THE ORDERS OF THE POSTNIKOV INVARIANTS OF THE THOM SPECTRUM MSO

BY

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The Thom space MSO_q has been introduced by Thom [18] for analysing the structure of the groups Ω_n of the cobordism classes of closed compact oriented differential manifolds of dimension n. We wish to study the Postnikov invariants of the Thom space MSO_q. The idea is to use the now known structure of the ring Ω: one obviously can so obtain the stable invariants only. Aside from the invariants which happen to be zero, it has not been possible to determine the classes which give these invariants, but only their orders. However this allows to sharpen a result of Thom relative to the realization of an integral homology class of a differential manifold by a differential submanifold.

I. Preliminaries

1. By what precedes, the object under study is really the Thom spectrum $MSO = \{MSO_q, f_q\}_{q \geq 1}$, i.e. the sequence of spaces $\{MSO_q\}$ and maps $\{f_q\}$, $q \geq 1$:

$$f_q : S(MSO_q) \to MSO_{q+1},$$

(S, reduced suspension) as defined by Thom [18].

$\pi_{n+q}(MSO_q) = \Omega_n$, if $q \geq n + 2$, and we will always make this assumption. $\Omega/\Omega_T = \bar{\Omega}(= \Omega \text{ mod torsion})$ is a polynomial algebra on the integers whose generators V^{4i} are characterized by

$$s_i[V^{4i}] = p \quad \text{if } 2i + 1 = p^s,\ p \text{ a prime},$$

$$s_i[V^{4i}] = 1 \quad \text{otherwise} \qquad \text{(see [4, Appendix])}.$$

In dimensions which are not multiples of four, Ω_n is a direct sum (occasionally empty) of groups isomorphic with Z_2; in dimensions which are multiples of four, it is a direct sum of groups isomorphic with Z and of groups isomorphic with Z_2, [14], [20].

A remark (with an elementary proof) by Dold [9] shows that Ω_n, $n \geq 8$, is never trivial. Moreover, Ω_{4i} has for its free part a direct sum of $\pi(i)$ groups isomorphic with Z, where $\pi(i)$ is the number of partitions of i in positive integers.

A general reference for the stable homotopy and the stable cohomology of MSO_q is [7].

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2. (Finite) Postnikov tower of a CW-complex. Let there be given a CW-complex X and a positive integer n. One can then construct a CW-complex $X(n)$ containing X as a subcomplex by attaching cells of dimensions greater than $n + 1$ with maps chosen such as to kill the homotopy groups of X in dimensions greater than n [22]. Then the inclusion $X \subset X(n)$ induces an isomorphism of the homotopy groups of dimensions at most n and $X(n)$ has the $(n + 1)$-homotopy type of X [23].

Proceeding as above one constructs a finite sequence of CW-complexes,

$$X \subset X(n) \subset X(n-1) \subset \cdots \subset X(1) \subset X(0),$$

in which the inclusion $X(i + 1) \subset X(i)$ induces an isomorphism of the homotopy groups in dimensions at most i, and the homotopy of $X(i)$ is trivial in dimensions greater than i. One can classically [5, (n.1)] turn the inclusions into fibrations in the sense of Serre [16] without changing the homotopy type of the $X(i)$'s. One may therefore consider the above sequence as a sequence of fibrations:

$$X \to X(n) \to X(n-1) \to \cdots \to X(1) \to X(0).$$

The fibre of the fibration $X(\mathrm{i}) \to X(i - 1)$ has $\pi_i(X)$ as its only non-trivial homotopy group, and is therefore a space of type $(\pi_i(X), i)$. The Postnikov invariants of X are the characteristic classes of these fibrations.

To apply this to a spectrum such as MSO, one takes $X = MSO_q$, with $q \geq n + 2$; X is then $(q - 1)$-connected. In such a case $X(n)$ denotes rather what has been called above $X(q + n)$, so that $X(0)$ is a space of type $(\pi_q(X), q)$. Thus the fibration $X(i) \to X(i - 1)$ has for fibre a space of type $(\pi_{q+i}(MSO_q), q + i)$.

3. Symmetric polynomials. We will need in Part III the following elementary facts about symmetric polynomials. (See [14].)

