INVARIANT ATTRACTORS IN TRANSFORMATION GROUPS

BY
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Introduction

In this paper, an attempt is made to generalize the notion of attraction due to Coddington and Levinson [3] which appears in the theory of ordinary differential equations so that it is meaningful in the context of the general notion of a transformation group.

Let (X, T) be a transformation group whose phase space is a separated uniform space. The generalized notion of attraction for (X, T) is defined in Section 1. In Section 2, the general definition of attraction in Section 1 is specialized. It should be pointed out that the specialization is itself a generalization of notions due to Ellis and Gottschalk [4] and the author [2]. Sections 3 and 4 are devoted to a brief study of the specialized notions introduced in Section 2.

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Standing Notation. Let (X, T) be a transformation group whose phase space X is always a separated uniform space. Let $\mathfrak{D}$ be the class of all nonvacuous invariant subsets of X, let $\mathfrak{U}$ be the uniformity of X, let $\mathfrak{B} \subset \mathfrak{P}T$, let $\mathfrak{K}$ be the class of all compact subsets of T, and for each $x \in X$, let $\mathfrak{N}_x$ be the neighborhood filter of x.

1. The general notion

In this section the general notion of attraction is defined for the transformation group (X, T) and illustrated for a case in which (X, T) is a one-parameter continuous flow, which has a singular point.

DEFINITION 1. Let $x \in X$ and let $D \in \mathfrak{D}$.

(1) x is said to be *$\mathfrak{B}$-attracted to D under* (X, T) provided that if $\alpha \in \mathfrak{U}$, then there exists $B \in \mathfrak{B}$ such that $xB \subset D\alpha$. The set of all points of X which are $\mathfrak{B}$-attracted to D under (X, T) is denoted by $S((X, T); \mathfrak{B}; D)$.

(2) x is said to be *regionally $\mathfrak{B}$-attracted to D under* (X, T) provided that if $\alpha \in \mathfrak{U}$ and $U \in \mathfrak{N}_x$, then there exists $y \in U$ and there exists $B \in \mathfrak{B}$ such that $yB \subset D\alpha$. The set of all points of X which are regionally $\mathfrak{B}$-attracted to D under (X, T) is denoted by $R((X, T); \mathfrak{B}; D)$.

Remark 1. Let $D, E \in \mathfrak{D}$ and let $\mathfrak{B}, \mathfrak{C} \subset \mathfrak{P}T$. Then the following statements hold:

(1) $S((X, T); \mathfrak{B}; D) \subset R((X, T); \mathfrak{B}; D)$.

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(2) If $D \subset E$, then

$S((X, T); \mathcal{B}\ D); \subset S((X, T); \mathcal{B}; E)$ and $R((X, T); \mathcal{B}; D) \subset R((X, T); \mathcal{B}; E)$.

(3) If $\mathcal{B} \subset \mathcal{C}$, then

$S((X, T); \mathcal{B}; D) \subset S((X, T); \mathcal{C}; D)$ and $R((X, T); \mathcal{B}; D) \subset R((X, T); \mathcal{C}; D)$.

Example. Let X denote n-dimensional euclidean space with its usual topology where n is some positive integer, let T be the additive group of real numbers with its usual topology and let (X, T) denote a one-parameter continuous flow which has the origin as a singular point. Let $D = \{\mathbf{0}\}$ where $\mathbf{0}$ is the origin in X. Let $\mathcal{B}$ be the class of replete semigroups of T. Then $\mathcal{B}$-attraction to D under (X, T) is equivalent to the notion of attraction defined in Coddington and Levinson [3, p. 376].

2. Specialization of the general notion

Let $D \in \mathfrak{D}$. In this section, the general notion of attraction is specialized to obtain four sets $P(D)$, $L(D)$, $M(D)$, and $Q(D)$, in such a way that the proximal and regionally proximal relations of Ellis and Gottschalk [4] and syndetically proximal and regionally syndetically proximal relations of the author [2] are obtained as a special case.

