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MICROLOCALIZATION OF $\mathcal{O}_X$ ALONG DIHEDRAL LAGRANGIANS

GIUSEPPE ZAMPIERI

1. Introduction

Let X be a complex manifold, $T^*X \xrightarrow{\pi} X$ the cotangent bundle to X, $\sigma = \sigma^{\mathbb{R}} + \sqrt{-1}\sigma^{\mathbb{I}}$ the canonical 2–form on $\dot{T}^*X$, Λ_1, Λ_2 two $\mathbb{R}$-Lagrangian conic submanifolds of $\dot{T}^*X$. We assume that the intersection $\Lambda_1 \cap \Lambda_2$ is regular in a neighborhood of a point p, and that the tangent planes $\lambda_i(p)$: $\overset{\text{def.}}{=} T_p\Lambda_i$ verify $\operatorname{codim}_{\lambda_1(p)}(\lambda_1(p) \cap \lambda_2(p)) = 1$. According to [D'A-Z 3] (which improves [S]), one can then find a complex symplectic transformation χ_1 which interchanges Λ_1, Λ_2 with the conormal bundles $T^*_{M_1}X, T^*_{M_2}X$ to two hypersurfaces $M_1, M_2 \subset X$ whose Levi-forms are positive-semidefinite at $q = \chi_1(p)$.

We prove here in Proposition 1.1 that we can find another symplectic transformation χ_2 such that the Levi-form of one hypersurface is positive-semidefinite, whereas the other has one negative eigenvalue. The choice of the hypersurface which carries the negative eigenvalue is not arbitrary; it relies on intrinsic geometric properties of the pair Λ_1, Λ_2. In case the intersection $\Lambda_1 \cap \Lambda_2$ is "clean" of codimension 1, the two cases occur according to the "positivity" $\Lambda_1 > \Lambda_2$ (resp. $\Lambda_2 > \Lambda_1$) in the sense of [D'A-Z 4]. In the first transformation χ_1 this is characterized by the inclusion $\Sigma_1 \supset \Sigma_2$ (resp. $\Sigma_1 \subset \Sigma_2$) (where Σ_i are the closed half-spaces with boundary M_i and inward conormal q. (In the second transformation χ_2 the inclusions are reverted.)

We put $\lambda_0(p) = T_p\pi^{-1}\pi(p)$, assume that $\dim(\lambda_i(p) \cap \lambda_0(p)) \equiv \text{const}$, and still suppose the intersection $\Lambda_1 \cap \Lambda_2$ regular and clean. We denote by Λ_1^+ (resp Λ_2^+) one half-part of Λ_1 (resp. Λ_2) with boundary $\Lambda_1 \cap \Lambda_2$, and set $\Lambda = \Lambda_1^+ \cup \Lambda_2^+$. In Theorem 1.2 we prove that Λ can be reduced to the conormal bundle $T^*_Y X$ to a C^1-manifold Y of X by one and only one of the tranformations χ_1, χ_2. This can be proved by a direct analysis of the shift of simple sheaves along the Λ_i's under the action of quantizations of the χ_i's.

We finally discuss the complex of microfunctions along Λ in the sense of [K-S 1], and show that its non-trivial cohomology ranges through an interval described by the numbers of negative Levi eigenvalues of the Λ_i^+'s. By these results we are able to state a strong improvement of our former theorem in [Z 2] on existence for $\bar{\partial}$ on dihedrons of $\mathbb{C}^n$.

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Section 1

Let X be a complex manifold of dimension n, π: $T^*X \to X$ the cotangent bundle to X, $\alpha = \alpha^{\mathbb{R}} + \sqrt{-1}\alpha^{\mathbb{I}}$ $(\sigma = \sigma^{\mathbb{R}} + \sqrt{-1}\sigma^{\mathbb{I}})$ the 1–form (2–form). We identify $T^*(X^{\mathbb{R}}) \simeq (T^*X)^{\mathbb{R}}$ with the aid of $\alpha^{\mathbb{R}}$. We let H: $T^*T^*X \xrightarrow{\sim} TT^*X$, (resp. $H^{\mathbb{R}}$: $T^*T^*X^{\mathbb{R}} \xrightarrow{\sim} TT^*X^{\mathbb{R}}$) be the Hamiltonian isomorphism associated to σ (resp. $\sigma^{\mathbb{R}}$). We take an $\mathbb{R}$-Lagrangian (i.e., Lagrangian for $\sigma^{\mathbb{R}}$) conic submanifold Λ in a neighborhood of a point $p \in \dot{T}^*X (\overset{\text{def.}}{=} T^*X \setminus T^*_XX)$, and put

$$
\begin{aligned}
(1.1)\quad & e(p) = T_pT^*X \quad \nu(p) = \mathbb{C}H(\alpha(p)) \quad \lambda(p) = T_p\Lambda \quad \lambda_0(p) = T_p\pi^{-1}\pi(p) \\
& \mu(p) = \lambda(p) \cap \sqrt{-1}\lambda(p) \quad c_{\lambda/\lambda_0}(p) = \dim_{\mathbb{R}}(\lambda(p) \cap \lambda_0(p)) \\
& \delta_\lambda(p) = \dim_{\mathbb{C}}(\mu(p)) \quad \gamma_{\lambda/\lambda_0}(p) = \dim_{\mathbb{C}}(\lambda(p) \cap \sqrt{-1}\lambda(p) \cap \lambda_0(p)).
\end{aligned}
$$

