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A NEW APPROACH TO THE CONSTRUCTION OF COMPLETE MINIMAL SURFACES DERIVED FROM THE GENUS TWO CHEN-GACKSTATTER EXAMPLE

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1. Introduction

C. C. Chen and F. Gackstatter [C-G] discovered in the early 80's two minimal surfaces with one end that are directly related to Enneper's surface, except that they possess one and two handles, respectively. Furthermore, both surfaces have the same symmetry group as in Enneper's example.

The genus-one Chen-Gackstatter surface has been characterized by F. J. López [L1], and independently by Bloss [B], as the only complete minimal once punctured torus in $\mathbb{R}^3$ with total curvature -8π.

H. Karcher [K] and F. J. López [L2] have obtained two different generalizations of this surface by increasing the genus and the order of the symmetry group. This technique was used the first time by D. Hoffman and W. H. Meeks in [H-M] to construct a family of surfaces from Costa's example [C] and it allows new examples of high genus surfaces to be obtained without the period problem increasing in difficulty.

On the other hand, F. J. López, F. Martín and D. Rodríguez [L-M-R] have proved that the genus-two Chen-Gackstatter example is the unique, complete, orientable minimal surface of genus two in $\mathbb{R}^3$ with total curvature -12π and eight symmetries. This result has been obtained by studying a family of quite symmetric minimal surfaces derived from the genus-two Chen-Gackstatter surface and it is a corollary of a more general uniqueness theorem for this infinite family.

E. C. Thayer [T], using a similar technique, has discovered a family of complete minimal surfaces with arbitrary even genus following on from the genus-two Chen-Gackstatter example. In this paper we study these surfaces using a new approach to the period problem. Furthermore, we prove that this problem has only one solution (Section 3) and get a uniqueness theorem for these surfaces in terms of their genus, symmetry and total curvature (Section 4). As a consequence we have obtained the uniqueness result for the second Chen-Gackstatter example which we have mentioned above.

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2. Preliminaries

In this section we summarize some results about complete minimal surfaces of finite total curvature.

If we consider $x\colon M \longrightarrow \mathbb{R}^3$ a minimal immersion of a orientable surface M in three dimensional Euclidean space the total curvature of x is denoted by $\mathcal{C}(M)$. In a natural way, using isothermal parameters, M is a Riemann surface and we have labeled (g, η) the Weierstrass data of x. One should remember that the Gauss map g of x is a meromorphic function on M, and η is a holomorphic 1-form on M (see [O]).

Moreover, $x = \text{Real} \int (\phi_1, \phi_2, \phi_3)$ where

$$\phi_1 = \frac{1}{2}\eta(1-g^2), \ \phi_2 = \frac{i}{2}\eta(1+g^2), \ \phi_3 = \eta g \tag{1}$$

are holomorphic 1-forms on M satisfying

$$\sum_{j=1}^{3} | \phi_j |^2 \neq 0. \tag{2}$$

In particular the 1-forms ϕ_j, $j = 1, 2, 3$, have no real periods on M.

In what follows we have assumed that M is complete and $\mathcal{C}(M) > -\infty$. Under these hypotheses, A. Huber proved (see [H]) that M is conformally diffeomorphic to a compact Riemann surface $\overline{M}$ punctured in a finite number of points $\{P_1, \ldots, P_r\}$ and R. Osserman [O] showed that (g, η) extends meromorphically to $\overline{M}$. Therefore, g has well defined degree and $\mathcal{C}(M) = -4\pi \deg(g)$.

L. P. Jorge and W. H. Meeks [J-M] proved that the asymptotic behavior of x around an end P_i is determined by the number:

$$I_i = \text{Maximum}\{\text{ord}(\phi_j, P_i), \quad j = 1, 2, 3\} - 1$$

where $\text{ord}(\phi_j, P_i)$ is the pole order of ϕ_j at P_i. Moreover,

$$2\deg(g) = -\chi(\overline{M}) + \sum_{i=1}^{r}(I_i + 1). \tag{3}$$

We have assumed that M is not the covering of any minimal surface and we have written $\text{Iso}(M)$, the isometry group of M. The subgroup of $\text{Iso}(M)$ consisting of those isometries which are the restriction of a rigid motion in $\mathbb{R}^3$ leaving $x(M)$ invariant is denoted by $\text{Sym}(M)$. Calabi proved that $\text{Iso}(M) = \text{Sym}(M)$ if and only if there exists $j \in \{1, 2, 3\}$ such that ϕ_j is not exact. A complete discussion about this subject can be found in [H-M].

In what follows the order $2n$ dihedral group is denoted by $\mathcal{D}(n)$.

We will need the following topological remarks. Let $\overline{M}$ be a compact Riemann surface of genus $k > 0$. Given $c_1,\ c_2 \in H_1(M, \mathbb{Z})$, we let the intersection number of

c_1 and c_2 be denoted by $c_1 \cdot c_2$. Consider $\mathcal{B} = \{a_1, \ldots, a_k, b_1, \ldots, b_k\}$, a homology basis of $\overline{M}$. By definition, $\mathcal{B}$ is a canonical homology basis if and only if $a_i \cdot b_j = \delta_{i\,j}$ and $a_i \cdot a_j = b_i \cdot b_j = 0, \forall i, j = 1, \ldots k$ ($\delta_{i\,j}$ means Kronecker's delta). We must use this kind of basis in Classical Riemann Bilinear Relations. For the details see [F-K]. We conclude these preliminaries recalling the definitions of Euler beta and gamma functions. For $v \in \mathbb{N}$ and $z \in \mathbb{C} - \{-1, -2, \ldots\}$, the gamma function is given by

$$\Gamma(z) = \lim_{v \to +\infty} \frac{v! v^z}{z(z+1)(z+2) \cdots (z+v)}.$$

Among classical properties of gamma function, we emphasize the following:

$$\Gamma(z+1) = z\Gamma(z), \quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \quad 2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = \sqrt{\pi}\,\Gamma(2z).$$

For $m, n \in \mathbb{C}$, $\mathrm{Re}(m) > 0$, $\mathrm{Re}(n) > 0$, we define the beta function by

$$\mathfrak{B}\ (m, n) = \int_0^1 t^{m-1}(1-t)^{n-1} dt.$$

This is related to the gamma function according to

$$\mathfrak{B}\ (m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

A complete reference for these topics is, for instance, [Str].

3. The family of examples

This section is devoted to a careful study of a family of complete orientable minimal surfaces with one end derived from the Chen-Gackstatter genus two surface. These surfaces were constructed first by E. C. Thayer in [T], as we mentioned in Section 1. We obtain an analytic uniqueness theorem (Theorem 1) for these examples. For a more geometric uniqueness theorem see Section 4.

Let $\overline{M}_{k\,a}$, $k \in \mathbb{N}$, $k \geq 2$, $a \in \mathbb{R}_+ - \{1\}$ be the compact Riemann surface of genus $2(k-1)$:

$$\overline{M}_{k\,a} = \left\{ (z, w) \in (\mathbb{C} \cup \{\infty\})^2 \colon\ w^k = \frac{z(z^2 - a^2)}{z^2 - 1} \right\}.$$

Write $\infty = (\infty, \infty)$, $0 = (0, 0)$, $\pm 1 = (\pm 1, \infty)$, $\pm a = (\pm a, 0)$. The following conformal mappings are defined by

$$H, T \colon \overline{M}_{k\,a} \longrightarrow \overline{M}_{k\,a}$$

$$T(z, w) = (-z, e^{\frac{\pi i}{k}} w) \quad H(z, w) = (\overline{z}, \overline{w}).$$

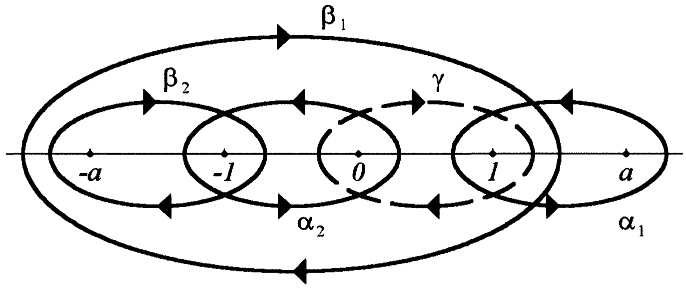

Figure 1. $a > 1$

Note that T has order $2k$ and H is an involution. They generate a group isomorphic to $\mathcal{D}(2k)$. Moreover, T^2 and H fix $0, \infty, \pm 1, \pm a$, and T fixes $0, \infty$.