Let s_ω be the smallest symmetric polynomial containing the monomial $t_1^{i_1} \cdots t_r^{i_r}$, $\omega = (i_1, \cdots, i_r)$, where the t_j's are indeterminates of the same degree. One can always take (and we will) r equal to the number of variables. Denote s_ω by $\sum^* t_1^{i_1} \cdots t_1^{i_r}$. It is known [19] that s_ω can be expressed as a polynomial with integral coefficients in the elementary symmetric polynomials in the t_j' s:

$$p_1 = \sum\nolimits^* t_1, \quad p_2 = \sum\nolimits^* t_1 t_2, \quad \cdots, \quad p_r = \sum\nolimits^* t_1 \cdots t_r.$$

Order the monomials in the t's by the lexicographic order; one gets a total ordering. If $t_1^{i_1} \cdots t_r^{i_r}$ is the F.T.L.O. (first term in the lexicographic cover) of s_ω, then $i_1 \geq i_2 \geq \cdots \geq i_r$; for if $i_2 > i_1$, say, $t_1^{i_2} t_2^{i_1} \cdots t_r^{i_r}$ would be before $t_1^{i_1} t_2^{i_2} \cdots t_r^{i_r}$. Henceforth we assume $i_1 \geq i_2 \geq \cdots \geq i_r$. Clearly this ordering induces an ordering of the partitions

$$\omega = (i_1, \cdots, i_r) \quad \text{of} \quad i = i_1 + i_2 + \cdots + i_r,$$

and of the polynomials s_ω by $s_\omega < s_{\omega'}$ if $\omega < \omega'$. It also induces an ordering of the monomials which we now introduce, the monomials $p_1^{i_1-i_2}p_2^{i_2-i_3} \cdots p_r^{i_r}$. Indeed each ω determines such a monomial, and given such a monomial, one can find without ambiguity the ω from whence it comes. These monomials are therefore naturally ordered in the order in which they appear when ω increases.

We recall that to obtain s_ω in terms of the p's one forms $s_\omega - p_1^{i_1-i_2}p_2^{i_2-i_3} \cdots p_r^{i_r}$; this is a homogeneous symmetric polynomial which no longer contains $t_1^{i_1} \cdots t_r^{i_r}$. (Use is made of the obvious fact that the F.T.L.O. of a product is the product of the F.T.L.O. of the factors.) It has a F.T.L.O., $ct_1^{j_1} \cdots t_r^{j_r}$, with $c \in Z$, $j_1 \geq j_2 \geq \cdots \geq j_r$, such that $t_1^{i_1} \cdots t_r^{i_r} < t_1^{j_1} \cdots t_r^{j_r}$. The polynomial $s_\omega - p_1^{i_1-i_2} \cdots p_r^{i_r} - cp_1^{j_1-j_2} \cdots p_r^{j_r}$ contains neither $t_1^{i_1} \cdots t_r^{i_r}$ not $t_1^{j_1} \cdots t_r^{j_r}$. Since it is clear that for a given monomial there exists only a finite number of monomials (here of the same degree) which succeed it in the ordering, s_ω is of the form

$$s_\omega = p_1^{i_1-i_2} \cdots p_r^{i_r} + cp_1^{j_1-j_2} \cdots p_r^{j_r} + \cdots ,$$

and the monomials in the p's appear in the order induced by the order of the ω's. Note that *the first coefficient is one* and that if ω' is the immediate successor of ω then $s_{\omega'}$ begins with the monomial which is the immediate successor of the first monomial of s_ω in the ordering of the monomials in p. Hence the s_ω's can be expressed in terms of the p's by the following triangular pattern. We illustrate for the case $i = 4$:

$$\begin{aligned}
s_4 &= p_1^4 - 4p_1^2 p_2 + 2p_2^2 + 4p_1 p_3 - 4p_4 \\
s_{31} &= p_1^2 p_2 - 2p_2^2 - p_1 p_3 + 4p_4 \\
s_{22} &= p_2^2 - 2p_1 p_3 + 2p_4 \\
s_{211} &= p_1 p_3 - p_4 \\
s_{1111} &= p_4 .
\end{aligned}$$

It is now clear that conversely the p_ω's can be expressed as polynomials in the s_ω's with integral coefficients, p_ω denoting $p_{i_1} p_{i_2} \cdots p_{i_r}$. Therefore:

LEMMA. *A Z-module over the p_n's is a Z-module over the s_ω's.*

We are now in a position to state our results.

II. Statement of the results and description of the method

One first looks at the connection between the Postnikov towers of MSO and MO. MO has the homotopy type of a cartesian product of spaces of type (Z_2, m) (Thom, [18]), i.e. the Postnikov invariants of MO are all the zero class. One easily obtains a map $MSO \to MO$ induced by the inclusion $SO \subset O$. This map induces a homomorphism

$$h : \Omega_n \to \mathfrak{N}_n ,$$

where $\mathfrak{N}_n$ is the cobordism group of non-oriented differential manifolds of dimension n. Clearly h is obtained by disregarding the orientation. Rohlin has shown that Ker $h = 2\Omega_n$ (Dold [8] has given an elementary proof, described also by Wall [21]).