DEFINITION 2. Let $x \in X$ and let $D \in \mathfrak{D}$.

(1) Let $\mathcal{B} = \mathcal{P}T$. Then x is said to be {*simply*} {*regionally*} *attracted to* D *under* (X, T) provided $\{x \in S((X, T); \mathcal{B}; D)\}$ $\{x \in ((X, T); \mathcal{B}; D)\}$. The set of all points of X which are {simply} {regionally} attracted to D under (X, T) is denoted by $\{P((X, T); D)\}$ $\{Q((X, T); D)\}$ or simply by $\{P(D)\}$ $\{Q(D)\}$ when there is no possibility of ambiguity.

(2) Let $\mathcal{B}$ be the class of syndetic subsets of T. Then x is said to be {*syndetically*} {*regionally syndetically*} *attracted* to D under (X, T) provided $\{x \in S((X, T); \mathcal{B}; D)\}$ $\{x \in R((X, T); \mathcal{B}; D)\}$. The set of all points of X which are {syndetically} {regionally syndetically} attracted to D under (X, T) is denoted by $\{L((X, T); D)\}$ $\{M((X, T); D)\}$ or simply by $\{L(D)\}$ $\{M(D)\}$ when there is no possibility of ambiguity.

(3) Let $L_*((X, T); D)$ denote $L((X, T); D)$ when the phase group T is given the discrete topology.

Remark 2. Let $R = P, Q, L,$ or M. For any transformation group (X, T), define $R^*(X, T) = R((X \times X, T); \Delta_X)$ where Δ_X denotes the diagonal of $X \times X$. Then

(1) $P^*(X, T)$ coincides with the simply proximal relation $P(X, T)$.

(2) $Q^*(X, T)$ coincides with the regionally simply proximal relation $Q(X, T)$.

(3) $L^*(X, T)$ coincides with the syndetically proximal relation $L(X, T)$.

(4) $M^*(X, T)$ coincides with the regionally syndetically proximal relation $M(X, T)$.

For the definition of $R(X, T)$ see [2, Definition 1].

Remark 3. Let $D \in \mathfrak{D}$. Then the following statements hold:

(1) $L(D) = \bigcap_\alpha \bigcup_K \bigcap_t (D\alpha)Kt$ where $\alpha \in \mathfrak{U}$, $K \in \mathfrak{K}$, and $t \in T$.

(2) $M(D) = \bigcap_\alpha (\overline{\bigcup_K \bigcap_t (D\alpha)Kt})$ where $\alpha \in \mathfrak{U}$, $K \in \mathfrak{K}$, and $t \in T$.

(3) $P(D) = \bigcap_\alpha (D\alpha)T$ where $\alpha \in \mathfrak{U}$.

(4) $Q(D) = \bigcap_\alpha \overline{(D\alpha)T}$ where $\alpha \in \mathfrak{U}$.

(5) $D \subset L(D) \subset P(D) \subset Q(D) \subset X$.

(6) $D \subset L(D) \subset M(D) \subset Q(D) \subset X$.

(7) $L(D)$ and $P(D)$ are invariant subsets of X but not necessarily closed.

(8) $M(D)$ and $Q(D)$ are invariant closed subsets of X.

Remark 4. Let $D \in \mathfrak{D}$ and let R denote P or Q or L or M. Then $R((X, T); D) = R((X, T); \bar{D})$.

3. Basic characterizations

Let $D \in \mathfrak{D}$. Then purpose of this section is to point out some general facts concerning the basic structure of the sets $P(D)$ and $L(D)$. The main results of this section are summarized in Theorem 1 and Corollary 1. Theorem 1, in characterizing $L(D)$, points out that $L(D)$ is essentially independent of the topology on T. This theorem and corollary are a generalization of [2, Theorem 3].

Standing Notation. For the remainder of this paper, if (X, T) is a transformation group, we shall use $\mathfrak{A}$ to denote the class of all syndetic subsets of the phase group T.