To construct a canonical homology basis of $\overline{M}_{k\,a}$ we distinguish two cases:

- Suppose $a > 1$. Let $\alpha_i(s)$, $\beta_i(s)$, $i = 1, 2$, $\gamma(s)$ be the oriented simple closed curves in the z-plane illustrated in Figure 1. We assume that $\alpha_1(0) \in \mathbb{R}$, $\alpha_1(0) > a$, $\alpha_2(0) \in \mathbb{R}$, $1 > \alpha_2(0) > 0$, $\beta_1(0) \in \mathbb{R}$, $a > \beta_1(0) > 1$, $\beta_2(0) \in \mathbb{R}$, $0 > \beta_2(0) > -1$, $\gamma(0) \in \mathbb{R}$, $a > \gamma(0) > 1$. Let $a_i(s)$ be the unique lift of $\alpha_i(s)$ to $\overline{M}_{k\,a}$ satisfying $w(a_i(0)) \in \mathbb{R}_+$, $i = 1, 2$. Analogously, we define $b_i(s), i = 1, 2, c(s)$ as the corresponding lifts of $\beta_i(s), i = 1, 2, \gamma(s)$ with initial conditions $\operatorname{Arg}(w(b_i(0))) = \frac{\pi}{k}$, $i = 1, 2$, $\operatorname{Arg}(w(c(0))) = \frac{\pi}{k}$, respectively.
- Suppose $0 < a < 1$. Now, $\alpha_i(s)$, $\beta_i(s)$, $\gamma(s)$ are the oriented closed curves in the z-plane of Figure 2. Here $\alpha_1(0) \in \mathbb{R}$, $\alpha_1(0) > 1$, $\alpha_2(0) \in \mathbb{R}$, $a > \alpha_2(0) > 0$ $\beta_1(0) \in \mathbb{R}$, $1 > \beta_1(0) > a$, $\beta_2(0) \in \mathbb{R}$, $0 > \beta_2(0) > -a$, $\gamma(0) \in \mathbb{R}$, $1 > \gamma(0) > a$. Let $a_i(s)$ be the unique lift of $\alpha_i(s)$ to $\overline{M}_{k\,a}$ satisfying $w(a_i(0)) \in \mathbb{R}_+$, $i = 1, 2$.

  Analogously, denote by $b_i(s)$, $i = 1, 2$, $c(s)$, the corresponding lifts of $\beta_i(s)$, $i = 1, 2$ and $\gamma(s)$ with initial conditions $\operatorname{Arg}(w(b_i(0))) = \operatorname{Arg}(w(c(0))) = \frac{\pi}{k}$, $i = 1, 2$, respectively.

For any closed curve d in $\overline{M}_{k\,a}$, we identify d and its homology class $[d]$. Then observe that

$$b_1 = b_2 + c, \;\; T_*(a_1) = -b_2, \;\; T_*(a_2) = -c. \tag{4}$$

At this point we distinguish two cases:

- If $a > 1$, we let

$$a_i^j = \sum_{l=0}^{j} (T^{2l})_*(a_i), \;\; b_i^j = (T^{2j})_*(b_i) \;\; i = 1, 2; \; j = 0, \dots, k-2$$

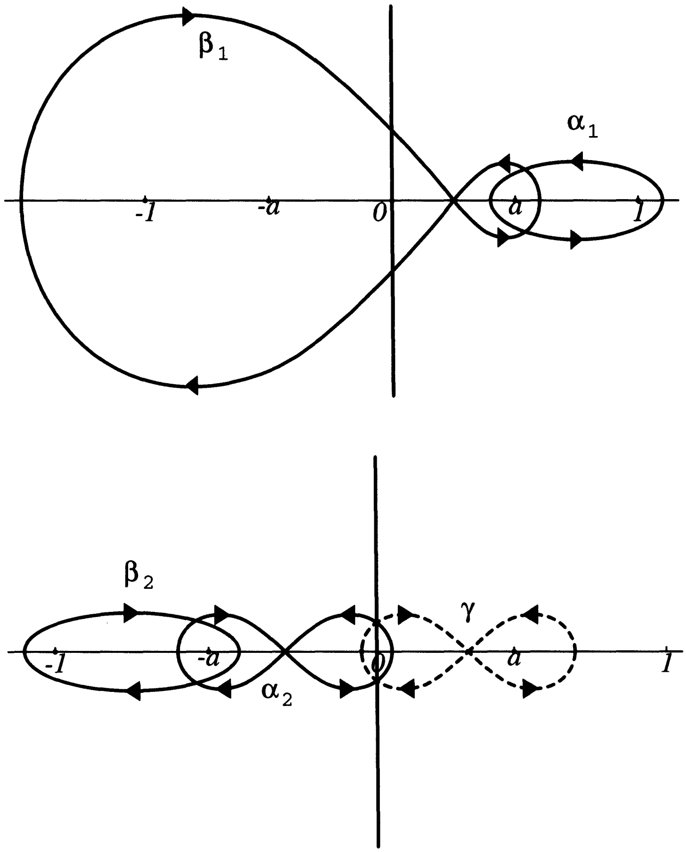

Figure 2. $0 < a < 1$

- If $a < 1$, we denote by

$$a_1^j = \sum_{l=0}^{j} (T^{2l})_*(a_1),$$

$$b_1^j = (T^{2j})_*(b_1 - a_1), \quad a_2^j = \sum_{l=0}^{j} (T^{2l})_*(a_2 - b_2), \quad b_2^j = (T^{2j})_*(b_2),$$

$j = 0, \dots, k-2.$

In both situations one has that

$$\mathcal{B} = \left\{ a_i^j,\ b_i^j \colon i = 1, 2\ j = 0, \dots, k-2 \right\}$$

is a canonical homology basis on $\overline{M}_{k\,a}$, where $(a_1^0, \ldots, a_1^{k-2}, a_2^0, \ldots, a_2^{k-2})$, $(b_1^0, \ldots, b_1^{k-2}, b_2^0, \ldots, b_2^{k-2})$ are the a, b-curves, respectively. Let τ_1, τ_2, σ_1, σ_2 be the following 1-forms on $\overline{M}_{k\,a}$:

$$\tau_1 = \frac{dz}{w^{k-1}}, \; \tau_2 = w^{k-1} dz, \; \sigma_1 = \frac{dz}{w^{k-1}(z^2-1)}, \; \sigma_2 = \frac{z\,dz}{(z^2-1)w}.$$

An easy computation gives

$$T^*(\tau_1) = e^{\frac{\pi i}{k}} \tau_1, \; T^*(\tau_2) = e^{-\frac{\pi i}{k}} \tau_2, \; T^*(\sigma_1) = e^{\frac{\pi i}{k}} \sigma_1, \; T^*(\sigma_2) = e^{-\frac{\pi i}{k}} \sigma_2. \tag{5}$$

Define the following functions on $\mathbb{R}_+ - \{1\}$:

$$f_1(a) = \frac{1}{\xi} \int_{b_2} \tau_1 \,, \; f_2(a) = \frac{1}{\xi} \int_c \tau_1 \,, \; g_1(a) = \frac{1}{\overline{\xi}} \int_{b_2} \tau_2 \,, \; g_2(a) = \frac{1}{\overline{\xi}} \int_c \tau_2,$$

$$h_1(a) = \frac{1}{\xi} \int_{b_2} \sigma_1 \,, \; h_2(a) = -\frac{1}{\xi} \int_c \sigma_1 \,, \; k_1(a) = -\frac{1}{\overline{\xi}} \int_{b_2} \sigma_2 \,, \; k_2(a) = -\frac{1}{\overline{\xi}} \int_c \sigma_2$$

where $\xi = 1 - e^{\frac{2\pi i}{k}}$. When $a > 1$ then $f_i(a), g_i(a), h_i(a), k_i(a) > 0$, $i = 1, 2$, and $a < 1$ implies $f_1(a), g_1(a) < 0$, $f_2(a), g_2(a) > 0$, $h_i(a), k_i(a) > 0$, $i = 1, 2$. From (4) and (5),

$$\int_{a_i} \tau_1 = \theta f_i(a), \quad \int_{a_i} \tau_2 = -\theta g_i(a), \quad i = 1, 2$$

$$\int_{a_i} \sigma_1 = (-1)^{i+1} \theta h_i(a), \quad \int_{a_i} \sigma_2 = \theta k_i(a), \quad i = 1, 2,$$

where $\theta = e^{\frac{\pi i}{k}} - e^{-\frac{\pi i}{k}}$. We define

$$f_3(a) = \begin{cases} f_1(a) + 2f_2(a) & \text{if } a > 1 \\ (1 + 2\cos(\frac{\pi}{k})) f_1(a) + 2f_2(a) & \text{if } 0 < a < 1 \end{cases}$$

$$g_3(a) = \begin{cases} g_1(a) + 2g_2(a) & \text{if } a > 1 \\ (1 + 2\cos(\frac{\pi}{k})) g_1(a) + 2g_2(a) & \text{if } 0 < a < 1 \end{cases}$$

$$h_3(a) = \begin{cases} h_1(a) - 2h_2(a) & \text{if } a > 1 \\ (1 + 2\cos(\frac{\pi}{k})) h_1(a) - 2h_2(a) & \text{if } 0 < a < 1 \end{cases}$$

$$k_3(a) = \begin{cases} k_1(a) + 2k_2(a) & \text{if } a > 1 \\ (1 + 2\cos(\frac{\pi}{k})) k_1(a) + 2k_2(a) & \text{if } 0 < a < 1. \end{cases}$$