THEOREM A. *The reductions mod two of the Postnikov invariants of MSO are null classes.*

COROLLARY 1A. *The components of the Postnikov invariants of MSO corresponding to the 2-torsion of Ω are null.*

Proof. We simply observe that these are cohomology classes over Z_2. ▌

COROLLARY 2A (Wall [20]). *$H^*(MSO;\, Z_2)$ is a direct sum of free A_2-modules (one generator for each summand Z_2 of Ω in the same dimension) and of A_2-modules with one relation* $\mathrm{Sq}^1\, u = 0$ *(one generator u for each summand Z of Ω in the same dimension).*

Proof. By Theorem A, $H^*(MSO; Z_2)$ is the cohomology mod 2 of a product of spaces of type $(Z_2\,,\, *)$ and of type $(Z,\, *)$ in the relevant dimensions. ▌

Proof of Theorem A. Let $X = MSO_q\,,\; Y = MO_q$ and

$$X \to X(n) \to X(n-1) \to \cdots \to X(0) = K(Z, q),$$

$$Y \to Y(n) \to Y(n-1) \to \cdots \to Y(0) = K(Z_2\,, q)$$

be Postnikov towers of the same height for X and Y. One wishes to construct maps $f_1 : X(i) \to Y(i)$ such that the diagrams

$$\begin{array}{ccc} X(i) & \xrightarrow{f_i} & Y(i) \\ \downarrow & & \downarrow \\ X(i-1) & \xrightarrow{f_{i-1}} & Y(i-1) \end{array}$$

are commutative, and which are extensions of $f : X \to Y$. The construction is by induction, starting with n. As the first step is very similar to the induction step, we do only the induction step. This is by classical obstruction theory; the obstruction to the extension of f_i is always zero, either because the coefficient group is zero, or because the cohomology group is zero. Details are left to the reader. Commutativity of the diagram follows since the fibrations are constructed from inclusions.

The characteristic class τu_i in $H^*(X(i-1), \Omega_i)$ corresponds to the characteristic class τv_i in $H^*(Y(i-1), \mathfrak{N}_i)$ by

$$\rho(\tau u_i) = f_{i-1}^*(\tau v_i),$$

where $\rho : H^*(X(i-1), \Omega_i) \to H^*(X(i-1), \mathfrak{N}_i)$ is the homomorphism induced by $h : \Omega_i \to \mathfrak{N}_i$. In view of Rohlin's Theorem, this is just reduction mod 2. Because $\tau v_i = 0$ and this construction can be made at any level of the Postnikov tower, Theorem A is proved. ▌

By using the construction recalled in I, M. Kervaire [12] has obtained an expression for the Postnikov invariants which we now describe. Consider the cohomology exact sequence of the pair $(X(i-1), X)$ over the coefficients Ω_i:

$$\cdots \xleftarrow{\delta} H^*(X;\Omega_i) \xleftarrow{j^*} H^*(X(i-1);\Omega_i) \xleftarrow{a^*} H^*(X(i-1), X;\Omega_i) \xleftarrow{\delta} \cdots .$$

Assume X is $(q-1)$-connected; in our case $X = MSO_q$. From the fact that $H_{q+i+1}(X(i-1), X) = \pi_{q+i+1}(X(i-1), X) = \pi_{q+i}(X) = \Omega_i$, it follows that there exists a fundamental class u in $H^{q+i+1}(X(i-1), X;\Omega_i) \cong \operatorname{Hom}(\Omega_i, \Omega_i)$. The result of Kervaire is that *the Postnikov invariant of X in dimension $q+i+1$ is a^*u.* One recalls that in dimension $4i+1$, the Postnikov invariant of MSO, τu_{4i}, has $\pi(i)$ components, each corresponding to one factor of the fibre K_{4i} of type $(Z, q+4i)$. Denote by k_ω the component corresponding to the partition ω. We apply the result of Kervaire to MSO; this allows us to prove:

THEOREM B. *The order of the component k_ω of the Postnikov invariant of MSO in dimension $4i+1$ is*

$$n_\omega = \{2i_1+1\}\cdot\{2i_2+1\}\cdot \cdots \cdot\{2i_r+1\},$$

where $\omega = (i_1, \cdots, i_r)$ is any partition of i, and $\{2i_t+1\} = p$ or 1 depending on whether $2i_t+1 = p^s$ (p a prime necessarily odd) or is divisible by at least two distinct primes.

