LIKELIHOOD RATIOS FOR STOCHASTIC PROCESSES RELATED BY GROUPS OF TRANSFORMATIONS II

BY

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We will make use of the notation established in *Likelihood Ratios for Stochastic Processes Related by Groups of Transformations*[2] (referred to as (I) in the following). Thus, X, S, and P are a set, a σ-algebra of subsets, and a probability measure on S. T_α is a one-parameter group of automorphisms of an algebra F of bounded, real-valued, S-measurable functions satisfying

(i) T_α preserves bounds, and $T_\alpha f(x)$ has a continuous derivative $D(T_\alpha f)(x)$ in α which is bounded uniformly in α and x for every f in F and x in X, and

(ii) if f_n is a uniformly bounded sequence from F with $\lim f_n(x) = 0$ for all x, then $\lim T_\alpha f_n(x) = 0$ for all x.

Examples of this situation will be found in (I).

We will write P_α for the measures which are the completions of

$$l_\alpha(f) = \int T_\alpha f \, dP,$$

$K_\sigma(\alpha)$ for the Gaussian kernel $(2\pi\sigma)^{-1/2} \exp(-\alpha^2/2\sigma)$, and P_α^σ for the measures which are the completions of

$$l_\alpha^\sigma(f) = \int_{-\infty}^{\infty} K_\sigma(\beta) \left(\int T_{\alpha+\beta} f \, dP \right) d\beta.$$

According to Theorem 4.2 of (I) the P_α^σ with $\sigma > 0$ and any α are mutually absolutely continuous, and for each positive σ there is a ϕ^σ in $L_1(P^\sigma)$ satisfying

$$\int \phi^\sigma T_\alpha f \, dP^\sigma = \frac{\partial}{\partial\alpha} \int T_\alpha f \, dP^\sigma$$

for f in F and

$$\log \frac{dP_\alpha^\sigma}{dP^\sigma} = \int_0^\alpha T_{-\beta} \, \phi^\sigma \, d\beta.$$

The theorem also asserts that the transformations $V^\sigma(\alpha)$ on $L_1(P^\sigma)$ defined by the equation $V^\sigma(\alpha)f = (dP_\alpha^\sigma/dP^\sigma)T_{-\alpha} f$ for f in F form a strongly continuous one-parameter group of isometries whose infinitesimal generator A^σ is defined on F and satisfies $A^\sigma f = \phi^\sigma f - Df$ there.

We note that $\bar{F}$, the set of uniform limits from F, contains the functions $f \wedge g = \min(f, g)$ and $f \vee g = \max(f, g)$ whenever it contains f and g,

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and that T_α can be extended to $\bar{F}$. We will assume as in (I) that F is dense in $L_1(P)$ since this can always be achieved by cutting down the size of S.

THEOREM 1. *If the P_α are mutually absolutely continuous, then T_α can be extended to all S-measurable finite functions, and the mappings $V(\alpha)$ on $L_1(P)$ defined by $V(\alpha)f = (dP_\alpha/dP)T_{-\alpha}f$ form a strongly continuous one-parameter group of isometries. The extension of T_α is linear and positive and satisfies the following*:

(1) *If f_n converges to 0 almost everywhere, so does $T_\alpha f_n$.*
(2) $T_\alpha(fg) = T_\alpha(f)T_\alpha(g)$.
(3) $T_\alpha(T_\beta f) = T_{\alpha+\beta}f$.
(4) $T_\alpha(dP^\sigma_\beta/dP^\tau_\gamma) = dP^\sigma_{\beta-\alpha}/dP^\tau_{\gamma-\alpha}$.
(5) *If either side of the equation*

$$\int T_\alpha h\, dP^\sigma_\beta = \int h\, dP^\sigma_{\beta+\alpha}$$

exists, so does the other side, and they are equal.

