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PERIODIC MAPPINGS OF COMPLEX PROJECTIVE SPACE WITH AN ISOLATED FIXED POINT

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ABSTRACT. If $p = 3$ or 5 and $3 \leq n < p + 3$, then the homotopy type of $\mathbb{C}P^n$ contains only finitely many PL homeomorphism types with a locally linear PL action of the cyclic group of order p fixing an isolated point.

1. Introduction

A PL cohomology complex projective n–space is a piecewise linear, closed, orientable $2n$-manifold, M^{2n}, such that there is a class $x \in H^2(M; \mathbb{Z})$ with the property that $H^*(M; \mathbb{Z}) = \mathbb{Z}[x]/\left(x^{n+1}\right)$. The Pontrjagin class of M^{2n}, $p_*\left(M^{2n}\right) \in H^*(M; \mathbb{Q})$, is *standard* if $p_*\left(M^{2n}\right) = (1 + x^2)^{n+1}$. Complex projective n-space, $\mathbb{C}P^n$, has a standard Pontrjagin class. Let p be an odd prime and let G_p denote the cyclic group of order p. If M^{2n} admits a locally linear PL G_p action, then the number of components of the fixed point set, M^{G_p}, is at most p ([2], p. 378). If M^{G_p} consists of two components, then the action is said to be of Type II. If $M^{G_p} = F_1^{2k_1} \cup F_2^{2k_2}$, then $k_1 + k_2 = n - 1$ ([2], p. 378) and we will say that the action is of Type II_k where $k = \min(k_1, k_2)$. Actions of Type II_0 fix an isolated point and a codimension–2 locally flat submanifold, F^{2n-2}.

An action of type II_0 is *regular* if its restriction to the normal block bundle of M^{G_p} is a multiple of one irreducible complex representation of M^{G_p}. The *degree* of F^{2n-2} in M^{2n} is the integer d if $i_*[F]$ is dual to dx where i: $F^{2n-2} \subset M^{2n}$ is the inclusion mapping. An action of Type II_0 is *standard* if it is regular and the degree of F^{2n-2} in M^{2n} is one.

THEOREM A. *Suppose that M^{2n} is a PL cohomology projective n-space which admits a locally linear PL G_p action of Type II_0 for $p = 3$ or 5. If $n < p + 5$, then the degree of the fixed codimension-2 submanifold is one. If $n < p + 3$, then the action is standard and the Pontrjagin class of M^{2n} is standard.*

It is not known if the bound on n in the first conclusion in Theorem A is best possible. It is known that in some cases, one is the only possible degree of the codimension-2 fixed submanifold. If n is odd and F^{2n-2} is a submanifold of $\mathbb{C}P^n$ which is fixed by a smooth G_p action of Type II_0, then the degree of F^{2n-2} is one

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([6], Theorem A). The bound on n in the statement in Theorem A about Pontrjagin classes is best possible. If p is any odd prime and $n \geq p+3$, then there are infinitely many PL homotopy complex projective n-spaces with nonstandard Pontrjagin classes which admit locally linear PL G_p actions of Type II_0 ([4], Proposition 0.3($\ell.\ell.$ PL)). If $n \geq 2p+9$, then there are infinitely many smooth homotopy complex projective n-spaces with nonstandard Pontrjagin classes which admit smooth G_p actions of Type II_0 ([4], Theorem 0.3 (diff.)). These two results imply that if $d(p)(d'(p))$ is the smallest integer n such that there exists a smooth (PL) homotopy complex projective n-space with a nonstandard Pontrjagin class and a smooth $(\ell.\ell.\ PL)$ G_p action of Type II_0, then $d'(p) \leq p+3$ and $d(p) \leq 2p+9$. *It follows from these inequalities and Theorem A that if $p = 3$ or 5, then $d'(p) = p+3$ and $p+3 \leq d(p) \leq 2p+9$.*

If M^{2n} is a smooth cohomology complex projective n-space, then there is a constant $c_{M^{2n}}$, which depends only on the Pontrjagin class of M^{2n}, such that if M^{2n} admits a smooth G_p action of Type II_0, then the action and the Pontrjagin class of M^{2n} are standard if $p \geq c_{M^{2n}}$ ([5], Theorem A). It is known that $c_{M^{2n}} \geq n+2$ ([5], Corollary 2.4). Theorem A improves these inequalities for $p = 3$ or 5. *If $p = 3$ or 5 and M^{2n} admits a locally linear PL G_p action of Type II_0, then the action and the Pontrjagin class of M^{2n} are standard if $p \geq n-2$.* In other words, if $p = 3$ and $n \leq 5$ or $p = 5$ and $n \leq 7$ and M^{2n} admits a locally linear PL G_p action of Type II_0, then the action and the Pontrjagin class of M^{2n} are standard. These results are new except for the cases $p = 3$ and $n = 3$ or 4 and $p = 5$ and $n = 4$ ([5], Theorem C, [6], Theorem E).

THEOREM B. *Suppose that M^{2n} is a PL cohomology projective n-space which admits a locally linear PL G_p action of Type II_0. If $n < p+3$ and the action is standard, then the Pontrjagin class of M^{2n} is standard. If $n < p+1$ and the action is regular, then the action is standard and the Pontrjagin class of M^{2n} is standard.*

The bound on n in the first statement in Theorem B is sharp because if $n \geq p+3$, then there are infinitely many PL homotopy complex projective n-spaces with nonstandard Pontrjagin classes which admit locally linear PL *standard* G_p actions of Type II_0 ([4], Proposition 0.3 ($\ell.\ell.$ PL)). All the known examples of G_p actions of Type II_0 are standard and it is known that in certain cases, a standard action is the only possibility. If $n \leq 4$ and M^{2n} is a smooth cohomology projective n-space with a smooth G_p action of Type II_0, then the action and the Pontrjagin class of M^{2n} are standard ([4], Theorem A(i) (ii) ($n \leq 3, p \geq 3, n = 4, p \geq 5$), [6], Theorem E ($n = 4$, $p = 3$)). It is not known if every locally linear PL G_p action of Type II_0 on $\mathbb{C}P^n$ is standard. If n is odd, then every smooth G_p action of Type II_0 on $\mathbb{C}P^n$ is standard ([13], Theorem B).

If $n \geq 3$, then there are only finitely many PL homotopy complex projective n-spaces with a standard Pontrjagin class [14], so Theorems A and B imply that there are only finitely many PL homotopy complex projective n-spaces with a locally linear PL G_p action of Type II_0 for certain values of p or for certain types of actions.

THEOREM C. *If $p = 3$ or 5 and $3 \leq n < p+3$, then there are at most $2^{[n/2]-1}$ PL homotopy complex projective n-spaces which admit a locally linear PL G_p action of Type II_0.*

THEOREM D. *If $3 \leq n < p+3$, then there are at most $2^{[n/2]-1}$ PL homotopy complex projective n-spaces which admit a standard locally linear PL G_p action of Type II_0. If $3 \leq n < p+1$, then there are at most $2^{[n/2]-1}$ PL homotopy complex projective n–spaces which admit a regular locally linear PL G_p action of Type II_0.*

Theorem B has been proved for smooth G_p actions of Type II_0 and the results used to study the constants $c_{M^{2n}}$ in some special cases ([13], Theorems C and D). It is useful to have Theorem B in the PL category because of the evident sensitivity of the inequality $n < p+3$ in this category and because not all PL homotopy complex projective n–spaces are smooth ([14], [11], Theorems 1.1, 1.2 and 1.3). We will also need Theorem B for the proof of Theorem A. This paper will contain other examples of extensions of techniques used in the smooth category to the PL category. We will also offer strengthened versions of parts of Theorem A. For example, the statement in Theorem A about the degree of the codimension-2 fixed submanifold in the case $p = 5$ follows from Theorem E. Theorem F applies our results to complex projective n-space.