In the course of the proof one needs the following theorem:

THEOREM C. *There exists a Milnor basis $\{V^{4i}\}_{i\in N}$, such that if $2i+1 = p^s$, p a prime, all the Pontrjagin numbers of V^{4i} are multiples of p.*

Using the notion of stable cohomology operation, we finally prove:

THEOREM D. *Given an integral cohomology class v of dimension m in an orientable differential manifold of dimension n, with $m < [n/2]$, one is assured that $f(m, n)\cdot v$ is realizable by a submanifold by taking $f(m, n)$ to be the product α_k of the denominators $\mu_1, \cdots, \mu_k$ of the Hirzebruch polynomials $L_1, \cdots, L_k$, k being the integral part of $m/4$ (for L_k, see [10]).*

Theorems B and C are proved in Part III, Theorem D in Part IV.

III. The computation of the orders

Let $X = MSO_q$; we will determine the image of

$$\delta : H^{q+4i}(X;\ \bar{\Omega}_{4i}) \to H^{q+4i+1}(X_{4i-1},\ X;\ \bar{\Omega}_{4i}),$$

where the last group is isomorphic with

$$\operatorname{Hom}(\Omega_{4i}\,{}_4\,{}^{\backprime}\bar{\Omega}_i) = \operatorname{Hom}(\bar{\Omega}_{4i}, \bar{\Omega}_{4i}).$$

By exactness Ker $a^* = \operatorname{Im} \delta$, hence we have a method for determining the orders of the components of a^*u, u the fundamental class of Hom $(\bar{\Omega}_{4i}, \bar{\Omega}_{4i})$. Clearly we can neglect the 2-torsion of X.

1. Determination of Im δ.

LEMMA 1B. *Let c be a cohomology class of MSO over any group of coefficients Γ. Then*

$$\delta c[\alpha] = \alpha_1^* \bar{c}[V^{4i}],$$

where V^{4i} is a representative of an element in $\bar{\Omega}_{4i}$, α is the corresponding homotopy class in $\pi_{4i}(MSO)$, $c = \varphi\bar{c}$, and α_1 is the homotopy class of the map $V^{4i} \to BSO$ induced by α.

Proof. Let c' be a $(q + 4i)$-cocycle representing

$$c \in H^{q+4i}(MSO_q\,; \Gamma).$$

$\delta c'$ is defined as the coboundary of any extension of c' to X_{4i-1} (see [17, p. 164]). We compute $\delta c'$ on the cycles which are images of the cube I^{q+4i+1} by maps

$$f: (I^{q+4i+1}, I^{q+4i}, J^{q+4i}) \to (X_{4i-1}, X, *),$$

with $[f \mid I^{q+4i}] = \alpha$; i.e. $f \mid (I^{q+4i}, \dot{I}^{q+4i})$ is a representative of $[V^{4i}]$. One has for any coefficients

$$\delta c'[fI^{q+4i+1}] = c'[fI^{q+4i}] = f^*c'[I^{q+4i}].$$

One has therefore to compute $f^*c'[I^{q+4i}]$, or equivalently $\alpha^*c'[I^{q+4i}] = \alpha^*c[S^{q+4i}]$ (by replacing the cube by the sphere, which is permissible because the boundary of the cube is carried onto the base point ∞ of MSO_q). Consider the diagram which follows, known to be commutative, [18], and in which φ is the Thom-Gysin isomorphism.

$$\begin{array}{ccc} H^{q+4i}(S^{q+4i}; \Gamma) & \xleftarrow{\alpha^*} & H^{q+4i}(MSO_q; \Gamma) \\ \uparrow{\scriptstyle \varphi'} & & \uparrow{\scriptstyle \varphi} \\ H^{4i}(V^{4i}; \Gamma) & \xleftarrow{\alpha_1^*} & H^{4i}(BSO_q; \Gamma) \end{array}$$

We obtain

$$\alpha^*(c) = \alpha^*\varphi(\bar{c}) = \varphi'\alpha_1^*(\bar{c}).$$

But $\varphi' = D_S^{-1}sD_V$, where D is the Poincaré duality (valid here over an arbitrary coefficient group because the manifolds are orientable) and s is the homology homomorphism induced by the inclusion $V \subset S$. Hence

$$\varphi'(\alpha_1^*(\bar{c}))[S^{q+4i}] = D_S^{-1}sD_V(\alpha_1^*(\bar{c}))[S^{q+4i}].$$