Proof. Let (f_n) be a decreasing sequence of nonnegative functions from $\bar{F}$. If $\lim_{n\to\infty} \int f_n\, dP = 0$, then the functions $\int T_\alpha f_n\, dP$ are uniformly bounded and converge to 0 for every α, so $\lim_{n\to\infty} \int f_n\, dP^\sigma_\beta = 0$ for every σ and β, so P^σ_β is absolutely continuous with respect to P and hence with respect to every P_α. If $\lim_{n\to\infty} \int f_n\, dP^\sigma = 0$, then $\int T_\alpha f_n\, dP$ must go to 0 for almost every α and hence for every α, so every P_α is absolutely continuous with respect to P^σ and hence with respect to every P^τ_β.

The mappings $V(\alpha)$ defined on $\bar{F}$ by $V(\alpha)f = (dP_\alpha/dP)T_{-\alpha}f$ clearly have isometric extensions to $L_1(P)$. $V(\alpha)(fg) = (V(\alpha)f)T_{-\alpha}g$ if f and g are in $\bar{F}$, and by an easy continuity argument this equation still holds if g is in $\bar{F}$ and f is in $L_1(P)$. A similar relation holds for $V^\sigma(\alpha)$. If f and g are in $\bar{F}$, then

$$\frac{dP^\sigma}{dP} V^\sigma(\alpha)\left(\frac{dP}{dP^\sigma} f\right)$$

is in $L_1(P)$, and

$$\begin{aligned}\int \frac{dP^\sigma}{dP} V^\sigma(\alpha)\left(\frac{dP}{dP^\sigma} f\right) g\, dP &= \int V^\sigma(\alpha)\left(\frac{dP}{dP^\sigma} f\right) g\, dP^\sigma \\ &= \int V^\sigma(\alpha)\left(\frac{dP}{dP^\sigma} fT_\alpha g\right) dP^\sigma = \int \frac{dP}{dP^\sigma} fT_\alpha g\, dP^\sigma \\ &= \int fT_\alpha g\, dP = \int V(\alpha)(fT_\alpha g)\, dP = \int (V(\alpha)f)g\, dP.\end{aligned}$$

Thus

$$V(\alpha)f = \frac{dP^\sigma}{dP} V^\sigma(\alpha)\left(\frac{dP}{dP^\sigma} f\right)$$

if f is in $\bar{F}$, and by continuity this holds for all f in $L_1(P)$. We have

$$\| V(\alpha)f - f \| = \int \left| \frac{dP^\sigma}{dP} V^\sigma(\alpha) \left(\frac{dP}{dP^\sigma} f\right) - f \right| dP$$

$$= \int \left| V^\sigma(\alpha) \left(\frac{dP}{dP^\sigma} f\right) - \frac{dP}{dP^\sigma} f \right| dP^\sigma,$$

so the strong continuity of $V(\alpha)$ follows from the strong continuity of $V^\sigma(\alpha)$. Also

$$V(\alpha + \beta)f = \frac{dP^\sigma}{dP} V^\sigma(\alpha + \beta) \left(\frac{dP}{dP^\sigma} f\right) = \frac{dP^\sigma}{dP} V^\sigma(\alpha) \left(V^\sigma(\beta) \left(\frac{dP}{dP^\sigma} f\right)\right)$$

$$= \frac{dP^\sigma}{dP} V^\sigma(\alpha) \left(\frac{dP}{dP^\sigma} V(\beta)f\right) = V(\alpha)(V(\beta)f),$$

which verifies the group property of $V(\alpha)$.

For f in $L_1(P)$ we define $T_\alpha f$ to be

$$V(-\alpha)f/V(-\alpha)1 = (dP/dP_{-\alpha})V(-\alpha)f.$$

Since $0 < dP/dP_{-\alpha} < \infty$ almost everywhere, this is a linear, positivity-preserving extension of T_α. For bounded f and g if we choose sequences (f_n) and (g_n) from F so that $V(-\alpha)(f_n g_n)$, $V(-\alpha)f_n$, and $V(-\alpha)g_n$ converge almost everywhere to $V(-\alpha)(fg)$, $V(-\alpha)f$, and $V(-\alpha)g$, we see that

$$T_\alpha(fg) = (dP/dP_{-\alpha})V(-\alpha)(fg) = \lim_{n\to\infty}(dP/dP_{-\alpha})V(-\alpha)(f_n g_n)$$

$$= \lim_{n\to\infty}[(dP/dP_{-\alpha})V(-\alpha)f_n][(dP/dP_{-\alpha})V(-\alpha)g_n] = T_\alpha(f)T_\alpha(g).$$

In particular, T_α takes characteristic functions into characteristic functions, and disjoint characteristic functions into disjoint characteristic functions.