THEOREM E. *Suppose that M^{2n} is a PL cohomology projective n-space which admits a locally linear PL G_5 action of Type II_0 fixing a codimension-2 submanifold of degree d. If $n \leq 9$ or $n = 11$, then $d = 1$. If $n = 10$, then $d = 1$ or 3.*

THEOREM F. *If $n \leq 9$ or $n = 11$, then $\mathbb{C}P^n$ admits a locally linear PL G_5 action of Type II_0 if and only if the action is standard.*

This paper is organized as follows. Section 2 contains congruences for the degree and the signatures of the self-intersections of a codimension-2 locally flat submanifold of a PL cohomology projective n-space and represents an extension of a smooth category technique to the PL category. This section is independent of the discussion of group actions. In Section 3, we return to group actions, associate parameters with locally linear PL G_p actions of Type II_0 and formulate a version of the Atiyah-Singer g-Signature Formula (ASgSF) for this kind of action in terms of these parameters. Section 4 contains a discussion of the properties of the algebraic numbers which occur in this version of the Atiyah-Singer g-Signature Formula. Section 5 contains the proofs of Theorems B and A when $p = 3$, Theorems E and A when $p = 5$, Theorems C and D, and Theorem F in that order.

2. Codimension-2 locally flat submanifolds

Suppose that M^{2n} is a PL cohomology complex projective n-space and that $i\colon K^{2n-2} \subset M^{2n}$ is the inclusion map of a closed, connected, orientable locally flat submanifold of codimension-2. We say that the degree of K^{2n-2} is d if $i_*[K]$ is dual to dx. The degree of K^{2n-2} depends on the choice of a generator $x \in H^2(M; \mathbb{Z})$ and orientations of K^{2n-2} and M^{2n}. The orientations of M^{2n} and K^{2n-2} are chosen as follows. A generator $x \in H^2(M; \mathbb{Z})$ is chosen and M^{2n} is oriented by requiring that $x^n[M]$ is positive and K^{2n-2} is oriented by requiring that $(i^*x)^{n-1}[K]$ is positive if it is nonzero. With these conventions, the degree of K^{2n-2} is positive if it is nonzero because $(i^*x)^{n-1}[K] = d$. We will use the notation K_{dx}^{2n-2} to indicate that K^{2n-2} has degree d. For example, K_x^{2n-2} is a codimension–2 submanifold of degree one.

If $K^{2n-2} \subset M^{2n}$ is a closed, connected, orientable locally flat submanifold and s is a nonnegative integer, then the s-fold self-intersection of K in M is defined inductively using transversality in the PL category: $K^{(0)} = M^{2n}$, $K^{(1)} = K$, and if $K^{(s)} \subset M$ and $j\colon K^{(s)} \longrightarrow M^{2n}$ is transverse to K, then $K^{(s+1)} = j^{-1}(K)$. The dimension of $K^{(s)}$ is $2(n-s)$. For example, $K^{(n)}$ is a set of points. There is a chain of locally flat submanifolds $K^{(n)} \subset K^{(n-1)} \subset \cdots \subset K \subset M^{2n}$. We will be particularly interested in the signatures Sign $K^{(s)}$. They are zero if $n-s$ is odd. If $n-s$ is even, then it will turn out to be convenient to keep track of these signatures in the fashion indicated in our first definition.

Definition 2.1. Suppose that M^{2n} is a PL cohomology projective n-space and that $K_{dx}^{2n-2} \subset M^{2n}$ is a locally flat submanifold of degree d. If $0 \le k \le [n/2]$, then

$$s_k(d) = \begin{cases} \operatorname{Sign} K_{dx}^{(2k)}, & n \text{ even}, \\ \operatorname{Sign} K_{dx}^{(2k+1)}, & n \text{ odd}. \end{cases} \tag{2.2}$$

Note that $s_0(d) = \operatorname{Sign} M^{2n} = +1$ if n is even and the orientations are chosen in the manner described above, and $s_0(d) = \operatorname{Sign} K_{dx}$ if n is odd. Also, $s_{[n/2]}(d) = \operatorname{Sign} K_{dx}^{(n)} = d^n$ and we agreed at the beginning of this section that d is positive if it is nonzero because of our choice of orientation for K_{dx}^{2n-2}. It turns out that for the more interesting values of k, $0 \le k < [n/2]$, $s_k(d)$ can be expanded in terms of certain universal polynomials in d with rational coefficients and the signatures of self-intersections of K_x^{2n-2} and hence depend only on M^{2n} and d ([12] p. 170). These expansions are generally not practical, but they lead to useful congruences and divisibility results. If n is a positive integer, then $f(n)$ is $n!$ divided by a maximal power of 2.

PROPOSITION 2.3. *Suppose that M^{2n} is a PL cohomology projective n-space and that $K_{dx}^{2n-2} \subset M^{2n}$ is a locally flat submanifold of degree d. If $0 \le k \le [n/2]$, then*

$$f(n)s_k(d) \equiv \begin{cases} f(n)d^{2k} \operatorname{Sign} K_x^{(2k)} \bmod (d^{2k}(1-d^2)), & n \text{ even}, \\ f(n)d^{2k+1} \operatorname{Sign} K_x^{(2k+1)} \bmod (d^{2k+1}(1-d^2)), & n \text{ odd}. \end{cases} \tag{2.4}$$

Proof. Formula (2.4) is one of the generalizations of a smooth category technique to the PL category promised in the introduction. It is the generalization to the

PL category of these same congruences which hold in the smooth category ([12], Theorem 1.1). The key point is that if $K_{dx}^{2n-2} \subset M^{2n}$ is a locally flat submanifold of degree d and s is a nonnegative integer such that $n - s$ is even, and $L(M)$ is the total Hirzebruch L-class of M, then

$$\text{Sign} K_{dx}^{(s)} = \left\{\tanh^s dx L(M)\right\}[M]. \tag{2.5}$$

This equation in the smooth category is found in the literature ([15], p. 84, take $N = M, d = 1$, and keep in mind that x is an arbitrary cohomology class in this context). To see that this equation holds in the PL category, recall that if $K_{dx}^{2n-2} \subset M^{2n}$ is a locally flat submanifold with normal block bundle ν, then ν is a real 2-plane bundle ([9], [10], p. 254), and so the equation $\tau M^{2n}|K_{dx}^{(s)} = \tau K_{dx}^{(s)} \oplus s(\nu|K_{dx}^{(s)})$ together with the equations $\text{Sign}\, K_{dx}^{(s)} = L(K_{dx}^{(s)})[K_{dx}^{(s)}]$ and $L(s\,(\nu|K_{dx}^{(s)})) = (dx|K_{dx}^{(s)})^s \coth^s(dx|K_{dx}^{(s)})$ and naturality yield (2.5). It follows that $s_k(d)$ is expandable in terms of rational polynomials in d and the signatures $s_k(1)$ ([12], (2.9)) and (2.4) follows from this expansion ([12], (2.10)). □

COROLLARY 2.6. *Suppose that M^{2n} is a PL cohomology projective n-space and that $K_{dx}^{2n-2} \subset M^{2n}$ is a locally flat submanifold of degree d. If $0 \le k \le [n/2]$, then*

$$\begin{cases} f(n)s_k(d) \equiv 0 \pmod{d^{2k}}, & n \text{ even}, \\ f(n)s_k(d) \equiv 0 \pmod{d^{2k+1}}, & n \text{ odd}. \end{cases} \tag{2.7}$$

Proof. Formula (2.7) follows immediately from formula (2.4). □

Formula (2.7) will be one of our main theoretical tools when we return to group actions and try to show that d is one in certain situations, for example, $n < p+5$, $p = 3$ or 5, in Theorem A. Once it has been established that d is one in a given situation, the Pontrjagin class of M^{2n} is studied via the next proposition which relates $p_*(M^{2n})$ and the integers $s_k(1)$, $0 \le k \le [n/2]$, the signatures of the self–intersections of a codimension-2 submanifold of M^{2n} of degree 1.

