we assume I is closed. The kernel of $\mathscr{L}_I$ is the ideal

$$J = \{f \in L^\infty \colon bf \in I \text{ for some Blaschke product } b\}.$$

From the proof of Axler's factorization theorem (see [18]) it follows that J is closed. The map $\mathscr{L}_I$ induces an algebraic isomorphism

$$L^\infty/J \to (H^\infty + C)/I$$

which is the inverse of the canonical homomorphism. Since the canonical homomorphism $(H^\infty + C)/I \to L^\infty/J$ is continuous, the open mapping theorem states that this isomorphism must be topological, and hence that $\mathscr{L}_I$ is continuous. The proof of the theorem is complete.

COROLLARY 3.10. *Let I be a closed ideal in H^∞ such that $h(I, H^\infty) \subset \mathscr{M}(L^\infty)$. Then there exists a unique continuous epimorphism $L^\infty \to H^\infty/I$ which is canonical on H^∞.*

Proof. An inner function has modulus 1 on the Šilov boundary $\mathscr{M}(L^\infty)$; therefore if it is not invertible in H^∞, then it must vanish somewhere else in $\mathscr{M}(H^\infty)\setminus\mathscr{M}(L^\infty)$. Consequently, the ideal I must have inner factor 1. Theorem 3.3 provides us with a continuous epimorphism

$$L_I\colon H^\infty + C \to H^\infty/I$$

which is canonical on H^∞, and if we let J denote its kernel, we obtain a closed ideal in $H^\infty + C$ with

$$h(J, H^\infty + C) = h(I, H^\infty) \subset \mathscr{M}(L^\infty).$$

Defining $L_{I, L^\infty} = \tilde{L}_I \circ \mathscr{L}_J$, where $\tilde{L}_I$ is the isomorphism $(H^\infty + C)/J \to H^\infty/I$ induced by L_I and $\mathscr{L}_J$ is as in Theorem 3.9, we get the desired epimorphism. For the uniqueness, observe that any such epimorphism must coincide with L_{I, L^∞} on quotients of inner functions, and since such functions span a dense subspace of L^∞ by the Douglas-Rudin theorem [8, pp. 192–195], it must by continuity coincide with L_{I, L^∞} everywhere. The proof is complete.

COROLLARY 3.11. *Let I be a closed ideal in H^∞ with $h(I, H^\infty) \subset \mathscr{M}(L^\infty)$. Then I is an intersection of maximal ideals, and $h(I, H^\infty)$ is a weak peak interpolation set for H^∞.*

Proof. Corollary 3.10 gives us an epimorphism $L_{I, L^\infty}\colon L^\infty \to H^\infty/I$, and since it is canonical on H^∞, the intersection of its kernel $J = \ker L_{I, L^\infty}$ and H^∞ equals I. Also, since H^∞/I and L^∞/J are canonically isomorphic, they have the same maximal ideal spaces $h(J, L^\infty) = h(I, H^\infty)$. By a theorem of Šilov [9, §36], J is an intersection of maximal ideals, and hence the same can be said about $J \cap H^\infty = I$. Thus the fact that H^∞/I and L^∞/J are canonically isomorphic can be restated as $H^\infty|_{h(I, H^\infty)} = L^\infty|_{h(I, H^\infty)}$. Now $L^\infty|_{h(I, H^\infty)} = C(h(I, H^\infty))$, making $h(I, H^\infty)$ an H^∞ interpolation set, and since H^∞ is logmodular on its Šilov boundary $\mathscr{M}(L^\infty)$, it follows that $h(I, H^\infty)$ is also a weak peak set [17, p. 216].

At this point, it is certainly reasonable to conjecture that if $I \neq \{0\}$ is a closed ideal in H^∞ with inner factor 1, such that $h(I, H^\infty) \subset \mathscr{M}(B)$ for a Douglas algebra B, then there exists a continuous epimorphism $L_{I, B}\colon B \to H^\infty/I$ which is canonical on H^∞. We shall give an example to show that this is not true in general. Before we do so, we introduce some new terminology and notation.