Let f be a measurable function, ξ and ζ the characteristic functions of the sets where $f > 0$ and where $f < 0$, and χ_n a sequence of characteristic functions increasing to 1 for which each $f\chi_n$ is in $L_1(P)$. Then

$$T_\alpha(f\xi\chi_n) = T_\alpha(f\chi_n)T_\alpha(\xi\chi_n) \quad \text{and} \quad -T_\alpha(f\zeta\chi_n) = -T_\alpha(f\chi_n)T_\alpha(\zeta\chi_n)$$

are nondecreasing sequences of functions whose supports are disjoint, so $T_\alpha(f\chi_n)$ is almost everywhere convergent. If η_n is any other sequence with the same properties and we set $\omega_n = (1 - \chi_n) \vee (1 - \eta_n)$, then

$$| T_\alpha(f\chi_n - f\eta_n) | \leqq 2T_\alpha(|f|\omega_n),$$

so the support of the difference is contained in the support of $T_\alpha \omega_n$. Since ω_n decreases to 0, $V(-\alpha)(\omega_n)$ decreases to 0 in $L_1(P)$ and hence almost everywhere, so $T_\alpha \omega_n$ decreases to 0, proving that $\lim_{n\to\infty} T_\alpha(f\chi_n)$ is independent of the particular sequence used.

We define $T_\alpha(f)$ to be $\lim_{n\to\infty} T_\alpha(f\chi_n)$ for any sequence χ_n having the properties given above. Clearly this is a positivity-preserving extension of T_α. If we choose χ_n so that both $f\chi_n$ and $g\chi_n$ are bounded, then

$$\begin{aligned} T_\alpha(af + bg) &= \lim_{n\to\infty} T_\alpha(af\chi_n + bg\chi_n) \\ &= \lim_{n\to\infty}(aT_\alpha(f\chi_n) + bT_\alpha(g\chi_n)) \\ &= aT_\alpha(f) + bT_\alpha(g), \end{aligned}$$

so T_α is linear, and

$$T_\alpha(fg) = \lim_{n\to\infty} T_\alpha(fg\chi_n) = \lim_{n\to\infty}(T_\alpha(f\chi_n)T_\alpha(g\chi_n)) = T_\alpha(f)T_\alpha(g),$$

so (3) is also satisfied. If the support of f is contained in a set with characteristic function χ, then the support of $T_\alpha f$ is contained in the set whose characteristic function is $T_\alpha \chi$. Hence, if f_n converges to 0 and χ_n is the characteristic function of the set where $\sup_{m\geqq n}| f_m(x) | \geqq \varepsilon$, then

$$| T_\alpha(f_n) | \leqq T_\alpha(\varepsilon\chi_n + |f_n|(1 - \chi_n)) \leqq \varepsilon + g_n$$

where g_n converges to 0 since its support is contained in $T_\alpha(1 - \chi_n)$. This gives $\limsup_{n\to\infty}| T_\alpha f_n | \leqq \varepsilon$ for any ε and thus proves (1).