PROPOSITION 2.8. *If M^{2n} is a PL cohomology projective n-space, then the Pontrjagin class of M^{2n} is standard if and only if $s_k(1) = 1, 0 < k < m$, if $n = 2m$, or if and only if $s_k(1) = 1, 0 \le k < m$, if $n = 2m+1$.*

Proof. If M^{2n} is a smooth cohomology projective n-space, oriented as above, then the Pontrjagin class of M^{2n} is standard if and only if $\text{Sign} K_x^{(s)} = 1$ for all s such that $n - s$ is even and $0 < s < n$ ([13], Lemma 2.20). This is true if M^{2n} is a PL cohomology projective n-space because the arguments used to prove the assertion involves only the determination of signatures via the Hirzebruch L-polynomials and the cohomological properties of the self-intersections $K_x^{(s)}$ ([13], Proof of Lemma 2.20) and these facts are the same in the PL category. The statement in Proposition 2.8 translates the condition $\text{Sign} K_x^{(s)} = 1$ for all s such that $n - s$ is even and $0 < s < n$ into the terminology of the functions $s_k(d)$. □

3. Parameters for Type II_0 G_p actions and the Atiyah-Singer g-Signature Formula

Suppose that M^{2n} is a PL cohomology projective n-space which admits a locally linear PL G_p action of Type II_0. We will associate two invariants with the action exactly as was done in the smooth case ([7], p. 388): the degree of F^{2n-2}, the codimension-2 component of the fixed point set and the complex representation of G_p determined by $\tau_* M^{2n}$, the restriction of the tangent block bundle to the isolated fixed point. We will assume that the orientations of M^{2n} and F^{2n-2} are chosen as indicated in the first paragraph of Section 2. Since the degree of F^{2n-2} satisfies $d \not\equiv 0 \pmod p$ ([2], pp. 378–383), d must be positive. This integer is the first invariant to be associated with a locally linear PL G_p action of Type II_0.

The orbit space M^{2n}/G_p is a manifold near the image of F^{2n-2} and the latter is a locally flat submanifold of codimension-2 and hence has a PL normal bundle with the structure of a real 2-plane bundle ([9], [10], p. 254). The pull back of this normal bundle to M^{2n} is therefore a PL bundle with the structure of an equivariant 2-plane bundle. We will assume that a generator g of the group G_p has been chosen so that the eigenvalue of the action of g on an equivariant complex structure of the normal bundle of F^{2n-2} in M^{2n} is $\lambda = \exp(2\pi i/p)$. The second invariant associated with a G_p action of Type II_0, $\tau_* M^{2n}$, is a complex representation of G_p of complex dimension n, and with the right choice of complex structure, this representation is a sum of 1-dimensional complex representations with eigenvalues contained in the set $\{\lambda^j\colon 1 \le j \le \mu\}$, where $\mu = (p-1)/2$. Let m_j be the multiplicity of the eigenvalue λ^j, $1 \le j \le \mu$. Note that $m_1 + m_2 + \ldots + m_\mu = n$. Taking the two invariants d and $\tau_* M^{2n}$ together, a $(\mu+1)$-tuple $(d; m_1, m_2, \ldots, m_\mu)$ of integers is associated with the G_p action of Type II_0. Let $DE_p\left(M^{2n}\right)$ (for *degrees* and *eigenvalues*) be the set of all such $(\mu+1)$-tuples of integers arising from locally linear PL G_p actions of Type II_0 on the PL cohomology complex projective n-space M^{2n}. Note that a G_p action of Type II_0 is *regular* if and only if its $(\mu+1)$-tuple has the form $(d; n, 0, \ldots, 0)$, that is, $m_1 = n$ and $m_j = 0$, $2 \le j \le \mu$, and it is *standard* if and only if its $(\mu+1)$-tuple has the form $(1; n, 0, \ldots, 0)$. Let $\alpha_j = (\lambda^j + 1)(\lambda^j - 1)^{-1}$, $1 \le j \le \mu$.

THEOREM 3.1. (*ASgSF for Locally Linear PL G_p Actions of Type II_0*). *Suppose that M^{2n} is a PL cohomology projective n-space and that p is an odd prime. If $(d; m_1, m_2, \ldots, m_\mu) \in DE_p(M^{2n})$, then*

$$(3.2)\quad \pm\,\alpha_1^{m_1}\alpha_2^{m_2}\cdots\alpha_\mu^{m_\mu} = \begin{cases} 1 + (\alpha_1^2 - 1)\sum\limits_{k=1}^{m} s_k(d)\alpha_1^{2k-2}, & n = 2m, \\ \alpha_1 + (\alpha_1^3 - \alpha_1)\sum\limits_{k=1}^{m} s_k(d)\alpha_1^{2k-2}, & n = 2m+1. \end{cases}$$

Proof. The ASgSF holds for *tame* actions ([16], p. 189) and any locally smooth G_p action in which the components of M^{G_p} are either 0–dimensional or of codimension-2 is tame ([4], Theorem 1.1), and so the ASgSF holds for locally linear PL G_p actions of Type II_0. The ASgSF for smooth G_p actions of Type II_0 on a smooth cohomology projective n-space M^{2n} is exactly the same as (3.2) except that $s_k(d)$ is replaced by Sign $F^{(2k)}$ if $n = 2m$ and by Sign $F^{(2k+1)}$ if $n = 2m + 1$, where F^{2n-2} is the codimension-2 component of M^{G_p} ([13], Theorem 2.1). Therefore (3.2) holds because of the above remarks about the validity of the ASgSF for locally linear PL G_p actions of Type II_0 and the fact that if the degree of F^{2n-2} is d, then the signatures of the self–intersections of F^{2n-2} in M^{2n} are the functions $s_k(d)$ (Definition 2.1). □

Formula (3.2) is a special case of a version of the ASgSF which exhibits the g-signature as an element of the ring $\mathbb{Z}[\alpha_1, \alpha_2, \dots, \alpha_{p-1}]$ and *separates* topology and number theory in the ASgSF in the sense that topology is concentrated in the *integer* coefficients in the expression for the g-signature and number theory is concentrated in the various *powers* of α_j, $1 \le j \le p-1$, present in the expression for the g-signature ([1],Theorem 2.2). Formula (3.2) exposes the two invariants we have associated with G_p actions of Type II_0, the degree of the codimension-2 fixed submanifold, d, and the representation at the isolated fixed point as determined by the string of multiplicities m_j, $1 \le j \le \mu$. Formula (3.2) leads to an ASgSF in terms of d, the signatures $s_k(1)$, and α_j, $1 \le j \le \mu$ ([7], Theorem 4.4, [13], Theorem 2.6 and Table 3.5).