Let $\alpha_1^*(\bar{c})[V^{4i}] = \gamma \in \Gamma$. Then

$$sD_V(\alpha_1^*(\bar{c})) = \gamma \cdot 1_S\,,$$

where 1_S is the generator of the homology of S in dimension zero. Hence:

$$\alpha^*(c)[S^{q+4i}] = D_S^{-1}sD_V(\alpha_1^*(\bar{c}))[S^{q+4i}] = \alpha_1^*(\bar{c})[V^{4i}].$$

One applies Lemma 1B with $\Gamma = \bar{\Omega}_{4i}$, $c = \varphi\bar{c}$, $\bar{c}$ being a linear combination of $\pi(i)$ Pontrjagin classes. $\alpha^*\bar{c}$ is the corresponding linear combination of $\pi(i)$ Pontrjagin classes on V^{4i}. We shall write $\delta c \in \mathrm{Hom}\,(\bar{\Omega}_{4i}, \bar{\Omega}_{4i})$ as a $\pi(i) \times \pi(i)$-matrix, where the $\omega\omega'$-coefficient is the value of the Pontrjagin class p_ω on the basis element $V_{\omega'}$ of $\bar{\Omega}_{4i}$. $\bar{c}$ is represented by a column vector. In fact instead of the Pontrjagin classes p_ω ($p_\omega = p_{i_1} \cdots p_{i_r}$, $\omega = (i_1, \cdots, i_r)$ a partition of i), we shall take the classes s_ω expressed in terms of the p_ω as explained in I, 3. The s_ω form also a basis for the free part of $H^{4i}(MSO)$. One is therefore led to consider matrices S, one for each i, with elements $S_{\omega\omega'} = s_\omega[V_{\omega'}]$; here one writes $V_{\omega'} = V^{4j_1} \times V^{4j_2} \times \cdots \times V^{4j_r}$ if $\omega' = (j_1, \cdots, j_r)$ is a partition of i. Let s be the column vector whose entries are the s_ω's; any vector will then be of the form As, with A a $\pi(i) \times \pi(i)$-matrix with integral entries. For a particular choice of the basis in $\bar{\Omega}_{4i}$, the S's are generators of $\mathrm{Im}\,\delta$ mod torsion. Clearly each element of $\mathrm{Im}\,\delta$ is of the form AS.

If e_ω denotes the $\pi(i) \times \pi(i)$-matrix whose only non-zero entry is one occurring on the diagonal in the line ω, $k_\omega = a^* e_\omega$ will be one component of the Postnikov invariant. Its order is the smallest number d_ω such that $a^*\, d_\omega\, e_\omega = 0$ or $d_\omega\, e_\omega \in \mathrm{Ker}\, a^* = \mathrm{Im}\,\delta$. If this is done for all ω, one sees that the orders under investigation are exactly the smallest possible values of the d_ω such that $AS = D$, for all possible A, D being the diagonal matrix with entries d_ω. The fact that these orders are finite shows that there exists a basis for $\bar{\Omega}_{4i}$ such that AS is diagonal.

2. Some lemmas on the values of the s_ω's on manifolds. (See [14].) On every differential manifold V^{4i}, s_ω determines an integer $s_\omega[V^{4i}]$ if $\omega = (i_1, \cdots, i_r)$ is a partition of i. A basis $\{V^{4i}\}$ for the polynomial algebra $\Omega \otimes Q$ may be characterized by the condition $s_i[V^{4i}] \neq 0$.

LEMMA 2B. $s_\omega[V^{4i_1} \times V^{4i_2}] = \sum_{(\omega_1\omega_2)=\omega} s_{\omega_1}[V^{4i_1}]\cdot s_{\omega_2}[V^{4i_2}]$, *the summation being extended over all pairs of partitions ω_1, ω_2 whose union is ω.*

Proof. See [18], where the same property is proved for Stiefel-Whitney numbers. The proof applies literally here if one makes the computation over the rationals (a field). But this suffices to prove the result over the integers.∎

LEMMA 3B. *If $\omega_s \notin \prod(i_s)$, then $s_{\omega_s}[V^{4i_s}] = 0$.*

Proof. Immediate, for dimensional reasons.∎

COROLLARY 4B. $s_\omega[V_1^{4j_1} \times V_2^{4j_2} \times \cdots \times V_s^{4j_s}] = \sum_{(\omega_1\cdots\omega_s)=\omega} s_{\omega_1}[V_1^{4j_1}]\cdot s_{\omega_2}[V_2^{4j_2}] \cdots s_{\omega_s}[V_s^{4j_s}]$ (*and $\omega_\alpha \in \prod(j_\alpha)$ by Lemma* 3B).