We often drop p in the above notations. We define

$$
(1.2)\qquad L_{\lambda/\lambda_0} = \sigma(u, v^c)|_{u,v\in\lambda_0^\mu},
$$

(where $\lambda_0^\mu = ((\lambda_0 \cap \mu^\perp) + \mu)/\mu$ with $\cdot^\perp$ denoting the symplectic orthogonal). Its kernel being $(\lambda \cap \lambda_0/\mu \cap \lambda_0)^{\mathbb{C}}$, one gets

$$
(1.3)\qquad \operatorname{rank}(L_{\lambda/\lambda_0}) = n - c_{\lambda/\lambda_0} - \delta_\lambda + 2\gamma_{\lambda/\lambda_0}.
$$

One also has

$$
(1.4)\qquad \operatorname{sgn}(L_{\lambda/\lambda_0}) = \frac{1}{2}\tau(\lambda, \sqrt{-1}\lambda, \lambda_0),
$$

where τ is the inertia index in the sense of [K-S 1]. We shall denote by $s^\pm_{\lambda/\lambda_0}$ the numbers of respectively positive and negative eigenvalues for L_{λ/λ_0}. Now let M be a C^2–submanifold of $X^{\mathbb{R}}$, T^*_MX the conormal bundle to M in X, p a point of $\dot{T}^*_MX$, z_o the projection $\pi(p)$. If ϕ is a C^2-function at z_o with $\phi|_M \equiv 0$ and $d\phi(z_o) = p$, then for $\lambda_M = TT^*_MX$, one gets

$$
(1.5)\qquad L_{\lambda_M/\lambda_0} \sim \partial\bar{\partial}\phi|_{T^{\mathbb{C}}M} \quad (T^{\mathbb{C}}M = TM \cap \sqrt{-1}TM),
$$

where "$\sim$" means equivalence in signature and rank (cf. [S] and also [D'A-Z 2] as for codim $M > 1$). We shall write $s^\pm_M$ instead of $s^\pm_{\lambda_M/\lambda_0}$, and similarly set $c_M = c_{\lambda_M/\lambda_0}$, $\gamma_M = \gamma_{\lambda_M/\lambda_0}$, $L_M = L_{\lambda_M/\lambda_0}$, and so on. Let

$$
d_{\lambda/\lambda_0} = \frac{1}{2}[c_{\lambda/\lambda_0} + n - \delta_\lambda - \operatorname{sgn}(L_{\lambda/\lambda_0})].
$$

By (1.3) one has $d_{\lambda/\lambda_0} = c_{\lambda/\lambda_0} + s^-_{\lambda/\lambda_0} - \gamma_{\lambda/\lambda_0} (= n - \delta_\lambda + \gamma_{\lambda/\lambda_0} - s^+_{\lambda/\lambda_0})$. Let $D^b(X)$ denote the derived category of the category of bounded complexes of sheaves and $D^b(X; p)$, $p \in \dot{T}^*X$, denote the localization of $D^b(X)$ by the null-system

$\{\mathcal{F}; \mathrm{SS}(\mathcal{F}) \not\ni p\}$ (cf. [K-S 1] for the definition of the microsupport SS). Let χ be a germ of a contact transformation between neighborhoods of p and $q = \chi(p)$ and let ϕ_K be a quantization of χ by a kernel K (i.e., a simple sheaf with shift n on Λ^a_χ the antipodal to the graph of χ). Assume that χ transforms Λ to Λ'. According to [K-S 1], if $\mathcal{F}$ is simple along Λ with shift b at p, then $\Phi_K(\mathcal{F})$ is simple along Λ' with shift $b - \frac{1}{2}(\operatorname{sgn} L_{\lambda/\lambda_0}(p) - \operatorname{sgn} L_{\lambda'/\lambda_0}(q)) = b + (d_{\lambda/\lambda_0}(p) - d_{\lambda'/\lambda_0}(q)) - \frac{1}{2}(c_{\lambda/\lambda_0}(p) - c_{\lambda' \cap \lambda_0}(q))$ at q.

PROPOSITION 1.1. *Let Λ_1 and Λ_2 be $\mathbb{R}$–Lagrangian conic submanifolds of $\dot{T}^*X$ in a neighborhood of p. We assume that $\Lambda_1 \cap \Lambda_2$ is $\mathbb{I}$–regular (i.e., regular for $\sigma^{\mathbb{I}}$) and that*

$$\operatorname{codim}_{\lambda_1(p)}(\lambda_1(p) \cap \lambda_2(p)) = 1.$$

We may then find two contact transformations χ, from neighborhoods of p and q, such that

$$\chi(\Lambda_i) = T^*_{M_i}X, \quad \operatorname{codim} M_i = 1, \ i = 1, 2, \tag{1.6}$$

and with one satisfying

$$s^-_{M_i}(q) = 0, \ i = 1, 2, \tag{1.7}$$

and the other satisfying

$$\text{(i)} s^-_{M_2}(q) = 1, \ s^-_{M_1}(q) = 0 \quad or \quad \text{(ii)} s^-_{M_1}(q) = 1, \ s^-_{M_2}(q) = 0. \tag{1.8}$$

(*Cf.* [*D'A-Z* 3] *for the point* (1.7).)