We now turn to some important special cases of (3.2). We present our results first and then the proofs which involve pure number theory of the numbers α_j, $1 \le j \le \mu$, which will not be used in the rest of the paper.

PROPOSITION 3.3. *(ASgSF for Regular Locally Linear PL G_p Actions of Type II_0.) Suppose that M^{2n} is a PL cohomology projective n-space and that p is an odd prime. If $(d; n, 0, \dots, 0) \in DE_p(M^{2n})$ and $m = [n/2]$, then*

$$\alpha_1^{2m} = 1 + (\alpha_1^2 - 1) \sum_{k=1}^{m} s_k(d) \alpha_1^{2k-2}. \tag{3.4}$$

Note that G_3 actions are automatically regular since $\mu = 1$ and so it is useful to record (3.4) in the case $p = 3$. It is also useful at this point to introduce the notation $D_p(M^{2n})$ for the projection of $DE_p(M^{2n})$ on its first factor, i.e., $D_p(M^{2n})$ is the set of *degrees* of codimension-2 submanifolds which are fixed by a locally linear PL G_p action of Type II_0 on M^{2n}. Note that $DE_3(M^{2n}) = D_3(M^{2n})$. If t is a positive integer, let $\varepsilon(t) = (3^t + (-1)^{t-1})/4$. The next corollary follows immediately from (3.4) and the fact that $\alpha_1 = -i/\sqrt{3}$ if $p = 3$.

COROLLARY 3.5. *(ASgSF for Locally Linear PL G_3 Action of Type II_0) Suppose that M^{2n} is a PL cohomology projective n-space. If $d \in D_3(M^{2n})$ and $m = [n/2]$, then*

$$\varepsilon(m) = \sum_{k=1}^{m} (-1)^{k-1} 3^{m-k} s_k(d). \tag{3.6}$$

It is clearly desirable to be able to simplify (3.2) for a given prime p to something like (3.4), a *single* equation for $n = 2m$ and $n = 2m + 1$ and an equation in which the $\pm$ ambiguity on the left hand side is gone. We will see that $p = 3$ is not the only prime for which this is possible.

Definition 3.7. We will say that an odd prime p has a special Type II_0 ASgSF if $(d; m_1, m_2, \ldots, m_\mu) \in DE_p(M^{2n})$ and $m = [n/2]$, then (3.2) reduces to

$$(3.8) \qquad \alpha_1^{r_1}\alpha_2^{r_2}\cdots\alpha_\mu^{r_\mu} = 1 + (\alpha_1^2 - 1)\sum_{k=1}^{m} s_k(d)\alpha_1^{2k-2},$$

where r_j is a nonnegative even integer, $1 \le j \le \mu$. If $n = 2m$, then $r_1 = m_1$, if $n = 2m + 1$, then $r_1 = m_1 - 1$, and, for every n, $r_j = m_j$, $2 \le j \le \mu$.

Note that if p has a special Type II_0 ASgSF and $r_1, r_2, \ldots, r_\mu$ satisfy the conditions in Definition 3.7, then, in particular, $r_j \ge 0$, $1 \le j \le \mu$, and $r_1 + r_2 + \cdots + r_\mu = 2m$. Since (3.4) holds for every d in $DE_3(M^{2n}) = D_3(M^{2n})$, it follows that $p = 3$ has a special Type II_0 ASgSF. Our next result shows that a few more primes have this property.

PROPOSITION 3.9. *If $p = 3, 5, 7, 11$ or 13, then p has a special Type II_0 ASgSF. In particular, if $p = 3, 5, 7, 11$ or 13 and $(d; m_1, m_2, \ldots, m_\mu) \in DE_p(M^{2n})$, then $m_1 \equiv n \pmod 2$ and m_j is even, $2 \le j \le \mu$.*

We are now ready to begin the proofs of Propositions 3.3 and 3.9. To do this, we need to review results about the image of the ASgSF in the ring of integers mod 4. Let $W = \mathbb{Z} + 2\mathbb{Z}[\lambda]$ and $W_{(p)} = \mathbb{Z}[1/p] + 2\mathbb{Z}[\lambda/p]$. Note that $\mathbb{Z}[\alpha_1, \alpha_2, \ldots, \alpha_{p-1}] \subset W_{(p)}$, i.e., $p\alpha_j \in W$, $1 \le j \le p - 1$ ([1], Lemma 4.3). The g-signature is therefore an element of $W_{(p)}$ ([1], Theorem 2.2).

LEMMA 3.10 ([1], Lemma 7.8). *There are 2^{p-2} distinct ring homomorphisms $\eta\colon W_{(p)} \longrightarrow \mathbb{Z}/4\mathbb{Z}$ satisfying the conditions $\eta(1) = 1$, $\eta(\alpha_j) = \pm 1$, and*

$$(3.11) \qquad \eta(\alpha_j) = 1 + \sum_{k=0}^{(p-3)/2} \eta\left(2\lambda^{(2k+1)j}\right),$$

if $1 \le j \le p - 1$.

The homomorphisms η are constructed by considering the 2^{p-2} distinct *additive* homomorphisms $\widehat{\eta}\colon \mathbb{Z}[\lambda]/2\mathbb{Z}[\lambda] \longrightarrow \mathbb{Z}/2\mathbb{Z}$ and then noting that the maps η can be constructed in such a way that $\eta(2\lambda^j) = 2\widehat{\eta}(\lambda^j)$ and (3.2) is satisfied ([1], proof of Lemma 7.8). We will see very soon that distinct mappings η on $W_{(p)}$ may agree on all the numbers α_j, $1 \le j \le p - 1$. Note that a given $\widehat{\eta}$ takes an odd number of the powers $\lambda, \lambda^2, \ldots, \lambda^{p-1}$ in $\mathbb{Z}[\lambda]/2\mathbb{Z}[\lambda]$ to 1 in $\mathbb{Z}/2\mathbb{Z}$ and hence the total number of such $\widehat{\eta}$'s is $\binom{p-1}{1}+\binom{p-1}{3}+\cdots+\binom{p-1}{p-2}=2^{p-2}$. We will associate with each $\widehat{\eta}$, and its induced η, a binary number of length $p-1$ with an odd number of 1's, a 1 in position k indicating that λ^k is sent to 1, subscripts indicating the positions of the 1's. For example, $\widehat{\eta}_1$ sends λ^1 to 1, others to 0. Note that for $p=5$, $\eta_4(\alpha_j)=\eta_{1,2,3}(\alpha_j)=+1$, $j=1,2$, and $\eta_4(\alpha_j)=\eta_{1,2,3}(\alpha_j)=-1$, $j=3,4$, i.e., for $p=5$, η_4 and $\eta_{1,2,3}$ agree on the ring $\mathbb{Z}[\alpha_1,\alpha_2,\alpha_3,\alpha_4]$ and yet are distinct as mappings on $W_{(5)}$. We will use this control over ± 1 in the proofs that follow.

