INEQUALITIES FOR SUBPERMANENTS[1]

BY[2]

MARVIN MARCUS AND WILLIAM R. GORDON

I. Introduction and Statement of Results

If A is an r-square complex matrix then the permanent of A is defined by

$$\text{per}\,(A) = \sum_{\sigma \in S_r} \prod_{i=1}^{r} a_{i\sigma(i)}$$

where the summation extends over the whole symmetric group S_r of degree r. This function has considerable significance in certain combinatorial problems [7, p. 24]. The problem of finding relationships between rather awkward combinatorial matrix functions such as the permanent, and the more classical algebraic invariants is one of considerable interest and importance.

In a paper in the Illinois Journal in 1957 [3] the first of the present authors obtained an upper bound for the sum of the squares of all $\binom{n}{r}^2$ r-square subdeterminants of an n-square matrix A. This work was very recently generalized and improved in an interesting paper by Ryff [6]. In the present paper we turn our attention to the substantially more difficult problem of obtaining a significant upper bound for the sum of the squares of the absolute values of all $\binom{n}{r}^2$ r-square subpermanents of an n-square complex matrix A. We then apply our main result to the case of an incidence matrix for a (v, k, λ) configuration (Theorem 3).

We shall use the following notation throughout the paper. If A has real eigenvalues, then $\lambda_1(A) \geqq \lambda_2(A) \geqq \cdots \geqq \lambda_n(A)$ will denote these. The singular values of A (defined to be the numbers $\lambda_j^{1/2}(A^*A) \geqq 0, j = 1, 2, \cdots, n$) will be designated by $\alpha_1(A) \geqq \alpha_2(A) \geqq \cdots \geqq \alpha_n(A)$. If $1 \leqq r \leqq n$, then $Q_{r,n}$ will denote the set of $N = \binom{n}{r}$ strictly increasing sequences ω, $1 \leqq \omega_1 < \omega_2 < \cdots < \omega_r \leqq n$; $G_{r,n}$ is the set of $\binom{n+r-1}{r}$ non-decreasing sequences ω, $1 \leqq \omega_1 \leqq \omega_2 \leqq \cdots \leqq \omega_r \leqq n$. If α and β are in $G_{r,n}$ then $A[\alpha \mid \beta]$ is the r-square matrix whose i, j entry is $a_{\alpha_i \beta_j}$, $i, j = 1, 2, \cdots, r$. If $a_1 \geqq a_2 \geqq \cdots \geqq a_n \geqq 0$ is any set of n non-negative numbers then there are $\binom{n+r-1}{r}$ homogeneous products $a_\omega = \prod_{t=1}^{r} a_{\omega_t}$, $\omega \in G_{r,n}$. Now, although $a_1^r \geqq a_1^{r-1}a_2$ are the two largest of these products, it is not true generally that the ordering according to magnitude and the lexicographic ordering of the a_ω, $\omega \in G_{r,n}$ coincide (e.g. $a_1^{r-2}a_2^2$ is not necessarily smaller than $a_1^{r-1}a_3$). We let $L_r(a_1, a_2, \cdots, a_n)$ designate the sum of the largest $N = \binom{n}{r}$ of the $\binom{n+r-1}{r}$ homogeneous products a_ω.

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[2] The authors would like to take this occasion to make the following correction in their paper *Generalizations of some combinatorial inequalities* by H. J. Ryser, this journal, vol. 7 (1963), pp. 582–592: On page 591, line 19, instead of "The matrix PP^T is nonnegative," read "The matrix $PP^T = H$ is nonnegative."

If $z_1, z_2, \cdots, z_n$ are complex numbers of modulus 1, then a matrix

$$A = P \operatorname{diag}(z_1, z_2, \cdots, z_n),$$

where P is a permutation matrix, is called a *generalized permutation matrix.*

Our main result is

THEOREM 1. *If $1 \leqq r \leqq n$ and A is an n-square complex matrix, then*

$$\begin{aligned} s_r(A) &= \textstyle\sum_{\alpha,\beta \in Q_{r,n}} |\operatorname{per} A[\alpha \mid \beta]|^2 \\ &\leqq L_r(\alpha_1^2(A), \alpha_2^2(A), \cdots, \alpha_n^2(A)). \end{aligned} \tag{1}$$

If $r = 1$ equality holds in (1). If $r > 1$ and A has no zero row (or no zero column), then equality holds in (1) if and only if $A = \delta R$ where $\delta > 0$ and R is a generalized permutation matrix.