It will be sufficient to prove (5) for nonnegative h. If h is in $\bar{F}$, the equation holds. If h is bounded, we can find a bounded sequence h_n from $\bar{F}$ converging almost everywhere to h and with $V(-\alpha)h_n$ converging almost everywhere to $V(-\alpha)h$, so that $T_\alpha h_n$ converges almost everywhere to $T_\alpha h$, and then the equation holds for h by continuity. Finally, by choosing χ_n so that $h\chi_n$ is bounded,

$$\int T_\alpha h \, dP_\beta^\sigma = \lim_{n\to\infty} \int T_\alpha(h\chi_n) \, dP_\beta^\sigma = \lim_{n\to\infty} \int h\chi_n \, dP_{\beta+\alpha}^\sigma ,$$

which proves (5). (4) now follows from

$$\int fT_\alpha \frac{dP_\beta^\sigma}{dP_\gamma^\tau} \, dP_{\gamma-\alpha}^\tau = \int T_{-\alpha} f \frac{dP_\beta^\sigma}{dP_\gamma^\tau} \, dP_\gamma^\tau = \int f \, dP_{\beta-\alpha}^\sigma .$$

LEMMA. *If there is a ϕ in $L_1(dP)$ satisfying*

$$(*) \qquad \int \phi T_\alpha f \, dP = \frac{\partial}{\partial\alpha} \int T_\alpha f \, dP$$

for all f in F, then each P_α is absolutely continuous with respect to each P_β^σ, and

$$\phi^\sigma = \int_{-\infty}^{\infty} K_\sigma(\beta) V^\sigma(\beta) \left(\frac{dP}{dP^\sigma} \phi\right) d\beta.$$

Proof. The existence of $dP_\alpha/dP_\beta^\sigma$ is proved in Theorem 4.2 of (I). For f in F,

$$\begin{aligned} &\int T_\alpha f \left(\int_{-\infty}^{\infty} K_\sigma(\beta) V^\sigma(\beta) \left(\frac{dP}{dP^\sigma} \phi\right) d\beta\right) dP^\sigma \\ &\qquad = \int_{-\infty}^{\infty} K_\sigma(\beta) \left(\int T_\alpha f V^\sigma(\beta) \left(\frac{dP}{dP^\sigma} \phi\right) dP^\sigma\right) d\beta \end{aligned}$$

$$= \int_{-\infty}^{\infty} K_\sigma(\beta) \left(\int T_\alpha f T_{-\beta} \left(\frac{dP}{dP^\sigma} \phi \right) dP_\beta^\sigma \right) d\beta$$

$$= \int_{-\infty}^{\infty} K_\sigma(\beta) \left(\int \phi T_{\alpha+\beta} f \, dP \right) d\beta$$

$$= \int_{-\infty}^{\infty} K_\sigma(\beta) \left(\frac{\partial}{\partial \alpha} \int T_{\alpha+\beta} f \, dP \right) d\beta$$

$$= \frac{\partial}{\partial \alpha} \int T_\alpha f \, dP^\sigma.$$

THEOREM 2. *If the P_α are mutually absolutely continuous and there is a ϕ in $L_1(P)$ satisfying $(*)$, then the generator A of $V(\alpha)$ contains F in its domain and is defined there by $Af = \phi f - Df$. $(dP_\alpha/dP)T_{-\alpha}\,\phi$ is almost always integrable on every finite interval, and the equation*

$$\frac{dP_\alpha}{dP} = 1 + \int_0^\alpha \frac{dP_\beta}{dP} T_{-\beta} \phi \, d\beta$$

defines a continuous version of the stochastic process dP_α/dP.

Proof. From the fact that

$$V(\alpha) f = \frac{dP^\sigma}{dP} V^\sigma(\alpha) \left(\frac{dP}{dP^\sigma} f \right)$$

and the above lemma we get

$$\phi^\sigma = \frac{dP}{dP^\sigma} \int_{-\infty}^{\infty} K_\sigma(\beta) V(\beta) \phi \, d\beta.$$

Also f is in the domain of A, and

$$Af = \frac{dP^\sigma}{dP} A^\sigma \left(\frac{dP}{dP^\sigma} f \right)$$

whenever $(dP/dP^\sigma) f$ is in the domain of A^σ. In particular, if f is in F,

$$A \left(\frac{dP^\sigma}{dP} f \right) = \frac{dP^\sigma}{dP} (\phi^\sigma f - Df).$$