We state the following lemmas:

LEMMA 1. *The asymptotic behavior of f_i, g_i, k_i, $i = 1, 2$ at 0, ∞ is given by*

(i)

$$\begin{aligned}
\lim_{a\to 0} f_1(a)a^{\frac{2k-3}{k}} &= -\frac{1}{2}\mathfrak{B}\left(\frac{1}{k}, \frac{2k-3}{2k}\right),\\
\lim_{a\to \infty} f_1(a)a^{-\frac{1}{k}} &= \frac{1}{2}\mathfrak{B}\left(\frac{1}{k}, \frac{2k-1}{2k}\right)\\
\lim_{a\to 0} f_2(a)a^{\frac{2k-3}{k}} &= \frac{1}{2}\mathfrak{B}\left(\frac{1}{k}, \frac{1}{2k}\right),\\
\lim_{a\to \infty} f_2(a)a^{\frac{2(k-1)}{k}} &= \frac{1}{2}\mathfrak{B}\left(\frac{2k-1}{k}, \frac{1}{2k}\right)
\end{aligned}$$

(ii)

$$\begin{aligned}
\lim_{a\to 0} g_1(a) &= -\frac{1}{2}\mathfrak{B}\left(\frac{1}{k}, \frac{4k-3}{2k}\right),\\
\lim_{a\to \infty} g_1(a)a^{\frac{1-2k}{k}} &= \frac{1}{2}\mathfrak{B}\left(\frac{1}{2k}, \frac{2k-1}{k}\right)\\
\lim_{a\to 0} g_2(a)a^{\frac{3-4k}{k}} &= \frac{1}{2}\mathfrak{B}\left(\frac{2k-1}{k}, \frac{2k-1}{2k}\right),\\
\lim_{a\to \infty} g_2(a)a^{\frac{2(1-k)}{k}} &= \frac{1}{2}\mathfrak{B}\left(\frac{2k-1}{2k}, \frac{1}{k}\right)
\end{aligned}$$

(iii)

$$\begin{aligned}
\lim_{a\to 0} k_1(a) &= \frac{1}{2}\mathfrak{B}\left(\frac{1}{k}, \frac{2k-3}{2k}\right),\\
\lim_{a\to \infty} k_1(a)a^{\frac{1}{k}} &= \frac{1}{2}\mathfrak{B}\left(\frac{1}{2k}, \frac{k-1}{k}\right)\\
\lim_{a\to 0} k_2(a)a^{\frac{3-2k}{k}} &= \frac{1}{2}\mathfrak{B}\left(\frac{k-1}{k}, \frac{2k-1}{2k}\right),\\
\lim_{a\to \infty} k_2(a)a^{\frac{2}{k}} &= \frac{1}{2}\mathfrak{B}\left(\frac{1}{k}, \frac{2k-1}{2k}\right)
\end{aligned}$$

where $\mathfrak{B}$ is the classical Beta Function. Furthermore

$$\lim_{a\to 1}\frac{f_1(a)}{a-1} = \lim_{a\to 1}\frac{g_1(a)}{a-1} = \frac{\pi(k-1)}{k\sin(\pi/k)},\ \lim_{a\to 1} f_2(a) = (2k-1)\cdot\lim_{a\to 1} g_2(a) = k.$$

Proof. From the definition of f_1 we obtain

$$f_1(a) = \int_1^a \left(\frac{1-z^2}{z(z^2-a^2)}\right)^{\frac{k-1}{k}} dz.$$

A suitable change of variable gives

$$f_1(a) = (a-1)\int_0^1 \left(\frac{t(2+(a-1)t)}{(1+a+(a-1)t)(1+(a-1)t)(1-t)}\right)^{\frac{k-1}{k}} dt.$$

Hence

$$\lim_{a\to 1}\frac{f_1(a)}{a-1} = \int_0^1 \left(\frac{t}{1-t}\right)^{\frac{k-1}{k}} dt = \frac{(k-1)\pi}{k\sin(\pi/k)}$$

$$\lim_{a\to\infty} f_1(a)a^{-\frac{1}{k}} = \int_0^1 \left(\frac{t}{1-t^2}\right)^{\frac{k-1}{k}} dt = \frac{1}{2}\mathfrak{B}\left(\frac{1}{k}, \frac{2k-1}{2k}\right).$$

To compute the limit at 0, put $\frac{1}{z} = \left(\frac{1}{a}-1\right)t+1$ and so

$$f_1(a) = a^{\frac{3-2k}{k}}(a-1)\int_0^1 \left(\frac{t(2a+(1-a)t)}{(1-t)(1+a+(1-a)t)}\right)^{\frac{k-1}{k}} \frac{dt}{(a+(1-a)t)^{\frac{1+k}{k}}}.$$

Then

$$\lim_{a\to 0} a^{\frac{2k-3}{k}} f_1(a) = -\int_0^1 \frac{t^{\frac{k-3}{k}}}{(1-t^2)^{\frac{k-1}{k}}}dt = -\frac{1}{2}\mathfrak{B}\left(\frac{1}{k}, \frac{2k-3}{2k}\right).$$

Similar arguments work for g_i and k_i, $i = 1, 2$. □

LEMMA 2. *The functions* $f_i,\ g_i,\ h_i,\ k_i, i \in \{1, 3\}$ *satisfy*

(i) $f_1g_3 + f_3g_1 = \dfrac{2\pi k(k-1)(a^2-1)}{(2k-1)\sin(\pi/k)}$

(ii) $f_1k_3 + f_3k_1 = \dfrac{k\pi}{\sin(\pi/k)}$

(iii) $g_1h_3 + g_3h_1 = \dfrac{k\pi}{(2k-1)\sin(\pi/k)}$

(iv) $h_1k_3 + h_3k_1 = 0.$

Proof. Using classical bilinear relations we obtain

$$\sum_{p=1}^{2}\sum_{j=0}^{k-2}\left(\int_{a_p^j}\tau_2\int_{b_p^j}\tau_1 - \int_{b_p^j}\tau_2\int_{a_p^j}\tau_1\right) = 2\pi i\,\mathrm{Residue}(f\tau_1,\infty)$$

where $\tau_2 = df$ locally around ∞. In the case $a > 1$, taking into account (4) and (5), the last equality becomes

$$k\left\{\frac{1}{\xi}\left(\int_{a_1}\tau_2\int_{b_2}\tau_1+\int_{a_1}\tau_2\int_{c}\tau_1+\int_{a_2}\tau_2\int_{b_2}\tau_1\right)\right.$$
$$\left.-\frac{1}{\bar{\xi}}\left(\int_{b_2}\tau_2\int_{a_1}\tau_1+\int_{c}\tau_2\int_{a_1}\tau_1+\int_{b_2}\tau_2\int_{a_2}\tau_1\right)\right\}=2\pi i\frac{2k^2(k-1)(1-a^2)}{2k-1}$$

For $a < 1$ one has

$$k\left\{\frac{1}{\xi}\left(\int_{a_1}\tau_2\int_{b_2}\tau_1+\int_{a_1}\tau_2\int_{c}\tau_1+\int_{a_2}\tau_2\int_{b_2}\tau_1-\int_{a_1}\tau_2\int_{a_1}\tau_1-\int_{b_2}\tau_2\int_{b_2}\tau_1\right)\right.$$
$$\left.-\frac{1}{\bar{\xi}}\left(\int_{b_2}\tau_2\int_{a_1}\tau_1+\int_{c}\tau_2\int_{a_1}\tau_1+\int_{b_2}\tau_2\int_{a_2}\tau_1-\int_{a_1}\tau_2\int_{a_1}\tau_1-\int_{b_2}\tau_2\int_{b_2}\tau_1\right)\right\}$$
$$=2\pi i\frac{2k^2(k-1)(1-a^2)}{2k-1}$$

Using the definitions of the functions $f_i,\ g_i,\ i\in\{1,3\}$ it is not hard to check (i). Applying the same argument to the pairs (σ_2,τ_1), (τ_2,σ_1) and (σ_2,σ_1) we obtain the equalities (ii), (iii) and (iv), respectively. □

LEMMA 3. *The following equalities hold*:

(i) $\dfrac{d f_i}{da}=\dfrac{1}{ka}f_i+\dfrac{2(k-1)}{ka}h_i$

(ii) $\dfrac{d g_i}{da}=\dfrac{2a(k-1)}{k}k_i$

(iii) $\dfrac{d h_i}{da}=\dfrac{1}{ka(a^2-1)}f_i-\dfrac{2(k-1)}{ka}h_i$

(iv) $\dfrac{d k_i}{da}=\dfrac{2k-1}{ka(1-a^2)}g_i+\dfrac{2k-3}{ka}k_i$

for $i\in\{1,3\}$.