Proof of Proposition 3.3. Suppose that $(d; n, 0, \ldots, 0) \in DE_p(M^{2n})$. It follows immediately from (3.2) that

$$\pm\alpha_1^{2m} = 1 + (\alpha_1^2 - 1)\sum_{k=1}^{m} s_k(d)\alpha_1^{2k-2}. \tag{3.12}$$

Assume that (3.12) holds with a minus sign on the left hand side. If *any* of the homomorphisms η of Lemma 3.10 are applied, the result is the contradiction $-1 \equiv 1 \pmod 4$ because $\eta(\alpha_1)=\pm 1$. Therefore, (3.12) must hold with the plus sign on the left hand side and this is (3.4). □

Proof of Proposition 3.9. The proof begins by noting that $p=3$ has a special Type II_0 ASgSF because of Proposition 3.3 and the fact that G_3 actions are automatically regular. As for $p=5, 7, 11$ or 13, the argument begins with the observation that for each of these primes, Lemma 3.10 implies that there is a map $\psi_0\colon W_{(p)} \longrightarrow \mathbb{Z}/4\mathbb{Z}$ such that $\psi_0(\alpha_j)=+1$, $1\le j\le \mu$, and for each j, $1 \le j \le \mu-1$, there is a map $\psi_j\colon W_{(p)} \longrightarrow \mathbb{Z}/4\mathbb{Z}$ such that $\psi_j(\alpha_j)=-1$ and $\psi_j(\alpha_k)=+1$, $j<k\le\mu$. The table below describes these ring homomorphisms by recording the exponents in the powers of λ sent to 1 (mod 2) by the inducing homomorphism $\widehat{\eta}$.

	5	7	11	13
ψ_0	4	4	8	1, 2, 3, 4, 5, 6, 7
ψ_1	3	5	1, 2, 6	11
ψ_2		6	7	1, 6, 9
ψ_3			10	8
ψ_4			9	10
ψ_5				12

For example, if $p=5$, then $\psi_0=\eta_4$ and $\psi_1=\eta_3$ in the notation described above.

Armed with these mappings, we can complete the proof of the proposition. The map ψ_0 can be used to show that (3.2) must hold with the plus sign, because assuming the minus sign leads to the contradiction $-1\equiv 1 \pmod 4$ after application of ψ_0. Similarly, ψ_j, $1\le j\le\mu-1$, can be used to show that $m_1 \equiv n \pmod 2$ and m_j is even, $2 \leq j \leq \mu - 1$. It follows that m_μ is even and so the primes $p = 5$ 7, 11 and 13 have a special Type II_0 ASgSF. $\square$

We remark that there are some general results of this type. If the first factor of the class number of p is odd, then for smooth actions of Type II_0, $m_1 \equiv n \pmod 2$ and m_j is even, $2 \leq j \leq \mu$ ([8], 3.11).

4. The algebraic numbers α_j

If p is an odd prime and $\mu = (p-1)/2$, the polynomial

$$m_p(x) = \sum_{k=0}^{\mu} \binom{p}{2k+1} x^{\mu-k}, \tag{4.1}$$

will play an important role when we apply the special signature formulas (3.4), (3.6), and (3.8) (for $p = 5, 7, 11$ or 13) to the study of Type II_0 G_p actions.

PROPOSITION 4.2. *If p is an odd prime and $\mu = (p-1)/2$, then $m_p(\alpha_j^2) = 0$, $1 \leq j \leq \mu$. In particular, if $1 \leq s \leq \mu$, then*

$$\sum_{1 \leq \iota_1, \iota_2, \cdots, \iota_s \leq \mu} \alpha_{\iota_1}^2 \alpha_{\iota_2}^2 \cdots \alpha_{\iota_s}^2 = (-1)^s p^{-1} \binom{p}{2s+1}, \tag{4.3}$$

where the summation in (4.3) *is taken over all possible $\binom{\mu}{s}$ products $\alpha_{\iota_1}^2 \alpha_{\iota_2}^2 \cdots \alpha_{\iota_s}^2$, $1 \leq \iota_1, \iota_2, \ldots, \iota_s \leq \mu$.*

Proof. The equation $(\alpha_j + 1)(\alpha_j - 1)^{-1} = \lambda^j$ implies that $(\alpha_j + 1)^p = (\alpha_j - 1)^p$ and so the binomial theorem implies that $m_p(\alpha_j^2) = 0$, $1 \leq j \leq \mu$. Equation (4.3) follows from the standard interpretation of the coefficients of a monic polynomial in terms of its roots. $\square$

Proposition 4.2 establishes that α_j, $1 \leq j \leq \mu$, is an algebraic number, and that α_j^{-1}, $1 \leq j \leq \mu$, is an algebraic integer. The polynomial $m_p(x)$ is irreducible over $\mathbb{Q}$ by Eisenstein's Criterion and is well known ([16], pp. 220–221).

COROLLARY 4.4. *The polynomial $m_p(x)$ is the minimal polynomial over the rationals for α_j^2, $1 \leq j \leq \mu$.*

The two extreme cases of (4.3), $s = 1$ and $s = \mu$, are especially interesting and we record them here. We will use the first of these in Section 5.

$$\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_\mu^2 = -\frac{(p-1)(p-2)}{6}, \tag{4.5}$$

$$\alpha_1^2\alpha_2^2 \cdots \alpha_\mu^2 = (-1)^\mu p^{-1}. \tag{4.6}$$

There is an interesting family of relations for the numbers α_j, $1 \leq j \leq \mu$, which contains elementary formulas like (4.5). If $x \in \mathbb{R}$, then the function $((x))$ is defined by the conditions $((x)) = x - [x] - \frac{1}{2}$ if $x \notin \mathbb{Z}$ and $((x)) = 0$ if $x \in \mathbb{Z}$. If $q \not\equiv 0 \pmod{p}$, then the *Dedekind sum* $s(q, p)$ ([3], p. 92) is defined by the equation

$$s(q, p) = \sum_{k=1}^{p} \left(\left(\frac{k}{p}\right)\right)\left(\left(\frac{kq}{p}\right)\right). \tag{4.7}$$

THEOREM 4.8 ([3], (3) p. 100). *If p is an odd prime and $q \not\equiv 0 \pmod{p}$, then*

$$\sum_{k=1}^{\mu} \alpha_k \alpha_{kq} = -2ps(q, p). \tag{4.9}$$

It follows from *Dedekind reciprocity* ([3], p. 93) that $s(1, p) = (p-1)(p-2)/12p$ and so (4.5) is (4.9) in the special case $q = 1$.

We close this section with two results about divisibility by the polynomial $m_p(x)$. The first is a result which is valid for any odd prime and is entirely elementary and will be stated without proof. The second is a special result for the polynomial $m_5(x)$.

LEMMA 4.10. *If $a(x) = \sum_{k=0}^{q} a_k x^k$ and $b(x) = \sum_{k=0}^{q-\mu} b_k x^k$ are rational polynomials where $q \geq \mu$ and $a(x) = m_p(x)b(x)$, then*

$$a_k = \sum_{\ell=0}^{q-\mu} \binom{p}{p - 2(k-\ell)} b_\ell, \quad 0 \leq k \leq q, \tag{4.11}$$

$$\sum_{k=0}^{q} a_k = 2^{p-1} \sum_{\ell=0}^{q-\mu} b_\ell. \tag{4.12}$$

If $a_k \in \mathbb{Z}, 0 \leq k \leq q-\mu$, then $b_\ell \in \mathbb{Z}$, $0 \leq \ell \leq q-\mu$ and $a_k \in \mathbb{Z}$, $q-\mu+1 \leq k \leq q$. In particular, if $a_k \in \mathbb{Z}, 0 \leq k \leq q-\mu$, then $a_k \in \mathbb{Z}, 0 \leq k \leq q$, and

$$\sum_{k=0}^{q} a_k \equiv 0 \pmod{2^{p-1}}. \tag{4.13}$$

LEMMA 4.14. *If $a(x) = \sum_{k=0}^{q} a_k x^k$ is a polynomial with rational coefficients such that $a(x) \equiv 0 \pmod{m_5(x)}$, then*

$$q = 2\colon \begin{cases} 9a_2 - 5a_1 + 5a_0 = 0, \\ 2a_2 - a_1 = 0. \end{cases} \tag{4.15}$$

$$q = 3\colon \begin{cases} 17a_3 - 9a_2 + 5a_1 - 5a_0 = 0, \\ 19a_3 - 10a_2 + 5a_1 = 0. \end{cases} \tag{4.16}$$

$$q = 4\colon \begin{cases} 161a_4 - 85a_3 + 45a_2 - 25a_1 + 25a_0 = 0, \\ 36a_4 - 19a_3 + 10a_2 - 5a_1 = 0. \end{cases} \tag{4.17}$$

$$q = 5\colon \begin{cases} 305a_5 - 161a_4 + 85a_3 - 45a_2 + 25a_1 - 25a_0 = 0, \\ 341a_5 - 180a_4 + 95a_3 - 50a_2 + 25a_1 = 0. \end{cases} \tag{4.18}$$