LEMMA 1. *Let $D \in \mathfrak{D}$ and let $x \in L(D)$. Then $\overline{xT} \subset L(D)$.*

Proof. Let $y \in \overline{xT}$. We show $y \in L(D)$. Let $\alpha \in \mathfrak{U}$. Choose $\beta \in \mathfrak{U}$ such that $\beta = \beta^{-1}$ and $\beta^2 \subset \alpha$. There exists $A \in \mathfrak{A}$ such that $xA \subset D\beta$. Choose $K \in \mathfrak{K}$ such that $T = AK$. Let $t \in T$. It is sufficient to show that $ytK^{-1} \cap D\alpha \neq \emptyset$. Choose $U \in \mathfrak{N}_y$ such that $Utk^{-1} \subset ytk^{-1}\beta$ for all $k \in K$. Choose $s \in T$ such that $xs \in U$, whence $xstk^{-1} \in ytk^{-1}\beta$ for all $k \in K$. Since $\beta = \beta^{-1}$, $ytk^{-1} \in xstk^{-1}\beta$ for all $k \in K$. Since $st \in T$, $stK^{-1} \cap A \neq \emptyset$, whence there exists $k_0 \in K$ such that $ytk_0^{-1} \in xstk_0^{-1}\beta \subset (D\beta)\beta \subset D\alpha$. Therefore $ytK^{-1} \cap D\alpha \neq \emptyset$. The proof is completed.

LEMMA 2. *Let R be a compact invariant subset of $P(D)$. Then*

$$R \subset L_*((X, T); D).$$

Proof. Let α be an open index of X. For each $x \in R$, there exists $t_x \in \mathfrak{U}$ such that $xt_x \in D\alpha$ and hence there exists $U_x \in \mathfrak{N}_x$ such that $U_x t_x \subset D\alpha$. Since R is compact, there exists a finite subset F of R such that $R \subset \bigcup_{x \in F} U_x$. Let $K = \{t_x \mid x \in F\}$ and define $A_y = \{t \mid yt \in D\alpha\}$ for each $y \in R$. Since K is finite and $yA \subset D\alpha$ for all $y \in R$, it is sufficient to show that for any $t \in T$, $tK \cap A_y \neq \emptyset$ for all $y \in R$. Let $y \in R$. Since R is invariant, $yt \in R$ and hence there exists $x \in F$ such that $yt \in U_x$. Now $yt \in U_x$, whence $ytt_x \in D\alpha$, $t_x \in K$, $tt_x \in A_y$, and $tK \cap A_y \neq \emptyset$. The proof is completed.

THEOREM 1. *Let X be compact. Then the following statements hold*:
(1) $L(D) = \{x \mid x \in X, \overline{xT} \subset P(D)\} = \bigcup\{\overline{xT} \mid x \in X, \overline{xT} \subset P(D)\}$.
(2) $L(D) = L_*((X, T); D)$.

Proof. Use Lemmas 1 and 2.

COROLLARY 1. *Let X be compact. Then the following statements hold*:
(1) *The following statements are equivalent*:
 (i) $P(D) = L(D)$.
 (ii) *If $x \in P(D)$, then $\overline{xT} \subset P(D)$.*
(2) *If $P(D)$ is closed, then $P(D) = L(D)$.*

Proof. Use Theorem 1.

4. Productivity

Let $D \in \mathfrak{D}$. In this section, the productivity of $L(D)$, $M(D)$, and $P(D)$ are studied. The main results are summarized in Theorem 2 and Corollary 2. These results are a generalization of [2, Theorems 4 and 5].

The results obtained in this section are an immediate consequence of the following lemma:

LEMMA 3. *Let T be a group, let n be a positive integer, and let $A_1, \cdots, A_n$, $K_1, \cdots, K_n$ be subsets of T such that $T = A_i K_i$ for $i = 1, \cdots, n$. Let K_0 be the identity element of T. Then*

$$T = \left(\bigcap_{i=1}^{n} A_i\left(\prod_{j=1}^{i-1} K_j\right)^{-1}\right) \prod_{i=1}^{n} K_i .$$

Proof. See [2, Lemma 3].