A sequence $\{z_n\}$ of points of $\mathbf{D}$ is said to be thin if

$$\lim_{n\to\infty} \prod_{k, k\neq n} \left|(z_n - z_k)\big/(1 - \bar{z}_k z_n)\right| = 1.$$

A Blaschke product associated to a thin sequence is called a thin Blaschke product. Any point $m \in \mathcal{M}(H^\infty) \setminus \mathbf{D}$ which is in the closure of a thin sequence is called a thin point. We let $\mathcal{F}$ denote the collection of all thin points in $\mathcal{M}(H^\infty)$. It is well known that $\mathcal{F}$ is a union of nontrivial Gleason parts (see for instance [12]). In [13], Kenneth Hoffman introduced for every $m \in \mathcal{M}(H^\infty)$ an analytic mapping L_m: $\mathbf{D} \to \mathcal{M}(H^\infty)$ varying continuously with m, the image of which is the Gleason part $\mathcal{P}(m)$ containing m. Hoffman showed among other things if $m \in \mathcal{F}$, then L_m is a homeomorphism. In fact, if b is a thin Blaschke product whose zero sequence captures m in its closure, then $\hat{b} \circ L_m(z) = \lambda z$, $z \in \mathbf{D}$, for some unimodular constant λ, which we can take to be 1 by a change of b. It is now also clear that $\hat{H}^\infty \circ L_m = H^\infty$, because if $f \in H^\infty$, then $f \circ b \in H^\infty$ is a function such that $f \circ \hat{b} \circ L_m = f$. We are now ready to give our example of a closed ideal I in H^∞ with inner factor 1 and a Douglas algebra B such that $h(I, H^\infty) \subset \mathcal{M}(B)$, but no continuous homomorphism of B onto H^∞/I that is canonical on H^∞ exists. Let k denote the singular inner function

$$k(z) = \exp\big((z+1)/(z-1)\big), \quad z \in \mathbf{D}.$$

Example 3.12. The ideal kH^∞ is closed in H^∞. Define the ideal I by

$$I = \left\{ f \in H^\infty \colon f \circ L_m \in kH^\infty \right\}$$

for some $m \in \mathcal{F}$; then I is a closed ideal in H^∞ with inner factor 1. Since $\hat{H}^\infty \circ L_m = H^\infty$, $\hat{I} \circ L_m = kH^\infty$. Let B be the smallest Douglas algebra containing the complex conjugates of all thin Blaschke products. In [12] it is shown that $\mathcal{M}(B) = \mathcal{M}(H^\infty) \setminus (\mathcal{F} \cup \mathbf{D})$. We first show that $h(I, H^\infty) \subset \mathcal{M}(B)$. Observe that I contains the closed ideal

$$J = \left\{ f \in H^\infty \colon \hat{f} \circ L_m \equiv 0 \right\},$$

so $h(I, H^\infty) \subset h(J, H^\infty)$, but since $\hat{I} \circ L_m = kH^\infty$,

$$h(I, H^\infty) \subset h(J, H^\infty) \setminus \mathcal{P}(m).$$

The formula

$$L_m(\phi)(f) = \phi\big(\hat{f} \circ L_m\big), \quad f \in H^\infty, \phi \in \mathcal{M}(H^\infty),$$

extends L_m to a continuous mapping $\mathcal{M}(H^\infty) \to \mathcal{M}(H^\infty)$. Our next step is to show that $h(J, H^\infty) = L_m(\mathcal{M}(H^\infty))$. To this end, let $\phi \in h(J, H^\infty)$. Since $\hat{H}^\infty \circ L_m = H^\infty$, the formula $\psi(\hat{f} \circ L_m) = \phi(f)$, $f \in H^\infty$, defines a nonzero complex homomorphism ψ on H^∞ such that $\phi = L_m(\psi)$, as desired. We will now show that every thin Blaschke product has modulus 1 on

$$h(J, H^\infty) \setminus \mathcal{P}(m) = L_m\big(\mathcal{M}(H^\infty) \setminus \mathbf{D}\big),$$

thereby ensuring that

$$h(I, H^\infty) \subset h(J, H^\infty) \setminus \mathcal{P}(m) \subset \mathcal{M}(B) \quad [8, \text{p. } 375].$$

By [12], for a thin Blaschke product b, $\hat{b} \circ L_m$ is identically a unimodular constant, or

$$\hat{b} \circ L_m(z) = \lambda \frac{z - \alpha}{1 - \bar{\alpha} z}, \quad z \in \mathbf{D},$$

for some $\alpha \in \mathbf{D}$, $\lambda \in \mathbf{T}$. In either case, $|\hat{b} \circ L_m| = 1$ on $\mathcal{M}(H^\infty) \setminus \mathbf{D}$.