Proof. By a formal derivation, we obtain

$$\frac{d}{da}(\tau_1)=\frac{1}{ka}\tau_1+\frac{2(k-1)}{ka}\sigma_1-\frac{1}{a}d\left(\frac{z}{w^{k-1}}\right),\quad \frac{d}{da}(\tau_2)=-\frac{2a(k-1)}{k}\sigma_2$$

$$\frac{d}{da}(\sigma_1)=\frac{1}{ka(a^2-1)}\tau_1-\frac{2(k-1)}{ka}\sigma_1+\frac{1}{a(1-a^2)}d\left(\frac{z}{w^{k-1}}\right)$$

$$\frac{d}{da}(\sigma_2) = \frac{2k-1}{ka(a^2-1)}\tau_2 + \frac{2k-3}{ka}\sigma_2 + \frac{1}{a(1-a^2)}\, d\left(\frac{z^2}{w}\right).$$

Integrating on the suitable curves, it is easy to deduce the statements of this lemma. □

LEMMA 4. *The following equations hold*:

(i) $f_1h_3 - f_3h_1 = -\dfrac{k\pi}{\sin(\pi/k)}a^{\frac{3-2k}{k}}$

(ii) $g_1k_3 - g_3k_1 = -\dfrac{k\pi}{\sin(\pi/k)(2k-1)}a^{\frac{2k-3}{k}}$.

Proof. From Lemma 3, we can check the following equations:

$$\frac{d}{da}(f_1h_3 - f_3h_1) = \frac{3-2k}{ka}(f_1h_3 - f_3h_1)$$

$$\frac{d}{da}(g_1k_3 - g_3k_1) = \frac{2k-3}{ka}(g_1k_3 - g_3k_1).$$

If we integrate these ordinary differential equations we obtain

$$f_1(a)h_3(a) - f_3(a)h_1(a) = \begin{cases} C_1a^{\frac{3-2k}{k}} & \text{if} \quad a > 1 \\ D_1a^{\frac{3-2k}{k}} & \text{if} \quad 0 < a < 1 \end{cases}$$

$$g_1(a)k_3(a) - g_3(a)k_1(a) = \begin{cases} C_2a^{\frac{2k-3}{k}} & \text{if} \quad a > 1 \\ D_2a^{\frac{2k-3}{k}} & \text{if} \quad 0 < a < 1. \end{cases}$$

Hence,

$$(f_1h_3 - f_3h_1)(g_1k_3 - g_3k_1) = \begin{cases} C_1 \cdot C_2 & \text{if} \quad a > 1 \\ D_1 \cdot D_2 & \text{if} \quad 0 < a < 1. \end{cases}$$

Expanding and using (ii), (iii) and (iv) in Lemma 2 we have $C_1 \cdot C_2 = D_1 \cdot D_2 = \dfrac{k^2\pi^2}{\sin^2(\pi/k)(2k-1)}$.

Now using Lemma 1 and the properties of beta and gamma functions it follows that

$$\begin{aligned} C_2 &= \lim_{a\to\infty} \frac{g_1(a)k_3(a) - g_3(a)k_1(a)}{a^{\frac{2k-3}{k}}} = 2\lim_{a\to\infty} \frac{g_1(a)k_2(a) - g_2(a)k_1(a)}{a^{\frac{2k-3}{k}}} \\ &= \frac{1}{2}\left[\mathfrak{B}\left(\frac{2k-1}{k}, \frac{1}{2k}\right)\mathfrak{B}\left(\frac{1}{k}, \frac{2k-1}{2k}\right) - \mathfrak{B}\left(\frac{1}{k}, \frac{2k-1}{2k}\right)\mathfrak{B}\left(\frac{k-1}{k}, \frac{1}{2k}\right)\right] \\ &= -\frac{k\pi}{(2k-1)\sin(\pi/k)} \end{aligned}$$

$$D_2 = \lim_{a\to 0} \frac{g_1(a)k_3(a) - g_3(a)k_1(a)}{a^{\frac{2k-3}{k}}} = 2\lim_{a\to 0} \frac{g_1(a)k_2(a) - g_2(a)k_1(a)}{a^{\frac{2k-3}{k}}}$$
$$= -\frac{1}{2}\left[\mathfrak{B}\left(\frac{1}{k}, \frac{4k-3}{2k}\right)\mathfrak{B}\left(\frac{k-1}{k}, \frac{2k-1}{2k}\right)\right] = -\frac{k\pi}{(2k-1)\sin(\pi/k)}$$

and so, $C_1 = D_1 = -\dfrac{k\pi}{\sin(\pi/k)}$. □

LEMMA 5. *The functions* f_1, f_3, g_1, g_3 *satisfy*

$$\text{(i)}\ \frac{d f_i}{d a} = \frac{a}{k(a^2-1)} f_i(a) + (-1)^{\frac{i-1}{2}} \frac{(2k-1)a^{\frac{3(1-k)}{k}}}{k(a^2-1)} g_i(a)$$
$$\text{(ii)}\ \frac{d g_i}{d a} = (-1)^{\frac{i-1}{2}} \frac{a^{\frac{3(k-1)}{k}}}{k(a^2-1)} f_i(a) + \frac{(2k-1)a}{k(a^2-1)} g_i(a)$$

for $i \in \{1, 3\}$.

Proof. From Lemmas 2 and 4, observe that the functions $h_i, k_i, i = 1, 3$, satisfy the linear systems

$$\left.\begin{aligned} -f_3h_1 + f_1h_3 &= -\tfrac{k\pi}{\sin(\pi/k)} a^{\frac{3-2k}{k}} \\ g_3h_1 + g_1h_3 &= \tfrac{k\pi}{(2k-1)\sin(\pi/k)} \end{aligned}\right\} \qquad \left.\begin{aligned} -g_3k_1 + g_1k_3 &= -\tfrac{k\pi}{(2k-1)\sin(\pi/k)} a^{\frac{2k-3}{k}} \\ f_3k_1 + f_1k_3 &= \tfrac{k\pi}{\sin(\pi/k)} \end{aligned}\right\}$$

Solving and using (i) in Lemma 2, we obtain new expressions for k_i and h_i, $i = 1, 3$, depending on f_i and g_i, $i = 1, 3$. Substituting them in the equalities (i) and (ii) in Lemma 3 we conclude the proof. □

To define a proper minimal immersion of $M_{k\,a} = \overline{M}_{k\,a} - \{\infty\}$ into $\mathbb{R}^3$ for every $k \geq 2$ and suitable $a \in \mathbb{R}_+ - \{1\}$, consider the Weierstrass data

$$g = A\, w^{k-1}, \quad \eta g = B\, dz,\ A \in \mathbb{R},\ B \in \mathbb{C},\ |B| = 1$$

on $\overline{M}_{k\,a}$. Then, defining ϕ_j, $j = 1, 2, 3$, as in (1), the inequality (2) holds. Moreover, ϕ_j, $j = 1, 2, 3$, have no real periods if and only if the immersion

$$x : M_{k\,a} \to \mathbb{R}^3$$

$$x = \text{Real} \int (\phi_1, \phi_2, \phi_3)$$

is well defined.

THEOREM 1. *For each $k \geq 2$ there exists only one $a_0 \in \mathbb{R}_+ - \{1\}$, depending of k, such that x is well defined.*

Proof. The immersion x is well defined if and only if Real $\left(\int_d \phi_j\right) = 0$ for every closed curve d in $M_{k\,a}$ and every $j \in \{1, 2, 3\}$. As ϕ_j has only one singularity at ∞, then Residue$(\phi_j, \infty) = 0$, $j = 1, 2, 3$. So, it is sufficient to prove

$$\text{Real}\left(\int_d \phi_j\right) = 0, \quad j = 1, 2, 3$$

for any closed curve lying in the homology basis $\mathcal{B}$ of $\overline{M}_{k\,a}$ defined at the beginning of this section.

If we put $\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$ then $T^*(\Phi) = \mathcal{R} \cdot \Phi$, where $\mathcal{R} \in \mathcal{O}(3)$ is the matrix

$$\mathcal{R} = \begin{pmatrix} \cos(\pi/k) & \sin(\pi/k) & 0 \\ -\sin(\pi/k) & \cos(\pi/k) & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Hence using the last equality and (4),

$$\text{Real}\left(\int_d \Phi\right) = 0 \;\; \forall d \in \mathcal{B} \Longleftrightarrow \text{Real}\left(\int_{b_2} \Phi\right) = \text{Real}\left(\int_c \Phi\right) = 0$$

Recall that ϕ_3 is exact, $\phi_1 = \frac{B}{2A}(\tau_1 - A^2\tau_2)$ and $\phi_2 = \frac{iB}{2A}(\tau_1 + A^2\tau_2)$. Using the definitions of f_i and g_i, $i = 1, 2$, the last equations hold if and only if $B^2 = 1$ and

$$\begin{aligned} f_1(a) &= A^2 g_1(a) \\ f_2(a) &= A^2 g_2(a) \end{aligned}$$

for some $A \in \mathbb{R}$ and $a \in \mathbb{R}_+ - \{1\}$.