Remark 5. Let (X, T) and (Y, T) be transformation groups. Let φ be a uniformly continuous homomorphism of (X, T) onto (Y, T). Then

$$R((X, T); D)\varphi \subset R((Y, T); D\varphi)$$

where R is P or Q or L or M.

Remark 6. Let I be a set. For $i \in I$, let (X_i, T) be a transformation group where X_i is a uniform space which is not necessarily compact, and let D_i be an invariant nonvacuous subset of X_i. For $j \in I$, let φ_j be the canonical homomorphism of $(\times_{i \in I} X_i, T)$ onto (X_j, T). Let R denote P or Q or L or M. Then

(1) $R((\times_{i \in I} X_i, T); \times_{i \in I} D_i) \subset \times_{i \in I} R((X_i, T); D_i)$.

(2) If $j \in I$, then $R((\times_{i \in I} X_i, T); \times_{i \in I} D_i)\varphi_j = R((X_j, T); D_j)$.

(3) According to (1) and (2), $R((\times_{i \in I} X_i, T); \times_{i \in I} D_i)$ is a subdirect product of $(R((X_i, T); D_i) \mid i \in I)$.

LEMMA 4. *Let n be a positive integer. For $i \in \{1, \cdots, n\}$, let (X_i, T) be a transformation group, let $x_i \in X_i$, and let D_i be a nonvacuous invariant com-*

pact subset of X_i. *Then the following statements hold*:

I. *The following statements are pairwise equivalent*:

(1) *For each* $i \in \{1, \cdots, n\}$, $x_i \in \{M((X_i, T); D_i)\}$.

(2) *If* α_i $(i \in \{1, \cdots, n\})$ *is an index of* X_i, *and if* U_i $(i \in \{1, \cdots, n\})$ *is a neighborhood of* x_i, *then there exist* $y_i \in U_i$ $(i \in \{1, \cdots, n\})$ *and* $A \in \mathfrak{A}$ *such that* $y_i A \subset D_i \alpha_i$ $(i \in \{1, \cdots, n\})$.

(3) $(x_i \mid i \in \{1, \cdots, n\}) \in M((\times_{i=1}^n X_i, T); \times_{i=1}^n D_i)$.

II. *The following statements are pairwise equivalent*:

(1) *For each* $i \in \{1, \cdots, n\}$, $x_i \in L((X_i, T); D_i)$.

(2) *If* α_i $(i \in \{1, \cdots, n\}$ *is an index of* X_i *then there exists* $A \in \mathfrak{A}$ *such that* $x_i A \subset D_i \alpha_i$ $(i \in \{1, \cdots, n\})$.

(3) $(x_i \mid i \in 1, \cdots, n) \in L((\times_{i=1}^n X_i, T); \times_{i=1}^n D_i)$.

Proof. We prove I. Assume (1). We prove (2). There exist $y_1 \in U_1$ and $A_1 \in \mathfrak{A}$ such that $y_1 A_1 \subset D_1 \alpha_1$. Choose $K_1 \in \mathfrak{K}$ such that $T = A_1 K_1$. There exists an index β_2 of X_2 such that $(D_2 \beta_2)K_1^{-1} \subset D_2 \alpha_2$. There exist $y_2 \in U_2$ and $A_2 \in \mathfrak{A}$ such that $y_2 A_2 \subset D_2 \beta_2$, whence $y_2 A_2 K_1^{-1} \subset D_2 \alpha_2$. Choose $K_2 \in \mathfrak{K}$ such that $T = A_2 K_2$. Choose an index β_3 of X_3 for which $(D_3 \beta_3)(K_1 K_2)^{-1} \subset D_3 \alpha_3$. There exist $y_3 \in U_3$ and $A_3 \in \mathfrak{A}$ such that $y_3 A_3 \subset D_3 \beta_3$, whence $y_3 A_3 (K_1 K_2)^{-1} \subset D_3 \alpha_3$. Choose $K_3 \in \mathfrak{K}$ such that $T = A_3 K_3$. This process is continued. Hence, there exist for each $i \in \{1, \cdots, n\}$, $y_i \in U_i$, $A_i \in \mathfrak{A}$ and $K_i \in \mathfrak{K}$ such that $y_i A_i (\prod_{j=1}^{i-1} K_j)^{-1} \subset D_1 \alpha_i$ and $T = A_i K_i$ (K_0 = identity element in T). Define