Proof. By induction on s.∎

COROLLARY 5B. $s_\omega[V_1^{4i_1} \times \cdots \times V_r^{4i_r}] = s_{i_1}[V_1] \cdots s_{i_r}[V_r]$, *if* $\omega = (i_1, \cdots, i_r)$.

Proof. For $s_\omega[V_s] = 0$ if ω is not a partition of i_s.∎

COROLLARY 6B. $s_\omega[V_1^{4j_1} \times \cdots \times V_s^{4j_s}] = 0$ *unless* ω *is a subpartition of* $(j_1, \cdots, j_s) = \bar{\omega}$.

Proof. Use Corollary 4B.∎

Corollary 6B is equivalent to the assertion that $s_\omega[V_{\bar{\omega}}] = 0$ if ω precedes $\bar{\omega}$, i.e. *the matrices* S *considered in* **1** *are triangular* with diagonal entry

$$s_{i_1}[V_1^{4i_1}] \cdots s_{i_r}[V_r^{4i_r}]$$

on the row $\omega = (i_1, \cdots, i_r)$. For what follows it is useful to keep in mind that the invertible triangular matrices with integral entries and zeros above the main diagonal form a group under multiplication.

3. The system $AS = D$. If any column of S is divisible by the entry of that column on the diagonal, then the answer to the question raised above about that system is straight-forward. Indeed, let $\tilde{S}$ denote the matrix obtained from S *after* these divisions; then $\tilde{S}$ is unimodular, and $\tilde{S}^{-1}$ has integral entries. We note that $S = \tilde{S}\bar{D}$, $\bar{D} = (\bar{d}_{\omega\omega'})$ being the diagonal matrix such that $\bar{d}_{\omega\omega} = S_{\omega\omega}$. $AS = D$ implies $D = (A\tilde{S})\bar{D}$, i.e.: the entries of D must be multiples of the corresponding entries of $\bar{D}$, which are therefore the smallest possible ones (then $A = \tilde{S}^{-1}$).

We will show that one is always in that simple case and this will complete the proof of Theorem B.

LEMMA 7B. *If* $s_i[V^{4i}]$ *divides* $s_{\omega_i}[V^{4i}]$ *for every partition* ω_i *of* i *and for all* i, *then* $s_\omega[V_\omega]$ *divides* $s_{\omega'}[V_\omega]$ *for all* $\omega' > \omega$(V^{4i} *is any generator of* $\Omega \otimes Q$).

Proof. By assumption, $s_{i_1}[V_1^{4i_1}]$ divides $s_{\omega_1}[V_1^{4i_1}]$, $\cdots$. Therefore $s_\omega[V_\omega]$ ($= s_{i_1}[V_1] \cdots s_{i_r}[V_r]$ by Corollary 5B) is such that each factor divides a distinct factor in each term of the sum

$$s_{\omega'}[V_\omega] = \sum\nolimits_{(\omega_1 \cdots \omega_r) = \omega'} s_{\omega_1}[V_1] \cdots s_{\omega_r}[V_r],$$

(ω_t a partition of i_t by Corollary 4B), hence divides the sum.∎

Since Theorem C states that the hypothesis of Lemma 7B is realized by a proper choice of a basis for Ω mod torsion, Theorem B will be completely proved once Theorem C is proved.

4. Proof of Theorem C.

LEMMA 1C. (Atiyah-Hirzebruch [3]). *The dimension of the space of relations mod* p (p *a prime*) *between the Pontrjagin numbers of an oriented differential manifold of dimension* $4i$ *is the number* $d'(i)$ *of partitions of* i *with at least one summand of the form* $(p^j - 1)/2(1 \leq j, (p^j - 1)/2 \leq i)$.

Let E_1 be the vector space over Z_p of the Pontrjagin classes p_ω of dimension $4i$, and let E_2 be the vector space over Z_p of the cobordism classes also of dimension $4i$ ($E_2 = \Omega_{4i} \otimes Z_p = \bar{\Omega}_{4i}/p\bar{\Omega}_{4i}$). One has $\dim E_1 = \dim E_2 = \pi(i)$. Between E_1 and E_2 there exists a bilinear pairing given by the value mod p of an element in E_1 on an element in E_2 (recall that the Pontrjagin numbers are invariants of the cobordism classes). The space of relations (mod p) R_1 in E_1 is the annihilator of E_2 in this pairing; let R_2 be the annihilator of E_1. R_2 is the set of cobordism classes for which all Pontrjagin numbers mod p are zero. The following lemma is trivial and well known.

LEMMA 2C. *With the above notations, $R_1 \subset E_1$ and $R_2 \subset E_2$ are vector subspaces and dim $R_1 = \dim R_2$.* (See e.g. [2].)