Proof. As remarked by A. D'Agnolo in [D'A-Z 3], we must have an inclusion $\lambda_1 \cap \sqrt{-1}\lambda_1 \subset \lambda_2 \cap \sqrt{-1}\lambda_2$ or $\lambda_1 \cap \sqrt{-1}\lambda_1 \supset \lambda_2 \cap \sqrt{-1}\lambda_2$. Assume we have the first inclusion. Let $(z, \zeta), z = x + \sqrt{-1}y, \zeta = \xi + \sqrt{-1}\eta$ be coordinates in $e = T_pT^*X$, and let $l_1 = \{\zeta = 0\}$. According to [T], the problem is reduced to find a $\mathbb{C}$-Lagrangian plane $l_0 \subset e, l_0 \supset \nu$:

$$\begin{cases} e = l_0 \oplus l_1, \ l_0 \cap \lambda_i = \nu^{\mathbb{R}} (\text{the real line spanned by } H^{\mathbb{R}}(\alpha^{\mathbb{R}})) \\ s^-_{\lambda_i/l_0} = 0 \ i = 1, 2 \text{ in case } (1.7) \\ s^-_{\lambda_2/l_0} = 1, s^-_{\lambda_1/l_0} = 0 \text{ or } s^-_{\lambda_1/l_0} = 1, s^-_{\lambda_2/l_0} = 0 \text{ in case } (1.8) \end{cases}$$

To this end we set $\mu = (\lambda_1 \cap \sqrt{-1}\lambda_1) + \nu$, and replace e by $e' = \mu^\perp/\mu$. This is the same as assuming L_{λ_1/l_0} is non-degenerate from the beginning. We then reason as in [S] and reduce the above problem in $\mathbb{C} \times \mathbb{C}$ with $\lambda_1 = \{(x; \sqrt{-1}\eta)\}, \lambda_2 = \{(0; \zeta)\}$ if $\lambda_2 \cap \sqrt{-1}\lambda_2 \neq 0$ (resp. $\lambda_2 = \{(x; \epsilon x + \sqrt{-1}\eta)\}$ with $\epsilon \neq 0$ if $\lambda_2 \cap \sqrt{-1}\lambda_2 = 0$). (Note that the case listed as (a) in [S] cannot happen due to the $\mathbb{I}$–regularity of $\Lambda_1 \cap \Lambda_2$.) In case $\lambda_2 = \{(0; \zeta)\}$ one takes $l_0 = \{(s\zeta; \zeta)\}$, $s \in \mathbb{R}^+$ (resp. $s \in \mathbb{R}^-$) and gets $s^-_{\lambda_1/l_0} = 0$ (resp. 1) with $s^-_{\lambda_2/l_0} = 0$ for both choices of s. This gives (1.7) (resp. (1.8) (ii)) in this case.

In the other case one remarks that if $\cdot^{c_{\lambda_i}}$ denotes the conjugation in $\lambda_i + \sqrt{-1}\lambda_i$, then $(z;\zeta)^{c_{\lambda_1}} = (\bar{z}; -\bar{\zeta})$, $(z; \epsilon z + \zeta)^{c_{\lambda_2}} = (\bar{z}; -\bar{\zeta} + \epsilon\bar{z})$. Thus if $l_0 = \{(s\zeta;\zeta)\}$, and if $u = (s\zeta;\zeta) \in l_0$, then

$$\begin{aligned} L_{\lambda_1/l_0}(u,u) &= 2s(\xi^2+\eta^2) \\ L_{\lambda_2/l_0}(u,u) &= (2s - 2\epsilon s^2)(\xi^2+\eta^2). \end{aligned}$$

(For the second we just put $u = (s\zeta; \epsilon s\zeta + (1-\epsilon s)\zeta)$.) If one takes $s \in \mathbb{R}^+$, $|s| \ll 1$, one gets (1.7). As for (1.8) distinguish these cases:

(i) When $\epsilon > 0$ one takes $s \in \mathbb{R}^+$, $|s| \gg 1$ and gets $s^-_{\lambda_1/l_0} = 0$, $s^-_{\lambda_2/l_0} = 1$.
(ii) When $\epsilon < 0$, one takes $s \in \mathbb{R}^-$, $|s| \gg 1$, and gets $s^-_{\lambda_1/l_0} = 1$, $s^-_{\lambda_2/l_0} = 0$.

Q.E.D.