$$A = \cap_{i=1}^n A_i (\textstyle\prod_{j=1}^{i-1} K_j)^{-1}.$$

Now $y_i A \subset D_i \alpha_i$ for $i \in \{1, \cdots, n\}$. By Lemma 3, $A \in \mathfrak{A}$. The proof that (1) implies (2) is completed.

Assume (2). We prove (3). Let α be an index of $\times_{i=1}^n X_i$ and let U be a neighborhood of $(x_1, \cdots, x_n)$. There exists an index α_i $(i \in \{1, \cdots, n\})$ in X_i such that $\times_{i=1}^n D_i \alpha_i \subset (\times_{i=1}^n D_i)\alpha$. There exists a neighborhood U_i $(i \in \{1, \cdots, n\})$ of x_i such that $\times_{i=1}^n U_i \subset U$. There exist $y_i \in U_i$ $(i \in \{1, \cdots, N\})$ and $A \in \mathfrak{A}$ such that $y_i A \subset D_i \alpha_i$ for each $i \in \{1, \cdots, n\}$. Now

$$(y_i \mid i \in \{1, \cdots, n\}) \in \times_{i=1}^n U_i \subset U$$

and

$$(y_i \mid i \in \{1, \cdots, n\})A \subset \times_{i=1}^n D_i \alpha_i \subset (\times_{i=1}^n D_i)\alpha.$$

The proof that (2) implies (3) is completed.

That (3) implies (1) follows from Remark 6. The proof of I is completed.

To prove II, we simply observe that if

$$x_i \in L((X_i, T); D_i) \cap M((X_i, T); D_i)$$

then in the proof of I we may take y_i $(i = 1, \cdots, n)$ to be equal to x_i $(i = 1, \cdots, n)$. The proof is completed.

LEMMA 5. *Let* $((X_i, T) \mid i \in I)$ *be a family of transformation groups. For* $i \in I$, *let* $x_i \in X_i$, *and let* D_i *be an invariant compact subset of* X_i. *Then the following statements hold:*

I. *The following statements are pairwise equivalent:*

(1) *For each* $i \in I$, $x_i \in M((X_i, T); D_i)$.

(2) *If* J *is a finite subset of* I, *if* α_j $(j \in J)$ *is an index of* X_j, *and if* U_j $(j \in J)$ *is a neighborhood of* x_j, *then there exist* $A \in \mathfrak{A}$ *and* $y_j \in U_j$ $(j \in J)$ *such that* $y_j A \subset D_j \alpha_j$ $(j \in J)$.

(3) *If* J *is a finite subset of* I, *then*

$$(x_j \mid j \in J) \in M((\times_{j \in J} X_j, T); \times_{j \in J} D_j).$$

(4) $(x_i \mid i \in I) \in M((\times_{i \in I} X_i, T); \times_{i \in I} D_i)$.

II. *The following statements are pairwise equivalent:*

(1) *For each* $i \in I$, $x_i \in L((X_i, T); D_i)$.

(2) *If* J *is a finite subset of* I, *and if* α_j $(j \in J)$ *is an index of* X_j, *then there exists* $A \in \mathfrak{A}$ *such that for each* $j \in J$, $x_j A \subset D_j \alpha_j$,

(3) *If* J *is a finite subset of* I, *then*

$$(x_j \mid j \in J) \in L((\times_{j \in J} X_j, T); \times_{j \in J} D_j).$$

(4) $(x_i \mid i \in I) \in L((\times_{i \in I} X_i, T); \times_{i \in I} D_i)$.