Thus x is well defined if and only if

$$f_1(a)g_3(a) - f_3(a)g_1(a) = 0. \tag{6}$$

Let us define $\psi\colon \mathbb{R}_+ - \{1\} \to \mathbb{R}$ by

$$\psi(a) = \frac{f_3(a)}{f_1(a)} - \frac{g_3(a)}{g_1(a)}$$

and note that $\psi(a_0) = 0$ if and only if a_0 satisfies (6).

Firstly we study the asymptotic behavior of ψ at 0, 1 and ∞.

From Lemma 1,

$$\lim_{a \to 0} \psi(a) = 2 \lim_{a \to 0} \frac{f_2(a)g_1(a) - f_1(a)g_2(a)}{f_1(a)g_1(a)} = -2\frac{\mathfrak{B}\left(\frac{1}{k}, \frac{1}{2k}\right)}{\mathfrak{B}\left(\frac{1}{k}, \frac{2k-3}{2k}\right)} < 0$$

$$\lim_{a\to 1^-} \psi(a) = \frac{4k^2\sin(\pi/k)}{(2k-1)\pi} \lim_{a\to 1^-} \frac{1}{a-1} = -\infty$$

$$\lim_{a\to 1^+} \psi(a) = \frac{4k^2\sin(\pi/k)}{(2k-1)\pi} \lim_{a\to 1^+} \frac{1}{a-1} = +\infty$$

$$\begin{aligned}\lim_{a\to+\infty} \psi(a)a^{\frac{1}{k}} &= 2\lim_{a\to+\infty}\left(\frac{1}{a^{\frac{2(k-1)}{k}}}\frac{f_2(a)a^{\frac{2(k-1)}{k}}}{f_1(a)a^{-\frac{1}{k}}} - \frac{g_2(a)a^{-\frac{2(k-1)}{k}}}{g_1(a)a^{\frac{1-2k}{k}}}\right)\\ &= -2\frac{\mathfrak{B}\left(\frac{1}{k},\frac{2k-1}{2k}\right)}{\mathfrak{B}\left(\frac{1}{2k},\frac{2k-1}{k}\right)} < 0.\end{aligned}$$

So there exists $a_0 \in]1,+\infty[$ such that $\psi(a_0) = 0$.

To see that a_0 is the only zero on $\mathbb{R}_+ - \{1\}$ we compute $\psi'(a)$. From Lemma 5,

$$\psi'(a) = \frac{a^{\frac{3(k-1)}{k}}(f_1(a)g_3(a) + f_3(a)g_1(a))}{f_1^2(a)g_1^2(a)k(a^2-1)}\left(f_1^2(a) - (2k-1)a^{\frac{6(1-k)}{k}}g_1^2(a)\right)$$

and using (i) in Lemma 2 we deduce

$$\psi'(a) = \frac{a^{\frac{3(k-1)}{k}}2(k-1)\pi(f_1(a)+\sqrt{2k-1}a^{\frac{3(1-k)}{k}}g_1(a))(f_1(a)-\sqrt{2k-1}a^{\frac{3(1-k)}{k}}g_1(a))}{f_1^2(a)g_1^2(a)(2k-1)\sin(\pi/k)}$$

Observe that $\psi'(a)/\varrho(a)$ is positive in $]1,+\infty[$ and negative in $]0,1[$, where $\varrho\colon \mathbb{R}_+ - \{1\} \to \mathbb{R}$ is given by $\varrho(a) = f_1(a) - \sqrt{2k-1}a^{\frac{3(1-k)}{k}}g_1(a)$. It follows from Lemma 1 that

$$\lim_{a\to 0}\varrho(a) = \lim_{a\to+\infty}\varrho(a) = +\infty$$

$$\lim_{a\to 1}\frac{\varrho(a)}{a-1} = \frac{(k-1)\pi(1-\sqrt{2k-1})}{k\sin(\pi/k)} < 0$$

Furthermore, if $\varrho(b) = 0$ for some $b \in \mathbb{R}_+ - \{1\}$ then from Lemma 5,

$$\varrho'(b) = \frac{f_1(b)(k-1)(b^2-3)}{kb(b^2-1)}$$

Hence, if $b \in]0,1[$ is a zero of ϱ then $\varrho'(b) < 0$. Assume that $\varrho(a) = 0$ for some $a \in]0,1[$. Let a_1 be the first zero of ϱ in $]0,1[$. Since $\varrho(1) = 0$, $\varrho'(1) < 0$ we deduce that there exists another point $a_2 \in]0,1[$, $a_2 > a_1$ such that $\varrho(a_2) = 0$ and $\varrho'(a_2) > 0$, which is clearly absurd. Thus $\varrho(a) > 0, \forall a \in]0,1[$.

Suppose a_1 is the lowest root of ϱ in $]1,+\infty[$. If $a_1 \in]1,\sqrt{3}[$ then $\varrho'(a_1) < 0$ which is contrary to the choice of a_1 and the facts $\varrho(1) = 0$, $\varrho'(1) < 0$. Therefore $\varrho(a) < 0$, $\forall a \in]1,\sqrt{3}[$. Assume that ϱ has at least three zeros a_1, a_2, a_3 in

$[\sqrt{3}, +\infty[$. Without loss of generality, we will suppose that $a_1 < a_2 < a_3$ and that these three points are the lowest roots of ϱ. Then $a_2, a_3 \in]\sqrt{3}, +\infty[$ and so $\varrho'(a_2), \varrho'(a_3) > 0$ which is absurd. Thus ϱ has at most two zeroes in $]1, +\infty[$.

The above remarks imply $\psi'(a) < 0 \; \forall a \in]0, 1[$ and taking into account the limit of ψ at 0 we get $\psi(a) < 0 \; \forall a \in]0, 1[$. Analogously, ψ' has at most two zeroes in $]1, +\infty[$. Assume that ψ has at least two zeros in $]1, +\infty[$. According to the limits of this function at 1 and $+\infty$ we conclude that ψ' has at least three roots, which is absurd. This contradiction completes the proof. $\square$

For the sake of simplicity we write M_k instead of $M_{k\, a_0}$.

4. The geometric characterization

The aim of this section is to characterize the surfaces M_k from amongst all the other minimal surfaces with the same topology, symmetry and total curvature.

Let $x : M \to \mathbb{R}^3$ be a complete orientable minimal surface with finite total curvature and one end, and let (η, g) be its Weierstrass representation. From Huber's Theorem [H] there exist a compact Riemann surface $\overline{M}$ and one point $P \in \overline{M}$ such that M is conformally equivalent to $\overline{M} - \{P\}$. We write $n = \text{genus}(\overline{M})$ and assume that $n \geq 2$, n even. Then we can put $n = 2(k-1)$, where $k \in \mathbb{N}$, $k \geq 2$.

A symmetry of M induces in a natural way a conformal automorphism of M which extends to $\overline{M}$, leaving P invariant. Since the subgroup of holomorphic transformations has index one or two in $\text{Sym}(M)$, then Hurwitz's Theorem (see [F-K]) implies that $\text{Sym}(M)$ is finite. Then, except for a suitable choice of the origin, $\text{Sym}(M)$ is given by a linear group of isometries of $\mathbb{R}^3$.

For $R > 0$ large enough, $\overline{D} = x^{-1}\left(\{(x_1, x_2, x_3) \;/\; x_1^2 + x_2^2 + x_3^2 \geq R\}\right) \cup \{P\}$ is a conformal disc in $\overline{M}$. We can identify $\overline{D} \equiv \overline{D}(0, 1)$ and $P \equiv 0$. As $\text{Sym}(M)$ leaves $\overline{D}$ invariant, then $\Lambda = \{S_{|\overline{D}} \;/\; S \in \text{Sym}(M)\}$ is a group of conformal automorphisms of the unit disc which fixes 0. This implies that Λ is either cyclic or generated by a rotation composed by a symmetry with respect to a straight line containing 0. In the last case Λ is isomorphic to the dihedral group $\mathcal{D}(d/2)$, where $d = \sharp(\Lambda)$ is the cardinal of $\text{Sym}(M)$.