Let χ be a germ of contact transformation between a neighborhood of p and a neighborhood of q: $\overset{\text{def.}}{=} \chi(p)$ which interchanges Λ_i to Λ'_i, put $d_i(p') = d_{\lambda_i/\lambda_0}(p')$, $d'_i(q') = d_{\lambda'_i/\lambda_0}(q')$, and similarly define $c_i(p') = c_{\lambda_i/\lambda_0}(p')$, $c'_i(q') = c_{\lambda'_i/\lambda_0}(q')$. We recall that when $c_i(p')$ and $c'_i(q')$ are constant for p' and q' close to p and q respectively, then $d_i(p') - d'_i(q')$ is also constant. Thus if χ satisfies (1.6) and (1.7), then

$$d_i(p') - d'_i(q') \equiv d_i(p) - 1 \ \forall i = 1,2 \quad (q' = \chi(p')).$$

(The above equality also holds for $i = 1$ (resp. $i = 2$), when χ satisfies (1.6) and (1.8) (i) (resp. (1.8) (ii)). We now assume that Λ_1^+, Λ_2^+ are $\mathbb{R}$-Lagrangian manifolds with boundary Σ in a neighborhood of p which intersect along Σ, and put $\Lambda = \Lambda_1^+ \cup \Lambda_2^+$; we call Λ a dihedral Lagrangian manifold. We extend Λ_i^+ to Λ_i, defined from both sides of Σ, and set $\Lambda_i^- = (\Lambda_i \setminus \Lambda_i^+) \cup \Sigma$, $\mathring{\Lambda}_i^\pm = \Lambda_i^\pm \setminus \Sigma$.

THEOREM 1.2. *Let* $\Lambda = \Lambda_1^+ \cup \Lambda_2^+$. *Assume that* $c_i(p') \equiv \text{const}\,\forall p' \in \Lambda_i$, *and that*

(1.9) $$\Lambda_1 \cap \Lambda_2 \text{is } \mathbb{I}\text{–regular, clean, of codim 1 in } \Lambda_i.$$

(*i*) *Then we may find a contact transformation* χ *between neighborhoods of p and* $q = \chi(p)$ *such that*

(1.10) $$\chi(\Lambda) = T^*_Y X \quad \text{where } Y \text{ is a } C^1\text{-hypersurface.}$$

Moreover Y is the union of two half–hypersurfaces $M_1^+ \cup M_2^+$ *with the* M_i*'s satisfying* (1.7) *or* (1.8).

(*ii*) *Let* $\mathcal{F}$ *be a simple sheaf with shift* $\frac{1}{2}c_1$ *in* $\mathring{\Lambda}_1^+$ *which satisfies* $\mathrm{SS}(\mathcal{F}) \subset \Lambda$. *By quantizing* χ *with a kernel K, we get*

(1.11) $$\Phi_K(\mathcal{F}) \simeq \begin{cases} \mathbb{Z}_Y[d_1(p)-1] & \textit{for } (1.7) \textit{ or } (1.8)\,(i) \\ \mathbb{Z}_Y[d_1(p)-2] & \textit{for } (1.8)\,(ii). \end{cases}$$

Proof. For χ_1 satisfying (1.7), we put $R = \pi(T^*_{M_1}X \cap T^*_{M_2}X)$, $T^*_{M_i}X^+ = \chi_1(\Lambda_i^+)$, $M_i^+ = \pi(T^*_{M_i}X)^+$, $q' = \chi_1(p')$; we also denote by $T^*_{M_i}X^-$ and M_i^- the other components of $T^*_{M_i}X \setminus (T^*_{M_1}X \cap T^*_{M_2}X)$ and $M_i \setminus R$ respectively. For χ_2 satisfying (1.8) we shall use similar notations $\tilde{R}$, $T^*_{\tilde{M}_i}X^\pm$, $\tilde{M}_i^\pm$, $\tilde{q}$ .... We recall that $s^-_{M_i}(q') - s^-_{\tilde{M}_i}(\tilde{q}')$ is constant for $q' \in T^*_{M_i}X$ near q, and that

$$s^-_{M_1} - s^-_{\tilde{M}_1} \neq s^-_{M_2} - s^-_{\tilde{M}_2}. \tag{1.12}$$

Clearly either $M_1^+ \cup M_2^+$ or $M_1^+ \cup M_2^-$ (resp. $\tilde{M}_1^+ \cup \tilde{M}_2^+$ or $\tilde{M}_1^+ \cup \tilde{M}_2^-$) is a C^1-hypersurface Y (resp. $\tilde{Y}$). But by (1.12), $\mathbb{Z}_Y$ is transformed, by a quantization of $\chi_2 \circ \chi_1^{-1}$, to a complex whose shifts are different in the two components of $T^*_{\tilde{M}_i}X^+ \setminus (T^*_{\tilde{M}_1}X \cap T^*_{\tilde{M}_2}X)$. Thus by [K-S 1, Prop. 6.2.1], in the extension stated in [D'A-Z 1], we have $\chi_2 \circ \chi_1^{-1}(T^*_Y X) \neq T^*_{\tilde{Y}}X$. In conclusion $\chi = \chi_1$ or $\chi = \chi_2$ satisfies (i).