Proof. The proof of the above equations is very elementary and very tedious. By Corollary 4.4, $a(x) \equiv 0 \pmod{m_p(x)}$ if and only if $a(\alpha_1^2) = 0$. It follows from Proposition 4.2 in the special case $p = 5$ that $\alpha_1^2 = \frac{-5-2\sqrt{5}}{5}$ and $\alpha_2^2 = \frac{-5+2\sqrt{5}}{5}$. Equations (4.15) through (4.18) follow from the equation $a(\frac{-5-2\sqrt{5}}{5}) = 0$ and the irrationality of $\sqrt{5}$. □

5. TypeII$_0$ G_3 and G_5 actions

In this section, we return to locally linear PL G_p actions of Type II$_0$ on a PL cohomology complex projective n-space. We will prove Theorems B, A in the case $p = 3$, E, A in the case $p = 5$, C and D, and F, in that order. We will rephrase these results using the notation of Sections 2 and 3. We begin by rephrasing Theorem B.

THEOREM 5.1. *Suppose that M^{2n} is a PL cohomology projective n-space and that p is an odd prime. If $n < p + 3$ and $(1; n, 0, \dots, 0) \in DE_p(M^{2n})$, then the Pontrjagin class of M^{2n} is standard. If $n < p + 1$ and $(d; n, 0, \dots, 0) \in DE_p(M^{2n})$, then $d = 1$ and the Pontrjagin class of M^{2n} is standard.*

Proof. Suppose $(d; n, 0, \dots, 0) \in DE_p(M^{2n})$. It follows from (3.4) that if $m = [n/2]$, then

$$\sum_{k=1}^{m} (s_k(d) - 1)\, \alpha_1^{2k-2} = 0. \tag{5.2}$$

Now suppose that $(1; n, 0, \dots, 0) \in DE_p(M^{2n})$. It follows that (5.2) holds with $d = 1$ and , since $s_m(d) = d^n$, we have $s_m(1) = 1$ and so

$$\sum_{k=1}^{m-1} (s_k(1) - 1)\alpha_1^{2k-2} = 0. \tag{5.3}$$

If $n < p+3$, then $m-2 < \mu$ and so (5.3) implies that $s_k(1) = 1$, $0 < k < m$, since $m_p(x)$ is the minimal polynomial over $\mathbb{Q}$ of α_1^2 (Corollary 4.4) and the degree of $m_p(x)$ is μ. If n is odd and $1 \in D_p(M^{2n})$ then $s_0(1) = \operatorname{Sign} K_x = \operatorname{Sign} F = +1$, if orientations are chosen as in Section 2 ([6], Lemma 4.1). It therefore follows from Proposition 2.8 that the Pontrjagin class of M^{2n} is standard and we have verified the first statement of the Theorem. To verify the second statement, suppose that $(d; n, 0, \ldots, 0) \in DE_p(M^{2n})$ and consider (5.2) again. If $n < p+1$ then $m-1 < \mu$ and so (5.2) and Corollary 4.4 imply that $s_m(d) = d^n = 1$ and so $d = 1$ and the Pontrjagin class of M^{2n} is standard by the first statement. □

Theorem 5.1 is the same as Theorem B in the introduction. Theorem 5.1 can be applied to PL homotopy complex projective n-spaces. If M^{2n} is a PL homotopy $\mathbb{C}P^n$ and $n \geq 3$, then the PL homeomorphism type of M^{2n} is determined by an $(n-2)$-tuple of *splitting invariants* $(\sigma_2, \sigma_3, \ldots, \sigma_{n-1})$ [14]. The splitting invariants are integers and mod 2 integers and σ_k often appears as s_{2k} ([16], p. 191). The splitting invariants with even subscript, $\sigma_2, \sigma_4, \ldots, \sigma_{2[(n-1)/2]}$, are integers which determine the Pontrjagin class of M^{2n} ([14], [11], Theorem 3.1). If $n-s$ is even, then $\operatorname{Sign} K_x^{(s)} = 1 + 8\sigma_{n-s}$ and $\sigma_0 = 0$ because $K_x^{(n)}$ is a point, and so, with the orientation conventions as in Section 2, we have $\operatorname{Sign} K_x^{(n)} = 1$ [14]. Theorem 5.1 was proved for smooth cohomology $\mathbb{C}P^n$ ([13], Theorem 2.21) but not all PL homotopy complex projective n-spaces are smooth ([11], Theorems 1.1, 1.2, and 1.3) and so our next result uses the validity of Theorem 5.1 for all PL homotopy complex projective n-spaces.

PROPOSITION 5.4. *Suppose that M^{2n} is a PL homotopy complex projective n-space where $n \geq 3$ and $m = [n/2]$. If M^{2n} has integral splitting invariants $\sigma_2, \sigma_4, \ldots, \sigma_{2[(n-1)/2]}$ and $(1; n, 0, \ldots, 0) \in DE_p(M^{2n})$, then if $n = 2m+1$, $\sigma_{2m} = 0$, and if n is arbitrary, then*

$$\sum_{k=1}^{m-1} \sigma_{2(m-k)} \alpha_1^{2k-2} = 0. \tag{5.5}$$

If $n \geq p+3$ and $\sigma_2, \sigma_4, \ldots, \sigma_{2[(n-1)/2]}$ is a collection of integers with the properties that if $n = 2m+1$, then $\sigma_{2m} = 0$ and, if n is arbitrary, then (5.5) *holds, then there exists a PL homotopy complex projective n-space M^{2n} with integral splitting invariants $\sigma_2, \sigma_4, \ldots, \sigma_{2[(n-1)/2]}$ and $(1; n, 0, \ldots, 0) \in DE_p(M^{2n})$.*

Proof. Suppose that M^{2n} has integral splitting invariants $\sigma_2, \sigma_4, \ldots, \sigma_{2[(n-1)/2]}$ and $(1; n, 0, \cdots, 0) \in DE_p(M^{2n})$. If $n = 2m+1$, then $\operatorname{Sign} K_x^{(1)} = \operatorname{Sign} F = +1$ ([6], Lemma 4.1) and so $\sigma_{2m} = 0$. If n is arbitrary and $(1; n, 0, \ldots, 0) \in DE_p(M^{2n})$, then (5.3) holds and (5.5) is just (5.3) witten in terms of the splitting invariants. If $n \geq p+3$ and $\sigma_2, \sigma_4, \ldots, \sigma_{2[(n-1)/2]}$ is a collection of integers with the properties that if $n = 2m+1$, then $\sigma_{2m} = 0$ and, if n is arbitrary, then (5.5) holds, then there exists a PL homotopy complex projective n–space M^{2n} with integral splitting invariants $\sigma_2, \sigma_4, \ldots, \sigma_{2([n-2)/2]}$ and $(1; n, 0, \ldots, 0) \in DE_p(M^{2n})$ ([4], Theorem 0.3 ($\ell.\ell.\ PL$). Note that (5.5) appears here as $P(L_1) = 0$ ([4], p. 494) and σ_k is written as s_{2k} and the equation is written in terms of α_1^{-2}). □