Up to rigid motions of $\mathbb{R}^3$ we can assume that $g(P) = \infty$. Let $T \in \text{Sym}(M)$ denote a symmetry whose restriction $T_{|\overline{D}}$ generates the subgroup of holomorphic transformations of Λ. It is obvious that $\text{ord}(T) \in \{d, d/2\}$. Observe that T extends conformally to $\overline{M}$ and as linear transformation it fixes the x_3-axis. If $d > 3$, T is either a rotation around the x_3-axis or a rotation followed by a symmetry with respect to the (x_1, x_2)-plane. Without loss of generality, we suppose the rotation determined by T is by angle $\frac{2\pi}{\text{ord}(T)}$. As the normal vector at the end is vertical, $x(M)$ intersects the x_3-axis in a finite number of points and therefore T fixes a finite set of points in $\overline{M}$. For each $Q \in \overline{M}$, define the isotropy group $H_Q = \{J \in \langle T \rangle \colon J(Q) = Q\}$, and the orbit of Q, $\text{orb}(Q) = \{Q, T(Q), T^2(Q), \ldots, T^{\text{ord}(T)-1}(Q)\}$. Note that $\text{orb}(P) = \{P\}$.

If we label $\mu(Q) = \sharp(H_Q)$, the Riemann-Hurwitz Formula gives:

$$6 - 4k = \mathrm{ord}(T) \cdot \chi\left(\overline{M}/\langle T\rangle\right) - \mathrm{ord}(T) + 1 - \sum_{Q \in M} (\mu(Q) - 1) \tag{7}$$

If we assume that $\sharp(\mathrm{Sym}(M)) = 4k$, then $\chi\left(\overline{M}/\langle T\rangle\right) \geq 0$ and so $\overline{M}/\langle T\rangle$ is a sphere or a torus. In particular, there exists $Q \in M$ such that $\mu(Q) > 1$. Label $u : \overline{M} \to \overline{M}/\langle T\rangle$ the natural projection. Denote by $\mathcal{A} = \{a_1, a_2, \ldots, a_s\}$ the set of singular values of u (i.e., $\mu(Q) > 1$ if $Q \in u^{-1}(\mathcal{A})$ and $\mu(Q) = 1$ if $Q \notin u^{-1}(\mathcal{A})$). Pick $Q_i \in u^{-1}(a_i)$ and define $m_i = \frac{\mathrm{ord}(T)}{\mu(Q_i)}$, $i = 1, \ldots, s$. It is clear that $m_i \in \mathbb{N}$, $1 \leq m_i \leq \mathrm{ord}(T)/2$, $i = 1, \ldots, s$ and $\{m_1, \ldots, m_s\}$ are relatively prime. Then

$$H_{Q_i} = \{T^{pm_i}\colon\ p = 0, 1, \ldots, \mu(Q_i) - 1\},\ \ \mathrm{orb}(Q_i) = \{Q_i, T(Q_i), \ldots, T^{m_i - 1}(Q_i)\}$$

and so

$$\sum_{Q \in M} (\mu(Q) - 1) = \sum_{i=1}^{s} (\mathrm{ord}(T) - m_i) \tag{8}$$

Given $Q \in \overline{M}$ whose normal vector $g(Q)$ is vertical, it is clear that

$$\mathrm{orb}(Q) \subseteq g^{-1}(g(Q)) \tag{9}$$

To see this observe that $g \circ T = \varphi g$, $\varphi^{\mathrm{ord}(T)} = 1$. Even more, a classical result asserts (see [N, §437]):

Let M be a minimal surface in $\mathbb{R}^3$ and $P_0 \in M$. Then the multiplicity of the Gauss map g at P_0 is $\nu - 1$ if and only if the tangent plane at this point intersects the surface along ν analytic curves $C_1, \ldots, C_\nu$ in a neighborhood of P_0. These curves intersect each other at P_0 forming angles different to 0 and π. They divide a neighborhood of P_0 into 2ν open sectors, such that M lies on one side of the tangent plane in one sector and on the other side in the next sector.

Thus, if Q is a fixed point of T^{m_i}, the tangent plane at Q is horizontal and the number of curves in the intersection between the tangent plane and the surface is

$$\mathrm{ord}(T^{m_i})l_i, \quad \text{where } l_i \in \mathbb{N}\,, \tag{10}$$

if T^{m_i} is a rotation, and

$$\frac{\mathrm{ord}(T^{m_i})}{2}\widetilde{l_i}, \quad \text{where } \widetilde{l_i} \in \mathbb{N},\ \widetilde{l_i} \text{ odd}\,, \tag{11}$$

if T^{m_i} is the composition of a rotation and a symmetry.

So, the multiplicity of Q as zero or pole of g is

$$\frac{\operatorname{ord}(T)}{m_i} l_i - 1, \quad \text{where } l_i \in \mathbb{N}\,, \tag{12}$$

T^{m_i} is a rotation, and

$$\frac{\operatorname{ord}(T)}{2m_i} \widetilde{l_i} - 1, \quad \text{where } \widetilde{l_i} \in \mathbb{N},\ \widetilde{l_i} \text{ odd}\,. \tag{13}$$

T^{m_i} is the composition of a rotation and a symmetry.

In what follows we denote by $[g]_0$ and $[g]_\infty$ the zero and polar divisor of g, respectively. It is evident that $\operatorname{Deg}([g]_0) = \operatorname{Deg}([g]_\infty) = \deg(g)$, where $\operatorname{Deg}(D)$ means the degree of a divisor D. For more details see [F-K].

At this point we will distinguish two cases: Λ cyclic and $\Lambda \cong \mathcal{D}(2k)$. In the first case we obtain the following

PROPOSITION 1. *If Λ is cyclic then $\overline{M}/\langle T\rangle$ is conformally equivalent to the Riemann sphere. Furthermore the Gauss map g satisfies* $\deg(g) > 3k - 3$.

Proof. Taking into account (7) and (8) we get

$$4k \cdot \chi\left(\overline{M}/\langle T\rangle\right) = 5 + \sum_{i=1}^{s}(4k - m_i) > 0$$

thus $\chi\left(\overline{M}/\langle T\rangle\right) = 2$ and

$$\sum_{i=1}^{s}(4k - m_i) = 8k - 5.$$

Therefore, one has $2 \le s \le 3$.

If $s = 2$ then $m_1 + m_2 = 5$. This implies that either $\{m_1, m_2\} = \{1, 4\}$ or $\{m_1, m_2\} = \{2, 3\}$. In both cases, using (12) and (13), it is clear that $\deg(g) \ge 4k - 4 > 3k - 3$.

If $s = 3$ then $m_1 + m_2 + m_3 = 4k + 5$. Up to relabelings, we can suppose $m_1 \le m_2 \le m_3$. If $m_3 \le \frac{4k}{3}$ then $m_1 + m_2 + m_3 \le 4k$, which is impossible. Thus, $m_3 = 2k$ and $m_1 + m_2 = 2k + 5$. By similar arguments, we obtain $m_2 > k$, and so $m_2 \in \{2k, \frac{4k}{3}\}$. There are two possibilities either $m_1 = 5$, $m_2 = m_3 = 2k$ or 3 is a divisor of k and $m_1 = \frac{2k}{3} + 5$, $m_2 = \frac{4k}{3}$, $m_3 = 2k$. The first option implies that k is a multiple of 5. In the second possibility , as $\frac{2k}{3} + 5$ is a divisor of $4k$, a straightforward arithmetical computation gives $k = 15$, $m_1 = 15$, $m_2 = 20$, $m_3 = 30$. In both situations, m_1, m_2, m_3 are not relatively prime, which is absurd. In summary, the case $s = 3$ is impossible. □

If Λ is not cyclic then $\mathrm{ord}(T) = 2k$. Using (7) and (8), we have

$$2k \cdot \chi\left(\overline{M}/\langle T\rangle\right) = 5 - 2k + \sum_{i=1}^{s}(2k - m_i) \geq 0 \tag{14}$$

which implies either $\chi(\overline{M}/\langle T\rangle) = 0$ or $\chi(\overline{M}/\langle T\rangle) = 2$.

PROPOSITION 2. *If $\Lambda \cong \mathcal{D}(2k)$ and $\chi\left(\overline{M}/\langle T\rangle\right) = 0$ then 5 is a divisor of k and* $\deg(g) \neq 3k - 3$.

Proof. Replacing $\chi\left(\overline{M}/\langle T\rangle\right)$ by 0 in (14), we deduce that $s = 1$ and $m_1 = 5$ (in particular, 5 is a divisor of k). From (9), (12) and (13), we have $\deg(g)$ a multiple of 5 and so $\deg(g) \neq 3(k-1)$. □

We can now present our final version of the main result of this section.

THEOREM 2. *Let $x : M \to \mathbb{R}$ be a complete minimal immersion with $C(M) = -4\pi(3k-3)$, $4k$ symmetries and one end. If k is not a multiple of 3 then M is, up to rigid motions and homotheties, the minimal surface M_k described in Theorem 1.*

Proof. By Propositions 1 and 2, one has $\Lambda \cong \mathcal{D}(2k)$ and $\chi(\overline{M}/\langle T\rangle) = 2$. It follows from (14) that

$$\sum_{i=1}^{s}(2k - m_i) = 6k - 5.$$

Hence, we infer $3 \leq s \leq 5$. In the following paragraphs we show that the case $s = 5$ is topologically impossible and that the case $s = 4$ leads to $\deg(g) \neq 3k - 3$, and finally that if $s = 3$ we obtain the surfaces M_k of Theorem 1.