As for (ii), if Φ_{K_1} (resp. Φ_{K_2}) is a quantization of χ_1 (resp. χ_2), then either $SS(\Phi_{K_1}(\mathcal{F})) = T^*_Y X$ or $SS\left(\Phi_{K_2}(\mathcal{F})\right) = T^*_{\tilde{Y}}X$. A direct computation of shifts then gives (1.11). Q.E.D.

Remark 1.3. Given Λ_1^+, Λ_2^+, Λ_2^- with $\Lambda_1^+ \cap \Lambda_2^+$ smooth $\mathbb{I}$–regular of codim 1 in Λ_i^+, it is easy to see that in order to transform both $\Lambda_1^+ \cup \Lambda_2^+$ and $\Lambda_1^+ \cup \Lambda_2^-$ (by different χ) to $T^*_Y X$ with $Y \in C^1$, the cleaness of $\Lambda_1 \cap \Lambda_2$ is necessary.

Remark 1.4. When the intersection $T^*_{M_1}X \cap T^*_{M_2}X$ (M_i hypersurfaces) is clean of codimension 1, then one easily checks that the order of contact of M_1 and M_2 along $R = \pi(T^*_{M_1}X \cap T^*_{M_2}X)$ is exactly 2. In fact if for real coordinates $t = (t_1, t_2, t')$, one writes $M_1 = \{t_1 = 0\}$, $R = \{t_1 = t_2 = 0\}$, $M_2 = \{t_1 = h(t_2)\}$ ($h = O(t_2^2)$), $p = (0; dt_1)$, one gets $\lambda_{M_2} = \{u; t_1, \partial^2_{t_2}h\ u_2, \ldots): u_1 = 0\}$. Then $\lambda_{M_1} \cap \lambda_{M_2} \subset T(R \times_{M_2} T^*_{M_2}X)$ means $\partial^2_{t_2}h \neq 0$. Thus $R = M_1 \cap M_2$, and if one denotes by Σ_i, $i = 1, 2$, the closed half-spaces with boundary M_i and interior conormal q, one has $\Sigma_1 \supset \Sigma_2$ or $\Sigma_1 \subset \Sigma_2$. Notice that in passing from (1.7), to (1.8) the inclusions are reverted; thus in (1.7), $\Sigma_1 \supset \Sigma_2$ (resp. $\Sigma_1 \subset \Sigma_2$) corresponds to (ii) (resp. (i)) of (1.8). We also mention that the inclusion $\Sigma_1 \supset \Sigma_2$ or $\Sigma_1 \subset \Sigma_2$ in (1.7), is related to an intrinsic notion of "positivity" $\Lambda_1 > \Lambda_2$ or $\Lambda_1 < \Lambda_2$ defined in [D'A-Z 4].

Remark 1.5. Let $\Lambda_1 = T^*_M X$, $\Lambda_2 = T^*_S X$ where S, M are C^2-submanifolds of X with $S \subset M$; then the intersection $\Lambda_1 \cap \Lambda_2$ is always clean. We also assume $\sqrt{-1}p \notin T^*_S X$ (i.e. $S \times_M T^*_M X$ regular) and $\mathrm{codim}_M S = 1$. Then the orientation of S in M, which determines the positive and negative components $\Omega^\pm$ of $M \setminus S$ and $\Lambda_1^\pm$ of $\Lambda_1 \setminus (\Lambda_1 \cap \Lambda_2)$, determines also, via the Hamiltonian isomorphism, the components $\Lambda_2^\pm$ of $\Lambda_2 \setminus (\Lambda_1 \cap \Lambda_2)$. (Note that in our general notations the sign $\pm$ in $\Lambda_2^\pm$ has no geometric meaning.) Then if $\Lambda = \Lambda_1^+ \cup \Lambda_2^+$ (resp. $\Lambda = \Lambda_1^+ \cup \Lambda_2^-$) the complex $\mathcal{F}$ which satisfies $SS(\mathcal{F}) \subset \Lambda$ and which is simple with shift $\frac{1}{2}$ codim M along M is $\mathbb{Z}_{\bar{\Omega}^+}$ (resp. $\mathbb{Z}_{\Omega^+}$). For these complexes Th. 1.2 applies.

Section 2

Let M_i^+, $i = 1, 2$ be C^2-hypersurfaces of $X \simeq \mathbb{C}^n$ with boundary R which verify $TM_1|_R = TM_2|_R$, and let $p \in R \times_{M_i} T^*_{M_i}X$, $\pi(p) = z_o$.