COROLLARY 1. *Under the same hypotheses, if $N = \binom{n}{r}$ then*

$$s_r(A) \leqq \alpha_1^{2r}(A) + (N-1)\alpha_1^{2(r-1)}(A)\alpha_2^2(A).$$

For $r > 1$ the equality statement is the same as in Theorem 1.

We are interested in applying these results to *doubly stochastic* matrices $A = (a_{ij})$: i.e. those satisfying

$$\begin{aligned} a_{ij} &\geqq 0, & i, j &= 1, 2, \cdots, n, \\ \textstyle\sum_{i=1}^n a_{ij} &= 1, & j &= 1, 2, \cdots, n, \\ \textstyle\sum_{j=1}^n a_{ij} &= 1, & i &= 1, 2, \cdots, n. \end{aligned}$$

THEOREM 2. *If A is n-square doubly stochastic and $N = \binom{n}{r}$, then*

$$s_r(A) \leqq 1 + (N-1)\alpha_2^2(A). \tag{2}$$

Equality holds in (2) for $r > 1$ if and only if A is a permutation matrix.

We remark that for a stochastic matrix A, $|\operatorname{per} A[\alpha \mid \beta]| \leqq 1$, α and β in $Q_{r,n}$. However, this estimate yields only the trivial inequality $s_r(A) \leqq N^2$ whereas (2) has the immediate

COROLLARY 2. *If A is an n-square doubly stochastic matrix, then*

$$s_r(A) \leqq N. \tag{3}$$

Equality holds in (3) for $r \geqq 1$ *if and only if A is a permutation matrix.*

The incidence matrix of a (v, k, λ) configuration [7, p. 102] is v-square normal and satisfies $AA^* = A^*A = (k - \lambda)I + \lambda J$ where I is the v-square identity matrix and J is the v-square matrix all of whose entries are 1. The numbers v, k and λ are positive integers and satisfy $0 < \lambda < k < v$, and $(k - \lambda) = k^2 - \lambda v$; $k^{-1}A$ is doubly stochastic and $\alpha_1^2(A) = k^2$, $\alpha_j^2(A) = k - \lambda$, $j = 2, \cdots, v$. Directly applying Corollary 1 we have

THEOREM 3. *If A is the incidence matrix of a (v, k, λ) configuration, then*

$$s_r(A) \leqq k^{2r} + (\binom{v}{r} - 1)k^{2(r-1)}(k - \lambda).$$

Equality holds for $r = 1$. The inequality is always strict for $r > 1$.

II. Proofs

Let $P_r(A)$ denote the $\binom{n+r-1}{r}$-square r^{th} induced matrix of A [4]: recall that if α and β are in $G_{r,n}$ ordered lexicographically, then the α, β entry of $P_r(A)$ is $(\operatorname{per} A[\alpha \mid \beta])/(\mu(\alpha)\mu(\beta))^{1/2}$, where $\mu(\alpha)$ is the product of the factorials of the multiplicities of the distinct integers in α, e.g. $\mu(1, 1, 1, 3, 3, 4, 4, 4) = 3!\ 2!\ 3!$. Let $Q_r(A)$ be the N-square principal submatrix of $P_r(A)$ whose α, β entry, for α, β in $Q_{r,n}$ ordered lexicographically, is $(P_r(A))_{\alpha,\beta} = \operatorname{per} A[\alpha \mid \beta]$. Observe that

$$s_r(A) = \operatorname{trace}((Q_r(A))^* Q_r(A)) = \| Q_r(A) \|^2,$$

where $\| \ \|$ indicates the Euclidean norm. The idea of the proof of the inequality part of Theorem 1 is to obtain an upper bound on the Euclidean norm of $Q_r(A)$ as a principal submatrix of $P_r(A)$.

The discussion of the case of equality is, as usual, difficult and requires the use of the symmetric product of vectors. We temporarily defer this unpleasant business.

We require the elementary

LEMMA 1. *Let Y be an m-square complex matrix and let X be a rank q, k-square principal submatrix of Y.* Then

$$\| X \|^2 \leqq \sum_{j=1}^{q} \alpha_j^2(Y) \leqq \sum_{j=1}^{k} \alpha_j^2(Y). \tag{4}$$

If

$$\| X \|^2 = \sum_{j=1}^{k} \alpha_j^2(Y)$$

then there exists an m-square permutation matrix Q such that

$$Y = Q^T(X \dotplus W)Q \tag{5}$$

where W is $(m - k)$-square.