By Theorem 4.2 of (I),

$$\int \left| \frac{dP^\sigma}{dP} - 1 \right| dP = \int \left| \frac{dP}{dP^\sigma} - 1 \right| dP^\sigma \leqq \left(\frac{2\sigma}{\pi} \right)^{1/2} \| \phi \|,$$

and f and Df are bounded, by hypothesis, so $(dP^\sigma/dP)f$ and $(dP^\sigma/dP)Df$ converge to f and Df in $L_1(P)$. Finally,

$$\left\| \frac{dP^\sigma}{dP} \phi^\sigma f - \phi f \right\| \leqq C \left\| \frac{dP^\sigma}{dP} \phi^\sigma - \phi \right\| \leqq C \int_{-\infty}^{\infty} K_\sigma(\beta) \, \| V(\beta)\phi - \phi \| \, d\beta \to 0$$

as $\sigma \to 0$, proving the first assertion. Taking $f = 1$ we have

$$\frac{dP_\alpha}{dP} = V(\alpha)(1) = 1 + \int_0^\alpha V(\beta)(\phi) \, d\beta = 1 + \int_0^\alpha \frac{dP_\beta}{dP} T_{-\beta} \phi \, d\beta.$$

By Fubini's theorem and the L_1 continuity of the integrand, $(dP_\beta/dP)T_{-\beta}\,\phi$ is almost always integrable on $[0, \alpha]$, and its integral is equal to the L_1 integral almost everywhere. Hence, almost every $(dP_\beta/dP)T_{-\beta}\,\phi$ is integrable on every finite β interval, and

$$1 + \int_0^\alpha \frac{dP_\beta}{dP}\, T_{-\beta}\phi\, d\beta = \frac{dP_\alpha}{dP} \qquad \text{for almost every } \alpha,$$

i.e., the pointwise integral is a version of $(dP_\alpha/dP - 1)$.

Theorem 2 asserts the continuity and almost everywhere differentiability of the "sample functions" dP_α/dP. The following example shows that no such smoothness can be expected in general.

X, S, and P are the real line, the Borel sets, and the measure $p(x)\,dx$ for any almost everywhere positive p of integral 1. T_α represents translation by α, and F is any algebra of sufficiently smooth functions. Here

$$(dP_\alpha/dP)(x) = p(x - \alpha)/p(x)$$

which has no smoothness properties at all. The assumption of Theorem 2 is equivalent here to the existence of a derivative of p which is in $L_1(dx)$, and, in this case, $\phi(x) = -p'(x)/p(x)$.

If we set

$$p(x) = c\exp(-1/(1 - x^2)) \quad \text{for} \quad |x| < 1,$$

$$p(x) = 0 \qquad\qquad \text{for} \quad |x| \geqq 1,$$

and replace T_α by translation mod 2, the assumptions of Theorem 2 are satisfied. $T_{-\alpha}\,\phi$ is not integrable on any interval $[0, \alpha]$, however, so that the equation of the theorem cannot be replaced by

$$\frac{dP_\alpha}{dP} = \exp \int_0^\alpha T_{-\beta}\phi\, d\beta.$$

If there is a solution ϕ of $(*)$ in $L_1(P)$, then it is uniquely determined in $L_1(P)$ but not necessarily in $L_1(P^\sigma)$. According to Theorem 4.2 of (I), if a ϕ exists in $L_1(P)$, then P is absolutely continuous with respect to P^σ. We will call ϕ a *normalized solution of* $(*)$ if it also vanishes almost everywhere (P^σ) on the set where dP/dP^σ vanishes. Any solution of $(*)$ can obviously be normalized. Since the P_α^σ are mutually absolutely continuous, Theorem 1 implies that T_α can be extended to all S-measurable functions.