LEMMA 2.1. *Let $Y = M_1^+ \cup M_2^+$ be a C^1-hypersurface, denote by Σ the closed half–space with boundary Y and interior conormal p, and assume*

$$s^-_{M_2}(p) = s^-_R(p) + 1. \tag{2.1}$$

Then

$$\mathcal{H}^1_\Sigma(\mathcal{O}_X)_{z_o} = 0. \tag{2.2}$$

Proof. Because of (2.1), R must be "generic"; i.e., in our case, $\dim(T^{\mathbb{C}}R) = n-2$. We take complex coordinates $z = x + \sqrt{-1}y$, $z = (z_1, z_2, z')$ such that $z_o = 0$ and

$$\begin{aligned} M_2 &= \left\{z\colon -y_1 = y_2^2 + \sum_{i=3}^{s^-+1} y_i^2 - \sum_{i=s^-+2}^{s^-+s^++1} y_i^2 + o(\cdot)^2\right\}, \\ R &= \{z \in M_2\colon\ y_2 = \phi(z', \bar{z}') + o(\cdot)^2\}, \end{aligned}$$

where ϕ is real valued, $\phi = O(|z'|)^2$ and where $\cdot$ denotes all arguments but y_1 (resp. y_1, y_2) in the first (resp. second) line. We also suppose that $p = -dy_1$ and $M_2^\pm = \{z \in M_2\colon \pm y_2 \geq \phi + o(\cdot)^2\}$. Fix $z_3 = \cdots = 0$ and in the (z_1, z_2)–plane define the sets

$$\begin{aligned} I &= \left\{y_1 = \epsilon,\ -\delta < y_2 \leq t\right\} \cup \left\{y_2 = t,\ \frac{-t^2}{2} < y_1 \leq \epsilon\right\} \\ J &= \left\{-\delta < y_2 < t,\ \epsilon - \frac{t}{3}(y_2 + \delta) < y_1 < \epsilon\right\}, \end{aligned}$$

with $\delta = \eta t$, $\epsilon = \frac{\eta t^2}{4}$, η small, $t \to 0$. Set $B_{ct} = \{|(x_1, x_2)| < ct\}$, $W = X \setminus \Sigma$; then $W \cap \mathbb{C}^2_{z_1,z_2} \supset B_{ct} \times I$ for any large c. By [K, Th. 5] the restriction

$$\mathcal{O}_X(B_{\frac{ct}{2}} \times J) \to \mathcal{O}_X(B_{ct} \times I)|_{B_{\frac{ct}{2}} \times (I \cap J)},$$

is surjective. In particular $(\mathcal{O}_X)_{z_o} \xrightarrow{\sim} \varinjlim_{B \ni z_o} \mathcal{O}_X(B \cap W)$. Q.E.D.

PROPOSITION 2.2. (*Cf.* [Z 4].) *Let W be a dihedron of $\mathbb{C}^n$ with C^1-boundary $Y = M_1^+ \cup M_2^+$, each "face" M_i^+ a C^2-manifold with boundary $R = M_1^+ \cap M_2^+$. Let $z_o \in Y$,* *denote by p_o the exterior conormal to W at z_o, and set $S^- = \sup s^-_{M_i}(p')$ for $i = 1, 2$ and for $p' \in (M_i^+ \times_X T^*_{M_i}X) \cap B$ (B a neighborhood of p_o). Then*

$$\varinjlim_{B \ni z_o} H^j(W \cap B, \mathcal{O}_X) = 0 \quad \forall j > S^-. \tag{2.3}$$

Proof. We choose coordinates such that $z_o = 0$, $p_o = (1, 0, \ldots)$. We can then describe Y (resp. W) as a graph $x_1 = g(\cdot)$ (resp. as a subgraph $x_1 < g(\cdot)$) with $g(\cdot) = o(\cdot)$ where "$\cdot$" denotes all arguments but x_1. We put

$$\phi = -\log(g - x_1) + c|z|^2,$$

and denote by $s^-_\phi(z)$ the number of negative eigenvalues of the Levi-form $\bar{\partial}\partial\phi(z)$. Let $S = \{z = z^* + \mathbb{R}p_o | z^* \in R\}$, and let $\overset{\circ}{W}_i$ be the components of $W \setminus S$. Clearly $g, \phi \in C^2(W \setminus S) \cap C^1(W)$ and for $z = z^* + rp_o \in \overset{\circ}{W}_i$ we have

$$\bar{\partial}\partial\phi(z)^t(\bar{w}, w) = (g - x_1)^{-2} \left(\partial(g - x_1) \cdot w \bar{\partial}(g - x_1) \cdot \bar{w}\right) - (g - x_1)^{-1} \bar{\partial}\partial g(z^*)^t(\bar{w}, w) + c|w|^2.$$

Thus if the projection w' of w on $T^{\mathbb{C}}M$ satisfies $-\bar{\partial}\partial g(z^*)^t \bar{w}' w' \geq 0$ and if c is large enough, then for suitable c',

$$\bar{\partial}\partial\phi(z)(\bar{w}, w) > c'|w|^2.$$

It follows that

$$s^-_\phi(z) = s^-_{M_i}(z^*) \quad \forall z \in \overset{\circ}{W}_i. \tag{2.4}$$

We make now a C^0-change of holomorphic derivatives ∂_{z_i} such that

$$T_z^{\mathbb{C}} S = \operatorname{Span}\{\partial_{z_1}, \ldots, \partial_{z_{n-1}}\} \quad \forall z \in S.$$