Proof. Let $X = (x_{ij})$, $Y = (y_{ij})$ and let $(i_1, i_2, \cdots, i_k) \in Q_{k,n}$ be the sequence for which $x_{st} = y_{i_s i_t}$, $s, t = 1, 2, \cdots, k$. Set $X_1 = PYP$, where P is the m-square projection matrix whose i_s, i_s entry is 1, $s = 1, 2, \cdots, k$, and whose remaining entries are 0. Then clearly

$$\| X \|^2 = \| X_1 \|^2 = \operatorname{trace}(X_1^* X_1).$$

Now, by an inequality due to A. Horn [2]

$$\begin{aligned} \prod_{j=1}^{p} \alpha_j(X_1) &= \prod_{j=1}^{p} \alpha_j(PYP) \\ &\leqq \prod_{j=1}^{p} \alpha_j^2(P)\alpha_j(Y) = \prod_{j=1}^{p} \alpha_j(Y), \quad p = 1, 2, \cdots, q. \end{aligned}$$

(The inequality $\prod_{j=1}^{p} \alpha_j(AB) \leqq \prod_{j=1}^{p} \alpha_j(A)\alpha_j(B)$ is proved first for $p = 1$, (the Cauchy-Schwarz inequality), and then for $p > 1$ by applying the $p = 1$ case to the compound matrix of AB [8].)

Thus

$$\sum_{j=1}^{p} \log \alpha_j(X_1) \leqq \sum_{j=1}^{p} \log \alpha_j(Y), \quad p = 1, 2, \cdots, q,$$

and by applying a lemma of G. Pólya [5], using the convex non-decreasing function $\varphi(t) = e^{2t}$, we have

$$\sum_{j=1}^{q} \varphi(\log \alpha_j(X_1)) \leqq \sum_{j=1}^{q} \varphi(\log \alpha_j(Y));$$

or

$$\| X \|^2 = \| X_1 \|^2 = \sum_{j=1}^{q} \alpha_j^2(X_1) \leqq \sum_{j=1}^{q} \alpha_j^2(Y) \leqq \sum_{j=1}^{k} \alpha_j^2(Y),$$

the inequality (4).

Suppose $\| X \|^2 = \sum_{j=1}^{k} \alpha_j^2(Y)$ and choose a permutation matrix Q such that

$$Y = Q^T \begin{pmatrix} X & U \\ V & W \end{pmatrix} Q.$$

Then

$$Y^*Y = Q^T \begin{pmatrix} X^*X + V^*V & X^*U + V^*W \\ U^*X + W^*V & U^*U + W^*W \end{pmatrix} Q.$$

A result of K. Fan [1] implies that

$$\sum_{j=1}^{k} \alpha_j^2(Y) = \sum_{j=1}^{k} \lambda_j(Y^*Y)$$

is at least as great as the sum of any k main diagonal elements of Y^*Y. Thus

$$\| X \|^2 + \| V \|^2 = \text{trace } (X^*X + V^*V) \leqq \sum_{j=1}^{k} \alpha_j^2(Y) = \| X \|^2,$$

and hence $V = 0$. Similarly, by examining YY^* we conclude that $U = 0$ and Y has the form (5).

LEMMA 2. *If A is an n-square complex matrix then*

$$s_r(A) \leqq L_r(\alpha_1^2(A), \alpha_2^2(A), \cdots, \alpha_n^2(A)). \tag{6}$$

If equality holds in (6), *then*

$$(P_r(A))_{\alpha,\beta} = \text{per } A[\alpha \mid \beta]/(\mu(\alpha)\mu(\beta))^{1/2} = 0$$

if $\alpha \in Q_{r,n}$ and $\beta \in G_{r,n}$, $\beta \notin Q_{r,n}$ or if $\alpha \in G_{r,n}$, $\alpha \notin Q_{r,n}$ and $\beta \in Q_{r,n}$.