Therefore R_2 is isomorphic with $(Z_p)^{d'(i)}$, $d'(i)$ as in Lemma 1C. The following two lemmas prove that for $s = 1$, the complex projective space of complex dimension $p - 1$, $PC(p - 1)$, can be chosen for a Milnor basis element as required by Theorem C.

LEMMA 3C. *The total Pontrjagin class of $PC(2i)$ is $(1 + \alpha^2)^{2i+1}$, where α is a generator of the integral cohomology of $PC(2i)$.*

Proof. See [10, p. 73].∎

LEMMA 4C. *$s_i[PC(2i)]$ is equal to $2i + 1$ and divides $s_{\omega'}[PC(2i)]$ for all ω' partitions of i if $2i + 1 = p$, p a prime. Equivalently $PC(p - 1)$ satisfies the assumption of Lemma* 7B.

Proof. First $s_i = \alpha^{2i} + \cdots + \alpha^{2i}$ (p terms) or $p\cdot$ (fundamental class) (use the definition of s_i).

Secondly $s_{\omega'}$ is a linear combination with integral coefficients of the p_ω's, and by Lemma 3C the total Pontrjagin class of $PC(p - 1)$ is 1 mod p. It follows that p divides every Pontrjagin class of $PC(p - 1)$, therefore also $p_\omega[PC(p - 1)]$, for all ω partitions of $(p - 1)/2$.∎

Theorem C is now proved by reproducing an argument of Conner and Floyd [7, pp. 113–115].∎

IV. The additivity of obstructions and the realizability of homology classes

Thom [18] has shown that for every integral homology class v of dimension m in a differential manifold M of dimension n, there exists an integer $f(m, n, v)$ such that $f(m, n, v)\cdot v$ is realizable by a submanifold. An upper bound $f(m, n)$ of the numbers $f(m, n, v)$ will be determined below for $m < [n/2]$. According to Thom, if c is the cohomology class dual to v, v is realizable if and only if there exists a map $f : M \to MSO_{n-m}$ such that $f^*(U) = c$, where U is the fundamental class of MSO_{n-m}. Our task is to find sufficient conditions for the existence of f. f will be constructed by successive extensions through the Postnikov tower of MSO_{n-m}.

LEMMA 1D. [6, Exposé 13]. *Let f_1, f_2, f_3 be three maps $X \to \Omega Y$ such that*

$f_3 = f_1 \circ f_2$, *the multiplication being that of loops*; X, Y *are topological spaces*, ΩY *is the loop space of* Y *based at* y_0. *Then if* η *is in the image of the suspension*

$$\sigma : H^*(Y) \to H^*(\Omega Y),$$

the following is true:

$$f_3^*(\eta) = f_1^*(\eta) + f_2^*(\eta).$$

Proof. See [6, Exposé 13].∎

A basic reference for what follows is Adams [1], especially Chapter II.

LEMMA 2D. *In the stable range, the higher obstructions are additive.*

Proof. Consider the stable Postnikov tower of any space X and two maps

$$f_i : M \to X(0)$$

such that $f_i^*(u_0) = c_i \ (i = 1, 2)$. We consider this tower as obtained from another tower by applying to it the functor Ω. By Theorem 3.4.9 [1] and an easy induction, the Postnikov invariants correspond in both towers by suspension, may be up to sign; Lemma 1D can be applied to them. Assume f_i has been lifted to $X(k-1)$ and call one such lifting f_i^{k-1}. If we continue to denote (improperly) by u_0 the extension of u_0 to $X(k-1)$, then Lemma 1D gives

$$(1) \qquad (f_1^{k-1} \circ f_2^{k-1})(u_0) = (f_1^{k-1})^*(u_0) + (f_2^{k-1})^*(u_0) = c_1 + c_2 .$$

Also if u is any class in the cohomology of $X(k-1)$,

$$(2) \qquad (f_1^{k-1} \circ f_2^{k-1})^*(u) = (f_1^{k-1})^*(u) + (f_2^{k-1})^*(u).$$

Now the class u determines a stable cohomology operation of the k^{th}-kind, say Φ_k, for which the tower is a universal example, with $\Phi_k(u_0) \in u$. By the naturality of Φ_k:

$$(f_1^{k-1} \circ f_2^{k-1})^* \Phi_k(u_0) = \Phi_k(f_1^{k-1} \circ f_2^{k-1})^*(u_0) = \Phi_k(c_1 + c_2),$$

by (1); and

$$(f_1^{k-1} \circ f_2^{k-1})^*(\Phi_k(u_0)) = \Phi_k(f_1^{k-1})^*(u_0) + \Phi_k(f_2^{k-1})^*(u_0) = \Phi_k(c_1) + \Phi_k(c_2)$$

by (2) applied to $\Phi_k(u_0)$.