Proof. We prove I. By Lemma 4, it is sufficient to prove (2) implies (4) and (4) implies (1). That (4) implies (1) is immediate by Remark 6.

Assume (2). We prove (4). Let α be an index of $\times_{i \in I} X_i$ and let U be a neighborhood of $(x_i \mid i \in I)$. Then there exist finite subsets J and J_1, a neighborhood U_j $(j \in J)$ of x_j, and an index α_k $(k \in J)$, of X_k such that $\bigcap_{j \in J} U_j \varphi_j^{-1} \subset U$ and $\bigcap_{j \in J} \alpha_j \vartheta_j^{-1} \subset \alpha$ where φ_j and ϑ_j are the canonical homomorphisms of $(\times_{i \in I} X_i, T)$ onto (X_j, T) and $((\times_{i \in I} X_i)^2, T)$ onto (X_j^2, T) respectively. We may assume without loss of generality that $J_1 = J$. There exist $y_j \in U_j$ and $A \in \mathfrak{A}$ such that $y_j A \subset D_j \alpha_j$. Now

$$(y_i \mid i \in I) \in \bigcap_{j \in J} U_j \varphi_j^{-1} \subset U$$

and

$$(y_i \mid i \in I)A \subset (\times_{i \in I} D_i)(\bigcap_{j \in J} \alpha_j \vartheta_j^{-1}) \subset (\times_{i \in I} D_i)\alpha,$$

whence

$$(x_i \mid i \in I) \in M((\times_{i \in I} X_i, T); \times_{i \in I} D_i).$$

The proof of I is completed.

We prove II. We observe that by Lemma 4 and Remark 6 it is sufficient to prove that (2) implies (4). Assume (2). We prove (4). Let α be an index of $\times_{i \in I} X_i$. Choose a finite subset J of I and an index α_j $(j \in J)$ of X_j such that $\bigcap_{j \in J} \alpha_j \vartheta_j^{-1} \subset \alpha$ where ϑ_j is as above. There exists $A \in \mathfrak{A}$ such that for each $j \in J$, $x_j A \subset D_j \alpha_j$ whence

$$(x_i \mid i \in I)A \subset (\times_{i\in I} D_i)(\bigcap_{j\in J} \alpha_j \vartheta_j^{-1}) \subset (\times_{i\in I} D_i)\alpha$$

and

$$(x_i \mid i \in I) \in L((\times_{i\in I} X_i, T); \times_{i\in I} D_i).$$

The proof is completed.

THEOREM 2. *Let $((X_i, T) \mid i \in I)$ be a family of transformation groups. For $i \in I$, let D_i be an invariant compact subset of X_i. Then*

(1) $L((\times_{i\in I} X_i, T); \times_{i\in I} D_i) = \times_{i\in I} L((X_i, T); D_i)$.

(2) $M((\times_{i\in I} X_i, T); \times_{i\in I} D_i) = \times_{i\in I} M((X_i, T); D_i)$.

Proof. Use Lemma 5.

COROLLARY 2. *Let $((X_i, T) \mid i \in I)$ be a family of transformation groups whose phase spaces are compact. For each $i \in I$, let D_i be a nonvacuous invariant compact subset of X_i. Then the following statements hold:*

I. *The following statements are equivalent:*

(1) *$P((\times_{i\in I} X_i, T); \times_{i\in I} D_i)$ is closed in $\times_{i\in I} X_i$.*

(2) *For each $i \in I$, $P((X_i, T); D_i)$ is closed in X_i.*

II. *If $P((X_i, T); D_i)$ is closed in X_i for each $i \in I$, then*

$$P((\times_{i\in I} X_i, T); \times_{i\in I} D_i) = \times_{i\in I} P((X_i, T); D_i).$$

Proof. Use Theorems 1 and 2 and Remark 6.

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