We also write $\bar{\partial}'$ instead of $\bar{\partial}_{z_1}, \ldots, \bar{\partial}_{z_{n-1}}$. By noticing that W is foliated by the (C^1) level surfaces $Y_r = \{g - x_1 = r\}$ $r \in \mathbb{R}^+$, one concludes that for $\phi_i = \phi|_{\bar{W}_i}$,

$$\partial\phi_1|_S \equiv \partial\phi_2|_S,$$

which implies

$$\bar{\partial}'\partial\phi_1|_S \equiv \bar{\partial}'\partial\phi_2|_S.$$

The argument which leads to (2.4) implies $s^-(\bar{\partial}'\partial'\phi_i) = s^-(\bar{\partial}'\partial'\phi_i|_{\mathbb{C}^{n-2}_{z_2 \ldots z_{n-1}}})$ and $s^+(\bar{\partial}'\partial'\phi_i) = n - 1 - s^-(\bar{\partial}'\partial'\phi_i)$; in particular, $\bar{\partial}'\partial'\phi_i$ is non-degenerate. It follows that we can diagonalize $\bar{\partial}\partial\phi_1$ (or $\bar{\partial}\partial\phi_2$) by a change of holomorphic derivatives preserving $\operatorname{Span}\{\partial'\}$. Thus in a suitable basis of the ∂_{z_i}'s,

$$\bar{\partial}\partial\phi_1|_{\bar{W}_1}, \ \bar{\partial}\partial\phi_2|_{\bar{W}_2} \text{ are diagonal.}$$

Thus from (2.4) (possibly by a permutation of the ∂_{z_i}'s) we get

$$\sum_{i\,j\le n} \partial_{z_i}\bar\partial_{z_j}\phi(z)\bar w_i w_j - \sum_{i\le S^-} \partial_{z_i}\bar\partial_{z_i}\phi(z)|w_i|^2 \ge c(z)|w''|^2 \tag{2.5}$$

($w'' = (w_{n-S^-+1}, \dots, w_n)$). Once (2.5) is established, we can adapt the calculus of L^2-norms with weight $e^{-\phi}$ by [H, Ch. 4–5], and get the conclusion (cf. [Z 4 Th. 2.2 and Remark 2.4]). Q.E.D.

Remark 2.3. Let W be a dihedron with transversal faces M_1^+, M_2^+, and suppose TW is non-convex. In this case, if we define $S^- = \sup(\sup_{N(W)^{oa}} s^-_{M_1}, \sup_{N(W)^{oa}} s^-_{M_2}, \sup_{N(W)^{oa}} s^-_R + 1)$, where $N(W)^{oa}$ is the exterior conormal cone to W, then (2.3) still holds true for this new S^-. In fact if we set $S_i = \{z \in W; z - z^* \in R \times_{M_i} T^*_{M_i}X\}$, then $\phi = -\log\delta + c|z|^2$ is C^1 on W and C^2 on W_i. Moreover $s^-_\phi \le S^-$ by the same argument as in Proposition 2.2. Thus [Z 4] still applies.

Let Λ_i^+, $i = 1, 2$ be $\mathbb{R}$-Lagrangian submanifolds of T^*X with boundary Σ which intersect along Σ, let $\Lambda = \Lambda_1^+ \cup \Lambda_2^+$, and let $c_i = c_{\lambda_i/\lambda_0}$. Recall the bifunctor $\mu\,\mathrm{hom}(\cdot,\cdot)$ from [K-S 1].

THEOREM 2.4. *Let* $c_i \equiv$ const *in* Λ_i *and suppose* $\Lambda_1 \cap \Lambda_2$ *is regular (for* $\sigma^{\mathrm{I\!I}}$*) clean of* codim 1 *in* Λ_i. *Let* $\mathcal{F}$ *be simple with shift* $\frac{1}{2}c_1$ *in* $\overset{\circ}{\Lambda}{}_1^+$ *and satisfy* $\mathrm{SS}(\mathcal{F}) \subset \Lambda$. *Then*

$$H^j\mu\,\mathrm{hom}(\mathcal{F}, \mathcal{O}_X)_p = 0 \quad \forall j \notin [d_1(p), \sup_{i,p'} d_i(p')] \quad \textit{with } p' \in \Lambda_i^+ \cap \pi^{-1}(B). \tag{2.6}$$

Proof. Let χ be the contact transformation between neighborhoods of p and q defined in Theorem 1.2. We have $\chi(\Lambda) = T^*_Y X$ where $Y = M_1^+ \cup M_2^+$ is a C^1-hypersurface with the M_i's satisfying (1.7) or (1.8). By quantization we transform

$$\Phi_K(\mathcal{F}) \xrightarrow{\sim} \begin{cases} \mathbb{Z}_Y[d_1 - 1] & \text{in (1.7) and in (1.8) (i)} \\ \mathbb{Z}_Y[d_1 - 2] & \text{in (1.8) (ii).} \end{cases}$$