Therefore

$$\Phi(c_1 + c_2) = \Phi(c_1) + \Phi(c_2).∎$$

To prove Theorem D we must insure that

$$f : M^n \to X(0) = K(Z, n - m)$$

can be lifted to $X(n)$ with $f^*(u_0) = \alpha_k c$. This will be achieved if at each level from $X(0)$ to $X(4k)$ one can insure that $0 \in \Phi_i(\alpha_k c)$ with Φ_i the cohomology operation determined by τu_i (see [1, Lemma 3.3.8]). But $\tau u_i \in \Phi_i(u_0)$ has for order the least common multiple of the numbers n of Theorem B. The following lemma gives that order.

LEMMA 3D. *The least common multiple of the numbers* n_ω *of Theorem* B *is* $\prod_{3\leq p} p^{[2i/(p-1)]}$, *$p$ any prime.*

Proof. Each prime factor of the least common multiple must have for exponent the largest exponent it has among all numbers n_ω. This is obviously $[2i/(p-1)]$ for the prime p.∎

For a proof that this is the denominator μ_i of the polynomial L_i introduced by Hirzebruch see [3].

Proof of Theorem D. Assume that

$$f_{i-1} : M \rightarrow X(4i-4)$$

exists, such that $f_{i-1}^*(u_0) = \alpha_{i-1}\, c$. Then there exists a map

$$f_i : M \rightarrow X(4i-4)$$

such that $f_i^*\, u_0 = \mu_i\, \alpha_{i-1}\, c = \alpha_i\, c$, for the obstructions for f_i are the obstructions for f_{i-1} multiplied by μ_i, therefore can be made zero. Now

$$\Phi_i f_i^*(u_0) = \mu_i\, \Phi_i(\alpha_{i-1}\, c) = \mu_i f_{i-1}^*(\Phi_i\, u_0)$$

contains $\mu_i f_{i-1}^*(\tau u_i)$, which is zero; here and above use has been made of the additivity of Φ_i. Therefore f_i can be lifted to $X(4i)$. Starting with $f_0 : M \rightarrow X(0)$ such $f_0^*\, u_0 = c$, we end up with $f_k : M \rightarrow X(4k)$ such that $f_k^*\, u_0 = \alpha_k\, c$. Then $\Phi_i(\alpha_k\, c)$ contains 0 for all i.

k is clearly the largest integer such that $4k \leq m$. The result is valid in the stable range, i.e. for $m < n - m - 1$ or $m < (n-1)/2$. If n is even, this is $m \leq n/2 - 1$. If n is odd, $(n-1)/2$ is an integer equal to $[n/2]$. In both cases $m < [n/2]$.∎

A formula for α_k. One has $\alpha_k = \prod_{3\leqq p} p^{\nu(k,p)}$, p running over all primes. We compute

$$\nu(k, p) = \sum_{1\leqq i\leqq k} \left[\frac{i}{\frac{p-1}{2}}\right].$$

We decompose this sum as follows:

$$\left(\left[\frac{1}{\frac{p-1}{2}}\right] + \left[\frac{2}{\frac{p-1}{2}}\right] + \cdots\right) + \left(\left[\frac{\frac{p-1}{2}}{\frac{p-1}{2}}\right] + \left[\frac{\frac{p-1}{2}+1}{\frac{p-1}{2}}\right] + \cdots\right)$$

$$+ \cdots + \left(\left[\frac{q\,\frac{p-1}{2}}{\frac{p-1}{2}}\right] + \cdots + \left[\frac{k}{\frac{p-1}{2}}\right]\right)$$

where

$$q = \left[\frac{k}{\frac{p-1}{2}}\right].$$

Each partial sum contains $(p - 1)/2$ terms, except the last one where the number of terms is r, remainder of the division of k by $(p - 1)/2$, plus one $(k = q(p - 1)/2 + r, r < (p - 1)/2)$. Now clearly

$$\begin{aligned}\nu(k, p) &= \frac{(p-1)}{2}(1 + 2 + \cdots + (q - 1)) + (r + 1)q \\ &= \frac{p-1}{2}\frac{q(q-1)}{2} + \left(k + 1 - \frac{p-1}{2}q\right)q,\end{aligned}$$

or

$$\nu(k, p) = q(k + 1) - \frac{q(q+1)}{2}\frac{p-1}{2} \quad \text{where} \quad q = \left[\frac{2k}{p-1}\right]. \blacksquare$$

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