Proof. We remark that for $r = 1$ the condition of equality is vacuous and (6) is always equality. The matrix $P_r(A)$ is a multiplicative function of A and the characteristic roots of $P_r(A)$ are the $\binom{n+r-1}{r}$ homogeneous products of degree r in the characteristic roots of A. Thus

$$(P_r(A))^*P_r(A) = P_r(A^*)P_r(A) = P_r(A^*A)$$

and the singular values of $P_r(A)$ are the homogeneous products of degree r in the singular values of A. It follows that the sum of the squares of the largest N singular values of $P_r(A)$ is just $L_r(\alpha_1^2(A), \alpha_2^2(A), \cdots, \alpha_n^2(A))$. Thus, if we take $Y = P_r(A)$, $X = Q_r(A)$, and $k = N$ in Lemma 1, then we have

$$\begin{aligned} s_r(A) &= \textstyle\sum_{\alpha,\beta\epsilon Q_{r,n}} |\operatorname{per} A[\alpha \mid \beta]|^2 \\ &= \| Q_r(A) \|^2 \leqq L_r(\alpha_1^2(A), \alpha_2^2(A), \cdots, \alpha_n^2(A)). \end{aligned}$$

This proves (6). If equality holds in (6), then by Lemma 1 we know that $P_r(A)$ is zero in the rows (columns) in which $Q_r(A)$ lies, outside the columns (rows) in which $Q_r(A)$ lies. In other words, $(P_r(A))_{\alpha,\beta} = 0$ if $\alpha \epsilon Q_{r,n}$ and $\beta \notin Q_{r,n}$, or if $\alpha \notin Q_{r,n}$ and $\beta \epsilon Q_{r,n}$. But

$$(P_r(A))_{\alpha,\beta} = \operatorname{per} A[\alpha \mid \beta]/(\mu(\alpha)\mu(\beta))^{1/2},$$

completing the proof of the lemma.

To complete our arguments we need the idea of the symmetric product of vectors. A coordinate definition of this is sufficient for our purpose.

Let

$$x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \quad \text{and let} \quad X = (x_{ij}), \qquad \begin{aligned} i &= 1, 2, \cdots, r, \\ j &= 1, 2, \cdots, n. \end{aligned}$$

Then $x_1{:}x_2{:} \cdots {:}x_r$, the symmetric product of $x_1, x_2, \cdots, x_r$, is defined to be the $\binom{n+r-1}{r}$-vector whose α^{th} component, $\alpha \epsilon G_{r,n}$ ordered lexicographically, is

$$\operatorname{per} X[1, 2, \cdots, r \mid \alpha]/(r!\,\mu(\alpha))^{1/2}.$$

We remark that $x_1{:}x_2{:} \cdots {:}x_r$ is symmetric in the x_i, i.e. for each permutation $\sigma \epsilon S_r$,

$$x_{\sigma(1)}{:}x_{\sigma(2)}{:}\cdots{:}x_{\sigma(r)} = x_1{:}x_2{:}\cdots{:}x_r.$$

If $(u, v) = \sum_{j=1}^m u_j\bar{v}_j$ denotes the usual inner product of two vectors $u = (u_1, u_2, \cdots, u_m)$, $v = (v_1, v_2, \cdots, v_m)$, then it is known [4] that

$$(7) \qquad (x_1{:}x_2{:} \cdots {:}x_r, y_1{:}y_2{:} \cdots {:}y_r) = \frac{1}{r!} \operatorname{per}((x_i, y_j)),$$

and moreover

$$P_r(A)(x_1{:}x_2{:}\cdots{:}x_r) = (Ax_1){:}(Ax_2){:}\cdots{:}(Ax_r).$$

Let $e_1, e_2, \cdots, e_n$ be the unit n-vectors, $e_i = (\delta_{i1}, \delta_{i2}, \cdots, \delta_{in})$, $i = 1, 2, \cdots, n$. Then from (7), the vectors $(e_{\alpha_1}{:}e_{\alpha_2}{:}\cdots{:}e_{\alpha_r})/(\mu(\alpha)/r!)^{1/2}$ are the unit $\binom{n+r-1}{r}$-vectors. It is easy to see that