Now suppose a continuous homomorphism L_I: $B \to H^\infty/I$ that is canonical on H^∞ exists. Let b be a thin Blaschke product such that $m(b) = 0$, normalized so that $\hat{b} \circ L_m(z) = z$, $z \in \mathbf{D}$. Then $b_\zeta = (b - \zeta)/(1 - \bar{\zeta} b)$ is thin for all $\zeta \in \mathbf{D}$ [12], and hence $\bar{b}_\zeta = b_\zeta^{-1} \in B$. Therefore

$$\left\| L_I\left(b_\zeta^{-1}\right) \right\| = \left\| \left(b_\zeta + I\right)^{-1} \right\| \leq \|L_I\| \cdot \|\bar{b}_\zeta\| = \|L_I\| = C$$

for all $\zeta \in \mathbf{D}$. Thus, there exists $h_\zeta \in H^\infty$ such that $\|h_\zeta + I\| \leq C$ and $h_\zeta b_\zeta - 1 \in I$. Without loss of generality, we may assume $\|h_\zeta\| \leq 2C$. Now

$$\begin{aligned}\left(\hat{h}_\zeta \hat{b}_\zeta\right) \circ L_m(z) - 1 &= \left(\hat{h}_\zeta \circ L_m(z)\right)(z - \zeta)/(1 - \bar{\zeta} z) - 1 \\ &= k(z) f_\zeta(z), \quad z \in \mathbf{D},\end{aligned}$$

for some $f_\zeta \in H^\infty$. But $\|\hat{h}_\zeta \circ L_m\| \leq 2C$, so $\|f_\zeta\| \leq 2C + 1$ for all $\zeta \in \mathbf{D}$. Plugging in $\zeta = z$, we see that the fact that $k(z) \to 0$ as $z \to 1$ along the real axis makes this impossible.

Remarks 3.13. (a) There is a wide variety of closed subalgebras of H^∞ containing QA for which the techniques of this section are applicable. For instance, the algebra $QA_B = \overline{B} \cap H^\infty$ for a Douglas algebra B is of this kind. More explicitly, the analogue of Theorem 3.4 states that every closed ideal of QA_B has the form $u(J \cap QA_B)$, where u is an inner function, and J is a closed ideal in the algebra $QA_B + C = \overline{B} \cap (H^\infty + C)$. This description is possible because an argument similar to that of Lemma 2.3 shows that QA_B has the f-property in the sense of Havin. However, the obvious analogue of Corollary 3.10, which would state that if I is a closed QA_B-ideal with inner factor 1 and $h(I, QA_B) \subset \mathcal{M}(Q_B)$, where $Q_B = B \cap \overline{B}$, there is a continuous epimorphism $Q_B \to QA_B/I$ that is canonical on QA_B, turns out not to be true in general. A counterexample can be produced along the lines of Example 3.12. Let COP be the closed subalgebra of H^∞ consisting of those functions that are constant on all Gleason parts other than $\mathbf{D}$. It is easy to check that

COP $+ C$ is a closed subalgebra of L^∞, knowing that $H^\infty + C$ is. In COP, every closed ideal with inner factor 1 has the form $J \cap$ COP, where J is a closed ideal in COP $+ C$. We cannot get the general statement for arbitrary closed ideals, because COP does not have the f-property [2].

(b) One may wonder if in Corollary 3.5 we may localize to the Gleason parts instead of the maximal antisymmetric sets for $H^\infty + C$. We shall see that this is not possible, and indeed, one may not even localize to the COP level sets, which are those sets in $\mathcal{M}(H^\infty)$ where COP functions are constant. Donald Sarason has observed that COP contains an infinite Blaschke product b (see [2]). Let J be the closed principal ideal $b^2(H^\infty + C)$, which is proper because b is not invertible in $H^\infty + C$, and put $I = J \cap H^\infty$, which has inner factor 1. Clearly, $\hat{b}|_S \in I|_S$ on every COP level set S, but since b isn't invertible in $H^\infty + C$, b cannot be in $J = b^2(H^\infty + C)$.

(c) An open problem of Norman Alling [1] asks for a complete description of all closed prime ideals in H^∞. It is conjectured that if P is a closed prime ideal in H^∞, other than $\{0\}$, then P is either maximal, or there exists an $m \in \mathcal{M}(H^\infty) \setminus \mathbf{D}$ with nontrivial Gleason part $\mathcal{P}(m)$ such that $P = \{f \in H^\infty: f|_{\mathcal{P}(m)} = 0\}$. Using Theorem 3.4 and certain facts about uniform algebras it is possible to show that the following is true. If P is a closed nonzero nonmaximal prime ideal in H^∞, then there exists a unique maximal antisymmetric set S for $H^\infty + C$ such that if $f|_S \in P|_S$, then $f \in P$ for any $f \in H^\infty$. A proof of this can be devised using Bishop's construction of maximal antisymmetric sets [5] together with Theorem 3.4 and the observation that if A_α is a uniform algebra on X_α and P_α is a closed prime ideal of A_α, then there is a unique $A_\alpha \cap \overline{A}_\alpha$ level set $F \subset X_\alpha$ such that $f(F) = 0$ implies $f \in P_\alpha$.

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