If $s = 5$, one obtains that $\sum_{i=1}^{5} m_i = 4k + 5$. We may suppose that $m_1 \leq m_2 \leq m_3 \leq m_4 \leq m_5$. If $m_3 \leq \frac{2k}{3}$ then $\sum_{i=1}^{5} m_i \leq 4k$, which is absurd. Thus $m_i > \frac{2k}{3}$, $i = 3, 4, 5$, that is, $m_3 = m_4 = m_5 = k$ and $m_1 + m_2 = k + 5$. By similar arguments, we obtain $m_2 > \frac{k}{2}$ and so , assuming that k is not a multiple of 3, $m_2 = k$, $m_1 = 5$. But observe that m_1, m_2, m_3, m_4, m_5 are not relatively prime, which is absurd.

For $s = 4$, one has $\sum_{i=1}^{4} m_i = 2k + 5$. We must distinguish two cases:

(i) T is a rotation;
(ii) T is the composition of a rotation and a symmetry.

If T *is a rotation* then, up to relabelings, we can suppose that

$$\bigcup_{i=1}^{r} \mathrm{orb}(Q_i) \subseteq g^{-1}(0) \quad \text{and} \quad \bigcup_{i=r+1}^{4} \mathrm{orb}(Q_i) \subseteq g^{-1}(\infty)$$

for some $r \in \{0, \dots, 4\}$. So, using (9) and (12) one has

$$\deg(g) = \mathrm{Deg}([g]_0) = \sum_{i=1}^{r}(2k - m_i) + 2lk$$

$$\deg(g) = \mathrm{Deg}([g]_\infty) = \sum_{j=r+1}^{4}(2k - m_j) + m_\infty(g) + 2\tilde{l}k$$

where $m_\infty(g)$ is the multiplicity of g at P. It is obvious that $r \neq 2$ implies $\deg(g) \geq 3k > 3k-3$. In the case $r = 2$, if we suppose $\deg(g) = 3k-3$, we obtain $l = 0$ and $m_1 + m_2 = k+3$. Hence, $m_3 + m_4 = k+2$ and by using the second expression for $\deg(g)$ we obtain $\deg(g) = 3k - 2 + m_\infty(g) + 2\tilde{l}k > 3k-3$, which is absurd.

If T *is a rotation followed by a symmetry*, let

$$\begin{aligned}
\mathcal{E}_1 &= \{m_i \text{ even: } \mathrm{orb}(Q_i) \subseteq g^{-1}(0)\}\\
\mathcal{E}_2 &= \{m_j \text{ odd: } \mathrm{orb}(Q_j) \subseteq g^{-1}(0)\}\\
\mathcal{E}_3 &= \{m_h \text{ even: } \mathrm{orb}(Q_h) \subseteq g^{-1}(\infty)\}\\
\mathcal{E}_4 &= \{m_l \text{ odd: } \mathrm{orb}(Q_l) \subseteq g^{-1}(\infty)\}
\end{aligned}$$

and let $r_i = \sharp(\mathcal{E}_i)$, $i = 1,2,3,4$. We can suppose that $\mathcal{E}_1 = \{m_1, \dots, m_{r_1}\}$, $\mathcal{E}_2 = \{m_{r_1+1}, \dots, m_{r_1+r_2}\}$, $\mathcal{E}_3 = \{m_{r_1+r_2+1}, \dots, m_{r_1+r_2+r_3}\}$, $\mathcal{E}_4 = \{m_{r_1+r_2+r_3+1}, \dots, m_4\}$. Observe that $\mathcal{E}_i$ can be empty for some $i \in \{1,2,3,4\}$ and $r_2 + r_4$ is odd. Taking (9), (12) and (13) into account, we have

$$\deg(g) = \mathrm{Deg}([g]_0) = \sum_{m_i \in \mathcal{E}_1}(2k - m_i) + \sum_{m_j \in \mathcal{E}_2}(k - m_j) + 2lk \tag{15}$$

$$\deg(g) = \mathrm{Deg}([g]_\infty) = \sum_{m_h \in \mathcal{E}_3}(2k - m_h) + \sum_{m_l \in \mathcal{E}_4}(k - m_l) + m_\infty(g) + 2\tilde{l}k \tag{16}$$

where $l, \tilde{l} \in \mathbb{N}$. If $l > 1$, by (15) we deduce that $\deg(g) \geq 4k > 3k-3$. If $l = 1$, $\deg(g) = 3k-3$ leads to $l = 1$, $r_1 = 0$, $r_2 = 1$ and $m_1 = 3$, contrary to our assumptions.

Then, we must study case $l = 0$ only by discussing the possible values of $r_1 + r_2$. If $r_1 + r_2 = 1$, by (15) again, it follows that $\deg(g) \leq 2k - 1 < 3k-3$.
If $r_1 + r_2 = 2$, one has several cases:

- $r_1 = 0, r_2 = 2$. Then $\deg(g) = 2k - m_1 - m_2 < 3k-3$.
- $r_1 = r_2 = 1$. In this case $\deg(g) = 3k - m_1 - m_2$. As $m_1 + m_2 \geq 5$ we obtain $\deg(g) \leq 3k - 5 < 3k - 3$.
- $r_1 = 2, r_2 = 0$. Then $\deg(g) = 4k - m_1 - m_2$ and assuming $\deg(g) = 3k-3$ we obtain $m_1 + m_2 = k+3$. Hence, using ideas similar to those in case $s = 5$, we conclude that either $m_1 = k$ or $m_2 = k$. This implies k even and so $k+3$ is odd, contrary to the choice of m_1 and m_2.

If $r_1 + r_2 = 3$, one has two cases:

- $r_1 = 2$, $r_2 = 1$. Then $r_3 = 1$, $r_4 = 0$. Using (15) and (16), we have $\deg(g) = \mathrm{Deg}([g]_0) = 5k - m_1 - m_2 - m_3 = 3k - 5 + m_4$. Assuming that $\deg(g) = 3k - 3$, we obtain $m_4 = 2$ and $m_1 + m_2 + m_3 = 2k + 3$. The last equality implies, in the same way as in the case s=5, that k is even and $m_1 = m_2 = k$, $m_3 = 3$, contrary to our hypothesis.
- $r_1 = 1$, $r_2 = 2$. In this case, by (15) once again, one has $\deg(g) = 4k - m_1 - m_2 - m_3 = 2k - 5 + m_4 \leq 3k - 5 < 3k - 3$.

Finally, we consider case $s = 3$. From (14) one obtains $m_1 + m_2 + m_3 = 5$. As k is not a multiple of 3, one can observe that $\{m_1, m_2, m_3\} = \{1, 2\}$. It is clear that at least two $i, j \in \{1, 2, 3\}$ satisfy either $\mathrm{orb}(Q_i) \cup \mathrm{orb}(Q_j) \subseteq g^{-1}(0)$ or $\mathrm{orb}(Q_i) \cup \mathrm{orb}(Q_j) \subseteq g^{-1}(\infty)$. Suppose T is a rotation. From (12), we deduce that $\deg(g) \geq 4k - m_i - m_j \geq 4k - 4 > 3k - 3$. Taking into account the former, our hypotheses imply that T is a rotation followed by a symmetry. Furthermore, since $\deg(g) = 3k - 3$ one has, up to relabelings, $m_1 = 1$, $m_2 = m_3 = 2$, $\mathrm{orb}(Q_1) \cup \mathrm{orb}(Q_2) = g^{-1}(0)$, $\mathrm{orb}(Q_3) \subset g^{-1}(\infty)$ and $m_\infty(g) = k - 1$.