$$(8) \qquad \begin{aligned} (\sqrt{\mu(\alpha)\mu(\beta)}/r!)(P_r(A))_{\alpha,\beta} &= (P_r(A)e_{\beta_1}{:}e_{\beta_2}{:}\cdots{:}e_{\beta_r}, e_{\alpha_1}{:}e_{\alpha_2}{:}\cdots{:}e_{\alpha_r}) \\ &= (Ae_{\beta_1}{:}Ae_{\beta_2}{:}\cdots{:}Ae_{\beta_r}, e_{\alpha_1}{:}e_{\alpha_2}{:}\cdots{:}e_{\alpha_r}) \\ &= (A^{(\beta_1)}{:}A^{(\beta_2)}{:}\cdots{:}A^{(\beta_r)}, e_{\alpha_1}{:}e_{\alpha_2}{:}\cdots{:}e_{\alpha_r}) \end{aligned}$$

where $A^{(t)}$ is the t^{th} column of A. According to Lemma 2, if equality holds in (6) then

$$(A^{(\beta_1)}:A^{(\beta_2)}:\cdots:A^{(\beta_r)}, e_{\alpha_2}:e_{\alpha_1}:\cdots:e_{\alpha_r}) = 0 \tag{9}$$

for all $\beta \in Q_{r,n}$ and all $\alpha \notin Q_{r,n}$.

LEMMA 3. *Assume $r > 1$. If $x_1, x_2, \cdots, x_n$ are non-zero n-vectors and*

$$(x_{\beta_1}:x_{\beta_2}:\cdots:x_{\beta_r}, e_{\alpha_1}:e_{\alpha_2}:\cdots:e_{\alpha_r}) = 0$$

for $\beta \in Q_{r,n}$ and $\alpha \notin Q_{r,n}$ then $x_t = z_t e_{\sigma(t)}$ where $\sigma \in S_n$ and $z_1, z_2, \cdots, z_n$ are complex numbers.

Proof. We show first that if $y_i = (y_{i1}, y_{i2}, \cdots, y_{in})$, $i = 1, 2, \cdots, r$, are non-zero vectors and $(y_1:y_2:\cdots:y_r, e_{\alpha_1}:e_{\alpha_2}:\cdots:e_{\alpha_r}) = 0$ for $\alpha \notin Q_{r,n}$, then for $s = 1, \cdots, n$, $y_{is} y_{js} = 0$ for $i \neq j$, $i, j = 1, 2, \cdots, r$. Let $Y = (y_{ij})$, $i = 1, 2, \cdots, r, j = 1, 2, \cdots, n$. If $Y^{(s)} \neq 0$ we can assume by the symmetry of the symmetric product that $y_{is} \neq 0$, $i = 1, 2, \cdots, p$, $y_{is} = 0$, $i = p + 1, p + 2, \cdots, r$. Since the sequence $(s, s, \cdots, s)$ is not in $Q_{r,n}$ it follows that

$$(y_1:y_2:\cdots:y_r, e_s:e_s:\cdots:e_s) = \prod_{i=1}^{r} y_{is} = 0$$

and hence $p < r$. It is shown in [4] that if $y_{p+1}:y_{p+2}:\cdots:y_r = 0$ then some $y_t = 0$, $p + 1 \leqq t \leqq r$. Hence there is at least one non-decreasing sequence $j_1 \leqq j_2 \leqq \cdots \leqq j_{r-p}$ such that

$$\operatorname{per} Y[p + 1, p + 2, \cdots, r \,|\, j_1, j_2, \cdots, j_{r-p}] \neq 0.$$

Consider $b = (y_1:y_2:\cdots:y_r, e_s:e_s:\cdots:e_s:e_{j_1}:e_{j_2}:\cdots:e_{j_{r-p}})$ where there are p occurrences of e_s. If $p > 1$ then $(s, s, \cdots, s, j_1, j_2, \cdots, j_{r-p})$ is not in $Q_{r,n}$ and hence $b = 0$. But

$$b = \frac{1}{r!} \operatorname{per} \left[\begin{array}{cccc|c} y_{1s} & y_{1s} & \cdots & y_{1s} & \\ \vdots & & & \vdots & * \\ y_{ps} & y_{ps} & \cdots & y_{ps} & \\ \hline 0 & 0 & & 0 & \\ \vdots & & & \vdots & Y[b + 1, \cdots, r \,|\, j_1, \cdots, j_{r-p}] \\ 0 & 0 & \cdots & 0 & \end{array}\right]$$

$$= \frac{p!}{r!}\left(\prod_{i=1}^{p} y_{is}\right) \operatorname{per} Y[p + 1, \cdots, r \,|\, j_1, \cdots, j_{r-p}] \neq 0.$$

Thus $p = 1$ and the above assertion is proved.