LEMMA. *If ϕ is a normalized solution of $(*)$, then*

$$T_{-\alpha}\left(\frac{dP_\beta}{dP_\gamma^\sigma}\right) = \frac{dP_{\beta+\alpha}}{dP_{\gamma+\alpha}^\sigma}, \quad V^\sigma(\alpha)\left(\frac{dP}{dP^\sigma}\right) = \frac{dP_\alpha}{dP^\sigma}, \quad \textit{and} \quad A^\sigma\left(\frac{dP}{dP^\sigma}\right) = \phi\,\frac{dP}{dP^\sigma}.$$

Proof. For any f in F,

$$\int fT_{-\alpha}\left(\frac{dP_\beta}{dP_\gamma^\sigma}\right) dP_{\gamma+\alpha}^\sigma = \int (T_\alpha f)\,\frac{dP_\beta}{dP_\gamma^\sigma}\,dP_\gamma^\sigma = \int f\,dP_{\beta+\alpha}$$

which proves the first assertion. The second is an immediate consequence of the first. For any f in F of absolute bound 1,

$$\left|\int \frac{1}{\alpha}\left(\frac{dP_\alpha}{dP^\sigma} - \frac{dP}{dP^\sigma}\right) - \phi \frac{dP}{dP^\sigma}\right\} f\, dP^\sigma\right|$$

$$= \left|\frac{1}{\alpha}\int_0^\alpha \left(\int \phi T_\beta f\, dP\right) d\beta - \int \phi f\, dP\right|$$

$$= \frac{1}{\alpha}\left|\int_0^\alpha \int \left(V^\sigma(\beta)\left(\phi \frac{dP}{dP^\sigma}\right) - \phi \frac{dP}{dP^\sigma}\right) f\, dP^\sigma\, d\beta\right|$$

$$\leqq \frac{1}{\alpha}\int_0^\alpha \left(\int \left|V^\sigma(\beta)\left(\phi \frac{dP}{dP^\sigma}\right) - \phi \frac{dP}{dP^\sigma}\right| dP^\sigma\right) d\beta.$$

The left-hand side of this inequality can be made to approach the $L_1(P^\sigma)$ norm of $(1/\alpha)(dP_\alpha/dP^\sigma - dP/dP^\sigma) - \phi(dP/dP^\sigma)$, and the right-hand side goes to 0 as α approaches 0.

THEOREM 3. *Let ϕ be a normalized solution of* $(*)$. *If, for some $\gamma > 0$ (or $\delta < 0$), $T_{-\beta}\,\phi$ is integrable on* $[0, \gamma]$ *(or* $[\delta, 0]$*) almost everywhere with respect to P^σ, then the P_α are mutually absolutely continuous, $T_{-\beta}\,\phi$ is almost always integrable on every finite interval, and*

$$\log \frac{dP_\alpha}{dP} = \int_0^\alpha T_{-\beta}\phi\, d\beta.$$

Proof. We will only deal with the case $\gamma > 0$. Since ϕ is normalized, the previous lemma implies that

$$\frac{dP_\alpha}{dP^\sigma} = \frac{dP}{dP^\sigma} + \int_0^\alpha T_{-\beta}\phi \frac{dP_\beta}{dP^\sigma}\, d\beta.$$

$T_{-\beta}\,\phi\,(dP_\beta/dP^\sigma) = V^\sigma(\beta)\,(\phi\,(dP/dP^\sigma))$ is L_1-continuous, so $T_{-\beta}\,\phi\,(dP_\beta/dP^\sigma)$ is integrable on $[0, \alpha]$ almost everywhere, and its pointwise integral is equal to its L_1 integral. For $\alpha \leqq \gamma$,

$$Q_\alpha = \frac{dP}{dP^\sigma} \exp \int_0^\alpha T_{-\beta}\phi\, d\beta$$

is defined almost everywhere, and

$$Q_\alpha = \frac{dP}{dP^\sigma} + \int_0^\alpha T_{-\beta}\phi Q_\beta\, d\beta,$$

so by a uniqueness argument for real-valued functions

$$\frac{dP_\alpha}{dP^\sigma} = \frac{dP}{dP^\sigma} \exp \int_0^\alpha T_{-\beta}\phi\, d\beta \qquad \text{for } \alpha \leqq \gamma.$$