At this point, we can describe the underlying complex structure of $\overline{M}$. Up to Möbius transformation, we put $u(P) = \infty$, $a_1 = u(Q_1) = 0$, $a_2 = u(Q_2) = b$, $a_3 = u(Q_3) = 1$, $b \in \mathbb{C} - \{0, 1\}$. If we define $N = M - \bigcup_{i=1}^{3} \mathrm{orb}(Q_i)$, then

$$u_{|N} : N \longrightarrow \mathbb{C} - \{0, 1, b\}$$

is a $2k$-fold unbranched cyclic cover, and the conformal structure of $\overline{M}$ is determined by the structure of N. Let $\omega_i(t)$, $i = 1, 2, 3$, be the counterclockwise circuits around 0, 1 and b, respectively, and let $\widetilde{\omega}_i(t)$ be their respective lifts to N. Since $T^{m_i}(Q_i) = Q_i$, $i = 1, 2, 3$, the end points of $\widetilde{\omega}_i(t)$ will differ by a deck transformation of the form $T^{h_i m_i}$, where $h_i \in \{1, \ldots, 2k/m_i\}$, and $\gcd(h_i, 2k/m_i) = 1$, $i = 1, 2, 3$. Even more, the choice of T gives $h_i \equiv \pm 1 \, mod(2k/m_i)$, $i = 1, 2, 3$. Without loss of generality, we put $h_i \in \{1, -1\}$, $i = 1, 2, 3$. The integers $\{h_1, h_2, h_3\}$ determine the induced map from $\Pi_1(\mathbb{C} - \{0, 1, b\})$ into $\mathbb{Z}_{2k}$ whose kernel corresponds to $u_*(\Pi_1(N))$. Now consider the complex curve

$$\overline{M}_1 = \left\{ (u, w) \in (\mathbb{C} \cup \{\infty\})^2 \colon w^{2k} = \frac{u(u - b)^2}{(u - 1)^2} \right\}.$$

The cyclic covering defined by the u-projection of $\overline{M}_1$ has the same properties as $u_{|N}$ described above, and so they are equivalent; that is, up to conformal transformations,

$$\overline{M} = \overline{M}_1, \quad T(u, w) = (u, e^{\frac{\pi i}{k}} w),$$

and since

$$[g]_0 = Q_1^{k-1} \cdot Q_2^{k-1} \cdot (T(Q_2))^{k-1}, \quad [g]_\infty = P^{k-1} \cdot Q_3^{k-1} \cdot (T(Q_3))^{k-1},$$

it is easy to see that the Weierstrass data are

$$g = A_0\, w^{k-1}, \quad \eta g = B_0 \frac{u-b}{(u-1)w^k}\, du,$$

for suitable constants $A_0, B_0 \in \mathbb{C} - \{0\}$.

Since $\mathrm{Sym}(M) \cong \mathcal{D}(2k)$, there exists an antiholomorphic transformation $H \in \mathrm{Sym}(M)$ satisfying $H^2 = \mathrm{Id}$, $T \circ H \circ T = H$. As $T^{m_i} \circ H = H \circ T^{-m_i}$ then H leaves invariant the set of fixed points of T^{m_i}, $i = 1, 2, 3$. Thus, H induces an antiholomorphic automorphism $\widetilde{H}$ of the u-plane $\overline{M}/\langle T\rangle$ that satisfies $u \circ H = \widetilde{H} \circ u$. Furthermore, $\widetilde{H}$ satisfies one, and only one, of the following assertions:

- $\widetilde{H}$ fixes ∞, 0 and interchanges $1 \leftrightarrow b$;
- $\widetilde{H}$ fixes ∞, 0, 1 and b.

The first assertion implies that $|b| = 1$ and $\widetilde{H}(u) = b\,\overline{u}$, and so

$$(w \circ H)^{2k} = \frac{b\,\overline{u}^2}{\overline{w}^{2k}}. \tag{17}$$

If we let $\zeta = \sqrt[2k]{b}\, \dfrac{\overline{w \circ H}}{w}$, then $\zeta \circ T = \zeta$, and thus there exists a holomorphic transformation $\widetilde{\zeta}$ of the u-plane $\overline{M}/\langle T\rangle$, such that $\widetilde{\zeta} \circ u = \zeta$. Observe that the zero divisor of ζ is

$$[\zeta]_0 = Q_3^2 \cdot T(Q_3)^2$$

which implies $\deg(\zeta) = 4$. As $\deg(\zeta) = \deg(\widetilde{\zeta}) \cdot \deg(u)$ and $\deg(u) = 2k$, then we obtain $k = 2$ and $\deg(\widetilde{\zeta}) = 1$. From (17), we have

$$H(u, w) = \left(b\overline{u},\ \sqrt[4]{b}\, \frac{(\overline{u} - 1)\overline{w}}{\overline{u} - \overline{b}} \right).$$

Then $g \circ H \neq L \circ g$, for any rigid motion $L \in \mathcal{O}(3)$ leaving the x_3-axis invariant. Hence, this case is impossible.

It is straightforward to check that the second condition leads to $\widetilde{H}(u) = \overline{u}$, and without loss of generality,

$$H(u, w) = (\overline{u}, \overline{w}).$$

In particular, $b \in \mathbb{R}$.

If we write $u = z^2$, up to a biholomorphism, one has

$$\overline{M} = \left\{ (z, w) \in (\mathbb{C} \cup \{\infty\})^2 \colon w^k = \frac{z(z^2 - b)}{z^2 - 1} \right\}$$

and

$$g = A\, w^{k-1}, \quad \eta g = B\, dz$$

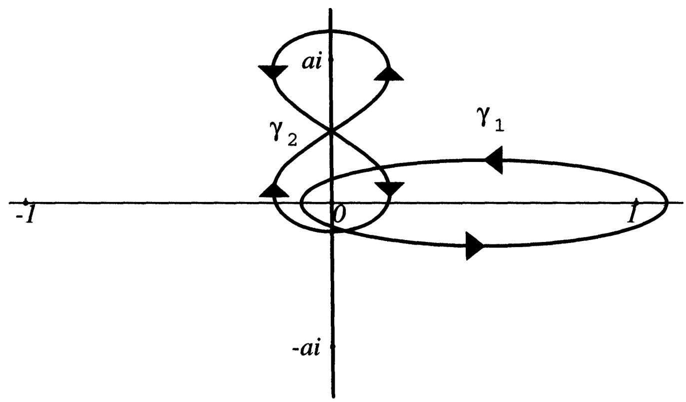

Figure 3. γ_1 and γ_2

where $b \in \mathbb{R} - \{0, 1\}$ and, up to homotheties and rigid motions, $A \in \mathbb{R}$, $B \in \mathbb{C}$, $|B| = 1$.

If $b > 0$, putting $a = \sqrt{b} > 0$, Theorem 1 leads to the surface M_k.

If $b < 0$, we write $a = \sqrt{-b}$. Let $\gamma_1(t)$, $\gamma_2(t)$ be the oriented simple closed curves in the z-plane illustrated by the Figure 3. Furthermore we take $\gamma_1(0) \in \mathbb{R}$, $\gamma_1(0) > 1$ and $\gamma_2(0) \in i\mathbb{R}$, $Im(\gamma_2(0)) > a$. Let $c_j(t)$ be the lift of $\gamma_j(t)$ to $\overline{M}$, $j = 1, 2$, with initial conditions $\mathrm{Arg}(w(c_1(0))) = 0$ and $\mathrm{Arg}(w(c_2(0))) = \frac{\pi}{2k}$. Using the notation from Theorem 1, we write $\tau_1 = \frac{dz}{w^{k-1}}$ and $\tau_2 = w^{k-1}dz$. Hence, it is not hard to check that

$$\int_{c_1} \tau_1 = \overline{\xi_1}\, F_1, \quad \int_{c_1} \tau_2 = \xi_1\, F_2, \quad \int_{c_2} \tau_1 = \xi_2\, G_1, \quad \int_{c_2} \tau_2 = -\overline{\xi_2}\, G_2$$

where $\xi_1 = e^{\frac{\pi i}{k}} - e^{-\frac{\pi i}{k}}$, $\xi_2 = e^{\frac{3\pi i}{2k}} - e^{-\frac{\pi i}{2k}}$ and $F_j, G_j \in \mathbb{R}$, $F_j, G_j > 0$, $j = 1, 2$. If we suppose that ϕ_1 and ϕ_2 have no real periods then

$$\frac{B}{A}\int_{c_j} \tau_1 = A\overline{B}\,\overline{\int_{c_j} \tau_2}, \quad j = 1, 2.$$

For $j = 1$ we obtain $B^2 = 1$ and similarly $j = 2$ implies $B^2 = -1$, which is absurd. $\square$

Remark 1. If k is divisible by 3 it is possible to find other algebraic curves $\overline{S}$, and Weierstrass data (g, η) on $\overline{S} - \{P\}$ with $\deg(g) = 3k - 3$ and the same group of symmetries.

To finish, we give an interesting consequence of Theorem 2. As we said in Section 1, this corollary has been obtained by F. J. López and the authors by studying another family of complete minimal surfaces which generalizes also the genus two Chen-Gackstatter example.

COROLLARY 1. *The only complete orientable genus two minimal surface in $\mathbb{R}^3$ with total curvature -12π and eight symmetries is the Chen-Gackstatter example.*

Proof. From the Huber Theorem [H], M is conformally equivalent to $\overline{M} - \{P_1, \ldots, P_r\}$, where $\overline{M}$ is a compact genus 2 Riemann surface. Furthermore from the Jorge-Meeks formula (3), $r = 1, 2$. If $r = 1$ then $I_1 = 3$ and $r = 2$ gives $I_1 = I_2 = 1$. The second possibility leads to the catenoid (see [Sch]) which is absurd. From Theorem 2 the first one corresponds to the Chen-Gackstatter genus two example. □

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