To complete the proof of the lemma, let X be the n-square matrix (x_{ij}) where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in})$. Now by the above, $X^{(s)}$ has at most one non-zero entry in it, $s = 1, 2, \cdots, n$. Thus X has at most n non-zero entries in it. On the other hand, if X were to have fewer than n non-zero entries then some row of X would have to be 0, i.e. some $x_i = 0$. Thus X has precisely n non-zero entries, one in each column and one in each row. This proves the lemma.

To prove Theorem 1 we argue as follows. Lemma 2 gives us the inequality. To discuss equality we note from (9) and Lemma 3 that

$$A = M \operatorname{diag}(z_1, z_2, \cdots, z_n)$$

where M is a permutation matrix and $z_1, z_2, \cdots, z_n$ are non-zero complex numbers. The singular values of A are clearly $|z_1|, |z_2|, \cdots, |z_n|$, and thus

$$L_r(\alpha_1^2(A), \alpha_2^2(A), \cdots, \alpha_n^2(A)) = L_r(|z_1|^2, |z_2|^2, \cdots, |z_n|^2).$$

On the other hand it is easy to compute that

$$s_r(A) = \sum_{\alpha,\beta\epsilon Q_{r,n}} |\operatorname{per} A[\alpha \mid \beta]|^2 = E_r(|z_1|^2, |z_2|^2, \cdots, |z_n|^2),$$

where E_r is the r^{th} elementary symmetric function of the indicated numbers. Thus, in the case of equality

$$s_r(A) = L_r(|z_1|^2, |z_1|^2, \cdots, |z_n|^2) = E_r(|z_1^2|^2, |z_2|^2, \cdots, |z_n|^2). \tag{10}$$

Lemma 4. *If* $a_1 \geqq a_2 \geqq \cdots \geqq a_n > 0$ *and* $r > 1$ *then*

$E_r(a_1, a_2, \cdots, a_n) = L_r(a_1, a_2, \cdots, a_n)$ *if and only if* $a_1 = \cdots = a_n$.

Proof. Suppose $a_k > a_{k+1}$ and equality holds. Then

$$\begin{aligned} E_r(a_1, \cdots, a_n) &= a_k a_{k+1} E_{r-2}(a_1, \cdots, a_{k-1}, a_{k+2}, \cdots, a_n) \\ &\quad + (a_k + a_{k+1})E_{r-1}(a_1, \cdots, a_{k-1}, a_{k+2}, \cdots, a_n) \\ &\quad + E_r(a_1, \cdots, a_{k-1}, a_{k+2}, \cdots, a_n) \\ &< a_k^2 E_{r-2}(a_1, \cdots, a_{k-1}, a_{k+2}, \cdots, a_n) \\ &\quad + (a_k + a_{k+1})E_{r-1}(a_1, \cdots, a_{k-1}, a_{k+2}, \cdots, a_n) \\ &\quad + E_r(a_1, \cdots, a_{k-1}, a_{k+2}, \cdots, a_n). \end{aligned}$$

This last expression is a sum of N homogeneous products of degree r in $a_1, \cdots, a_n$ and hence is no greater than $L_r(a_1, a_2, \cdots, a_n)$. This contradiction completes the proof.

From (10) and Lemma 4 we conclude in the case of equality in Theorem 1 that $\delta = |z_1|^2 = \cdots = |z_n|^2$ and hence

$$A = \delta M \operatorname{diag}(e^{i\theta_1}, e^{i\theta_2}, \cdots, e^{i\theta_n}), \qquad \theta_i \text{ real}, i = 1, 2, \cdots, n;$$

thus $A = \delta R$ where R is a generalized permutation matrix. Conversely if $A = \delta R$ it follows immediately that the equality holds in (1). This proves Theorem 1.

Corollary 1 is an immediate consequence of

$$L_r(a_1, \cdots, a_n) \leqq a_1^r + (N-1)a_1^{r-1}a_2, \qquad a_i = \alpha_i^2(A), i = 1, \cdots, n.$$

If A is doubly stochastic no row or column is 0 and moreover $\alpha_1(A) = 1$. Thus the inequality (2) follows from Corollary 1. The only doubly stochastic generalized permutation matrices are the permutation matrices and hence the case of equality in Theorem 2 follows. Corollary 2 follows similarly.

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UNIVERSITY OF CALIFORNIA
SANTA BARBARA, CALIFORNIA