P_α is thus absolutely continuous with respect to P for $0 \leqq \alpha \leqq \alpha_0$, but since

$T_{-\varepsilon}(dP_\alpha/dP) = dP_{\alpha+\varepsilon}/dP_\varepsilon$, P_α is absolutely continuous with respect to P_β for all α and β. Integrability of $T_{-\alpha}\phi$ follows from

$$\int_a^b |T_{-\beta}\phi|\, d\beta = \sum_{n=0}^{N-1} T_{-a-(n/N)(b-a)} \int_0^{\alpha/N} T_{-\beta}\phi\, d\beta,$$

and the final equation from

$$\log\frac{dP_\alpha}{dP} = \sum_{n=0}^{N-1}\log\frac{dP_{((n+1)/N)\alpha}}{dP_{(n/N)\alpha}} = \sum_{n=0}^{N-1} T_{-(n/N)\alpha}\log\frac{dP_{\alpha/N}}{dP} \qquad \text{for } \alpha > 0,$$

$$\log\frac{dP_\alpha}{dP} = -T_{-\alpha}\log\frac{dP_{-\alpha}}{dP} \qquad \text{for } \alpha < 0.$$

The final theorem is a version of the Cramer-Rao inequality.

THEOREM 4. *Suppose the P_α are mutually absolutely continuous and that a ϕ exists in $L_2(P)$ satisfying $(*)$. If e is any random variable with*

$$\int_J\left[\int e^2\, dP_\alpha\right]^{1/2} d\alpha < \infty$$

for some interval J containing the origin, and if we define the bias $b(\alpha)$ of the estimate e by $\alpha + b(\alpha) = \int e\, dP_\alpha$, then at almost every point of J, $b(\alpha)$ has a derivative, and

$$\int (e-\alpha)^2\, dP_\alpha \geqq \left(1+\frac{db}{d\alpha}\right)\Big/\int \phi^2\, dP.$$

If, in addition, $T_\beta e$ is continuous in $L_2(P)$ on J, then $b(\alpha)$ has a continuous derivative and satisfies the inequality at every point.

Proof. Suppose first that e is bounded. Then

$$\alpha + b(\alpha) = \int e\, dP_\alpha = \int eV(\alpha)1\, dP = \int_0^\alpha\left(\int eV(\beta)\phi\, dP\right) d\beta,$$

and, since

$$\int V(\alpha)\phi\, dP = \frac{\partial}{\partial\alpha}\int V(\alpha)1\, dP = 0,$$

we have

$$\left(1+\frac{db}{d\alpha}\right)^2 = \left(\int (e-\alpha)V(\alpha)\phi\, dP\right)^2 \leqq \int (e-\alpha)^2\, dP_\alpha \int (T_{-\alpha}\phi)^2\, dP_\alpha,$$

which proves the theorem in this case. In general, if we define e_N to be e when $|e| \leqq N$ and 0 elsewhere, then

$$\alpha + b(\alpha) = \alpha + \lim b_N(\alpha) = \lim\int e_N\, dP_\alpha = \lim\int_0^\alpha\left(\int e_N V(\beta)\phi\, dP\right) d\beta$$

$$= \lim\int_0^\alpha\left(\int T_\beta e_N\phi\, dP\right) d\beta = \int_0^\alpha\left(\int T_\beta e\phi\, dP\right) d\beta.$$

Hence b has a derivative equal to $\lim (db_N/d\alpha)$ at almost every point in J, and

$$\int (e - \alpha)^2 \, dP_\alpha \geqq \lim \left(1 + \frac{db_N}{d\alpha}\right)^2 \Big/ \int \phi^2 \, dP = \left(1 + \frac{db}{d\alpha}\right)^2 \Big/ \int \phi^2 \, dP$$

there. If $T_\beta \, e$ is L_2 continuous, both sides of the inequality are continuous, which completes the proof.

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