We are now ready to turn to the proof of Theorem A. We begin with the special case $p = 3$ of Theorem A. Note that when Theorem B (Theorem 5.1) is applied to $p = 3$, we learn that if M^{2n} admits a standard $\ell.\ell.\ PL\ G_3$ action of Type II_0 and $n < 6$, then the Pontrjagin class of M^{2n} is standard. If we establish that $n < 8$ implies that the degree of any codimension-2 submanifold fixed by an $\ell.\ell.\ PL\ G_3$ action of Type II_0 is one, we will have established the second statement and the first statement of Theorem A in the case $p = 3$ because G_3 actions of Type II_0 are automatically regular and hence standard if the degree is one. If $f(n)$ is $n!$ divided by a maximal power of 2 (Proposition 2.3) and $\varepsilon(t) = (3^t + (-1)^{t-1})/4$ (Corollary 3.5), then let $a(n) = f(n)\varepsilon([n/2])$. Theorem A in the case $p = 3$ follows from our next result and Theorem 5.1 in the case $p = 3$.

THEOREM 5.6. *Suppose that M^{2n} is a PL cohomology projective n-space. If $d \in D_3(M^{2n})$, then d^2 divides $a(n)$ if n is even and d^3 divides $a(n)$ if n is odd. In particular, if $n < 8$, then $d = 1$.*

Proof. The first statement follows from a multiplication of both sides of (3.6) by $f(n)$ and then applying (2.7). A computation shows that if $n < 8$, then these divisibility conditions and the fact that $d \not\equiv 0 \pmod 3$ ([2], pp. 378–383) imply that $d = 1$. □

The divisibility conditions in Theorem 5.6 are extensions of divisibility conditions for the smooth category to the PL category. The possible divisors of d have been computed for $n \leq 22$ ([12], p. 175).

We now turn to the proof of Theorem A in the case $p = 5$. As in the case $p = 3$, our strategy will be to take advantage of Theorem B (Theorem 5.1) in the case $p = 5$. This result asserts that if $n < 8$, the existence of a standard action produces a standard Pontrjagin class. We will prove Theorem A in the case $p = 5$ by first showing that if $n < 8$, every action is regular and then proving Theorem E which implies that if $n < 10$, then the degree of the fixed codimension-2 submanifold is one. This will complete our proof of Theorem A in the case $p = 5$: we will have the statement about the degree and the fact that $n < 8$ implies every action is standard, and so the Pontrjagin class is standard by Theorem 5.1. We begin with the regularity.

THEOREM 5.7. *Suppose that M^{2n} is a PL cohomology projective n-space. If $(d; m_1, m_2) \in DE_5(M^{2n})$, then $m_2 \equiv 0 \pmod 8$. In particular, if $n < 8$, then $m_2 = 0$, that is, $(d; m_1, m_2) = (d; n, 0)$ or the action is regular.*

Proof. It follows from Proposition 3.9 in the case $p = 5$, formula (3.8), and formula (4.5) in the case $p = 5$, that if $(d; m_1, m_2) \in DE_5(M^{2n})$, then there are nonnegative even integers r_1 and r_2 such that $r_2 = m_2$ and, if $m = [n/2]$, then $r_1 + r_2 = 2m$ and

$$\alpha_1^{r_1}(-\alpha_1^2 - 2)^{r_2/2} = 1 + (\alpha_1^2 - 1)\sum_{k=1}^{m} s_k(d)\alpha_1^{2k-2}. \tag{5.8}$$

It follows from (5.8) and (4.13) that $(-3)^{r_2/2} \equiv 1 \pmod{16}$ and this is $3^{m_2/2} \equiv (-1)^{m_2/2} \pmod{16}$ since $r_2 = m_2$, and so $m_2 \equiv 0 \pmod 8$. Since $n \geq m_2$ it is clear that $n < 8$ forces $m_2 = 0$. $\square$

We now turn to the task of showing that if $n < 10$ and $(d; m_1, m_2) \in DE_5(M^{2n})$, then $d = 1$. The proof is elementary but a good deal more tedious than the proof of the regularity in Theorem 5.7. The idea is to use (5.8), for appropriate values of m, together with Lemma 4.14. The homogeneous expressions in the coefficients a_k, $0 \leq k \leq q$, in Lemma 4.14, will lead to inhomogeneous expressions in $s_k(d)$, $0 \leq k \leq m$, with integer coefficients to which we can apply formula (2.7) to conclude that $d = 1$. We will take advantage of the facts that (5.8) is indexed by $m = [n/2]$ and the second statement of Theorem B in the case $p = 5$ and Theorem 5.7 imply that actions are standard, and hence Pontrjagin classes are standard by Theorem B, if $n < 6$ or $m < 3$. Our computations will start at $m = 3$.

LEMMA 5.9. *Suppose that M^{2n} is a cohomology projective n-space. If $(d; m_1, m_2) \in DE_5(M^{2n})$ and (r_1, r_2) is the pair of nonnegative integers in* (5.8) *with $m = [n/2]$ corresponding to (m_1, m_2), then the equations below hold where $s_k = s_k(d)$, $1 \leq k \leq m$.*

$$m = 3 (n = 6, 7),\ (r_1, r_2) = (6, 0):$$

$$(r_1, r_2) = (6, 0):\ \begin{cases} 9s_3 - 5s_2 + 5s_1 = 9, \\ 2s_3 - s_2 = 1. \end{cases} \tag{5.10}$$

$$m = 4 (n = 8, 9),\ (r_1, r_2) = (8, 0),\ (0, 8):$$

$$(r_1, r_2) = (8, 0):\ \begin{cases} 17s_4 - 9s_3 + 5s_2 - 5s_1 = 8, \\ 19s_4 - 10s_3 + 5s_2 = 14. \end{cases} \tag{5.11}$$

$$(r_1, r_2) = (0, 8):\ \begin{cases} 123s_4 - 65s_3 + 35s_2 - 25s_1 = 68, \\ 55s_4 - 29s_3 + 15s_2 - 5s_1 = -36. \end{cases} \tag{5.12}$$

$$m = 5 (n = 10, 11),\ (r_1, r_2) = (10, 0),\ (2, 8):$$

$$(5.13)\quad (r_1, r_2) = (10, 0)\colon \begin{cases} 161s_5 - 85s_4 + 45s_3 - 25s_2 + 25s_1 = 121, \\ 36s_5 - 19s_4 + 10s_3 - 5s_2 = 22. \end{cases}$$

$$(5.14)\quad (r_1, r_2) = (2, 8)\colon \begin{cases} 233s_5 - 123s_4 + 65s_3 - 35s_2 + 25s_1 = 21, \\ 521s_5 - 275s_4 + 145s_3 - 75s_2 + 25s_1 = -19. \end{cases}$$