If Σ is the closed half-space with boundary Y and interior conormal q, then

$$\mu\,\mathrm{hom}(\mathcal{F}, \mathcal{O}_X) \simeq R\Gamma_\Sigma(\mathcal{O}_X)[-d_1 + 1] \text{ (or } [-d_1 + 2]).$$

Now $H^j R\Gamma_\Sigma(\mathcal{O}_X)[+1] = 0\,\forall j < 0$ and even $\forall j \le 0$ in (1.8) (due to Lemma 2.1). Thus we get 0 in (2.6) $\forall j < d_1(p)$. As for the vanishing for large j, we remark that for $q' = \chi(p')$,

$$s^-_{M_i}(q') \equiv d_i(p') - d_i(p) \quad (\text{resp.} \equiv d_i(p') - d_i(p) + 1)$$

if $s^-_{M_i}(q) = 0$ (resp. $s^-_{M_i}(q) = 1$). The conclusion then follows from Proposition 2.2. Q.E.D.

For $A \subset X$ locally closed, we put $\mu_A(\mathcal{F}) \overset{\text{def.}}{=} \mu\hom(\mathbb{Z}_A, \mathcal{F})$.

COROLLARY 2.5. (*Cf.* [D'A-Z 3].) *Let $S \subset M$ be (C^2)-submanifolds of X with* $\operatorname{codim}_M S = 1$, $\sqrt{-1}p \notin T^*_S X$, *let Ω ($\bar{\Omega}$) be an open (closed) component of $M \setminus S$, and put $\Lambda_1 = T^*_M X$, $\Lambda_2 = T^*_S X$. Then*

$$H^j\mu_\Omega(\mathcal{O}_X)_p = 0 \quad \forall j \notin [d_1(p), \sup_{p'} d_1(p') \vee (\sup_{p'} d_2(p') - 1)] \quad (p' \in \Lambda_i^+ \cap \pi^{-1}(B))$$

$$(2.7)\quad H^j\mu_{\bar{\Omega}}(\mathcal{O}_X)_p = 0 \quad \forall j \notin [d_2(p), \sup_{i,p'} d_i(p')] \quad (p' \in \Lambda_i^+ \cap \pi^{-1}(B)).$$

PROPOSITION 2.6. (*Cf.* [Z].) *Let $W \subset X \simeq \mathbb{C}^n$ be a dihedron with transversal faces M_1^+, M_2^+ and with generic "edge" $R = M_1^+ \cap M_2^+$ in a neighborhood of $z_o \in \partial W$. Denote by $N(W)^{oa}$ the exterior conormal cone to W and put*

$$s^- = \begin{cases} \inf_{p' \in N_{z_o}(W)^{oa}} s_R^-(p') & \text{if } TW \text{ is convex} \\ \inf_{p' \in N_{z_o}(W)^{oa}} s_R^-(p') + 1 & \text{if } TW \text{ is non-convex} \end{cases}$$

Then if B ranges through a fundamental system of neighborhoods of z_o, we have

$$(2.8)\qquad \varinjlim_{B \ni z_o} H^j(W \cap B, \mathcal{O}_X) = 0 \quad \forall j < s^-,$$

(and $\varinjlim H^0(B, \mathcal{O}_X) \twoheadrightarrow \varinjlim H^0(W \cap B, \mathcal{O}_X)$ is surjective if $s^- \geq 1$).

Proof. Assume first TW is convex. Let p_1, p_2 be the unitary conormals to M_1, M_2 at z_o exterior to W; one has $\mu_W(\mathcal{O}_X) \xrightarrow{\sim} \mu_{\overset{\circ}{M_i}{}^+}(\mathcal{O}_X)[+1]$ at p_i $i = 1, 2$, and $\mu_W(\mathcal{O}_X) \xrightarrow{\sim} \mu_R(\mathcal{O}_X)[2]$ at $p \in \dot{N}_{z_o}(W)^{oa}/\mathbb{R}^+$, $p \neq p_1, p_2$. From the distinguished triangle in $D^b(X)$,

$$(\mathcal{O}_X)_{\bar{W}} \to R\Gamma_W(\mathcal{O}_X) \to R\dot{\pi}_*\mu_W(\mathcal{O}_X),$$

one concludes that (2.8) is 0 for $j < s^-$, $j \neq 0$, and that $(\mathcal{O}_X)_{\bar{W}} \twoheadrightarrow R\Gamma_W(\mathcal{O}_X)$ is surjective when $s^- \geq 1$.

Now let TW be non-convex. In this case one has $\mu_W(\mathcal{O}_X) \simeq \mu_{M_i^+}(\mathcal{O}_X)[1]$ at p_i, $i = 1, 2$, and $\mu_W(\mathcal{O}_X) \simeq \mu_R(\mathcal{O}_X)[+1]$ at $p \in \dot{N}_{z_0}(W)^{oa}/\mathbb{R}^+$, $p \neq p_1, p_2$. By Corollary 2.5 we get the conclusion in the same way as in the preceding case. Q.E.D.

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Dipartmento di Matematica, Univesità v. Belzoni 7, 35131 Padova, Italy
zampieri@galileo.math.unipd.it