Proof. Formula (5.10) follows by considering (5.8) with $m = 3$ and $(r_1, r_2) = (6, 0)$ and then applying (4.15) with $a_{k-1} = s_k - 1$, $1 \le k \le 3$. Formula (5.11) follows from (5.8) with $m = 4$ and $(r_1, r_2) = (8, 0)$ and (4.16) with $a_{k-1} = s_k - 1$, $1 \le k \le 4$. Formula (5.12) follows from (5.8) with $m = 4$ and $(r_1, r_2) = (0, 8)$ and (4.17) with $a_4 = s_4 - 1$, $a_3 = s_3 - s_4 - 8$, $a_2 = s_2 - s_3 - 24$, $a_1 = s_1 - s_2 - 32$ and $a_0 = 1 - s_1 - 16$. Formula (5.13) follows from (5.8) with $m = 5$ and $(r_1, r_2) = (10, 0)$ and (4.17) with $a_{k-1} = s_k - 1$, $1 \le k \le 5$. Formula (5.14) follows from (5.8) with $m = 5$ and $(r_1, r_2) = (2, 8)$ and (4.18) with $a_5 = s_5 - 1$, $a_4 = s_4 - s_5 - 8$, $a_3 = s_3 - s_4 - 24$, $a_2 = s_2 - s_3 - 32$, $a_1 = s_1 - s_2 - 16$, and $a_0 = 1 - s_1$. □

THEOREM 5.15. *Suppose that M^{2n} is a PL cohomology projective n-space and that $d \in D_5(M^{2n})$. If $n \le 9$ or $n = 11$, then $d = 1$. If $n = 10$, then $d = 1$ or 3.*

Proof. If $n < 6$ and $d \in D_5(M^{2n})$, then $d = 1$ because by Theorem B and Theorem 5.7, actions are standard and so, in particular, $d = 1$. If $n = 6$ or 7, then we know that $(r_1, r_2) = (6, 0)$ by Theorem 5.7, and so (5.10) holds. If $n = 6$ and the second equation listed in (5.10) is multiplied by $f(6) = 45$ and then (2.7) is invoked, the conclusion is that d^4 divides 45 and hence $d = 1$. If $n = 7$, multiply this same equation by $f(7) = 315$ and conclude that d^5 divides 315 by (2.7) and so $d = 1$. For $n = 8$ or 9, Theorem 5.7 says that $(r_1, r_2) = (8, 0)$ or $(0, 8)$ and so we must consider (5.11) and (5.12). First $(r_1, r_2) = (8, 0)$ and (5.11): for $n = 8$ and $n = 9$ multiply the second equation in (5.11) by $f(8)$ and $f(9)$, respectively, and conclude by (2.7) that d^4 divides $14f(8)$ and d^5 divides $14f(9)$, respectively. In both cases, the conclusion is $d = 1$. Now $(r_1, r_2) = (0, 8)$ and (5.12): for $n = 8$ and $n = 9$, add the first equation listed in (5.12) to -5 times the second equation listed in (5.12) to eliminate s_1 and multiply the resulting equation by $f(8)$ and $f(9)$, respectively. If (2.7) is invoked, the conclusions are d^4 divides $31f(8)$ and d^5 divides $31f(9)$, respectively, and so $d = 1$. For $n = 10$ or 11, Theorem 5.7 says that $(r_1, r_2) = (10, 0)$ or $(2, 8)$, and so we must consider (5.13) and (5.14). First $(r_1, r_2) = (10, 0)$ and (5.13): for $n = 10$ and $n = 11$, multiply the second equation by $f(10)$ and $f(11)$, respectively, and conclude that d^4 divides $22f(10)$ and d^5 divides $22f(11)$, respectively, by (2.7). In the case $n = 10$, the conclusion is $d = 1$ or 3 and in the case $n = 11$, we conclude that $d = 1$. Now $(r_1, r_2) = (2, 8)$ and (5.14): for $n = 10$ and $n = 11$, subtract the second equation from the first in (5.14), simplify, and multiply by $f(10)$ and $f(11)$, respectively, and conclude that d^4 divides $5f(10)$ and d^5 divides $5f(11)$, respectively, by (2.7). In the case $n = 10$, we conclude that $d = 1$ or 3, and if $n = 11$, we conclude that $d = 1$. □

Table 5.17

n	d	(m_1, m_2)	Pontrjagin Class
1	1	$(1, 0)$	—
2	1	$(2, 0)$	standard
3	1	$(3, 0)$	standard
4	1	$(4, 0)$	standard
5	1	$(5, 0)$	standard
6	1	$(6, 0)$	standard
7	1	$(7, 0)$	standard
8	1	$(8, 0), (0, 8)$	standard and exotic
9	1	$(9, 0), (1, 8)$	standard and exotic
10	$1, 3$	$(10, 0), (2, 8)$	standard and exotic
11	1	$(11,0), (3, 8)$	standard and exotic

Note that Theorem 5.15 is the same as Theorem E. We are now ready to state Theorem A in the case $p = 5$ in terms of the notation of Section 3 and prove it using Theorems B, 5.7 and 5.15.

THEOREM 5.16. *Suppose that M^{2n} is a cohomology projective n-space and that $(d; m_1, m_2) \in DE_5(M^{2n})$. If $n < 10$, then $d = 1$. If $n < 8$, then $(d; m_1, m_2) = (1; n, 0)$ and the Pontrjagin class of M^{2n} is standard.*

Proof. The first statement is contained in Theorem 5.15. If $(d; m_1, m_2) \in DE_5(M^{2n})$ and $n < 8$, then $m_2 = 0$ by Theorem 5.7 and $d = 1$ by Theorem 5.15, that is, $(d; m_1, m_2) = (1; n, 0)$. In other words, if $n < 8$ and M^{2n} admits an $\ell.\ell.$ PL G_5 action of Type II_0, then the action is standard and so the Pontrjagin class of M^{2n} is standard by Theorem B (Theorem 5.1). □

We offer a summary table which contains the possible values of $(d; m_1, m_2) \in DE_5(M^{2n})$, $n \leq 11$. We remark that our table only gives an upper bound for $DE_5(M^{2n})$ as only the standard action $(1; n, 0) \in DE_5(\mathbb{C}P^n)$ is known to exist. The classifications *standard* and *exotic* in the table for the Pontrjagin class for $1 \leq n \leq 11$ means that the classes must be standard, $2 \leq n \leq 7$, and can be either standard or exotic if $8 \leq n \leq 11$. The case $n = 1$ is included for completeness.

Proofs of Theorems C and D. Both theorems follow from the fact that the Pontrjagin class of a PL homotopy complex projective n-space is standard if and only the integral splitting invariants of M^{2n} vanish (Proposition 2.8). The number of mod 2 splitting invariants at level n is $n - 2 - [(n-1)/2] = [n/2] - 1$, and so the number of distinct PL homotopy complex projective n–spaces with a standard Pontrjagin class is $2^{[n/2]-1}$. $\square$

Proof of Theorem F. The proof will be complete if we show that if $n \leq 9$ or $n = 11$ and $(d; m_1, m_2) \in DE_5(\mathbb{C}P^n)$, then $d = 1$, $m_1 = n$ and $m_2 = 0$. If $n < 8$, then this follows from Theorem 5.16. If $n = 8, 9$ or 11, then $d = 1$ by Theorem 5.15 and so $s_k(d) = s_k(1) = 1$, $1 \leq k \leq [n/2]$, by Proposition 2.8. Equations (5.12) and (5.14) are not satisfied by these values of s_k and so $m_1 = n$ and $m_2 = 0$. $\square$

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