INVARIANT IDEALS OF POSITIVE OPERATORS IN $C(X)$. I

BY

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This paper presents the first part of a systematic study of the family of closed ideals invariant under a given positive linear operator T on $C(X)$, where X is a compact Hausdorff space. (The special relationship of positive operators and ideals of the algebra $C(X)$ is due to the identity of closed algebraic ideals and closed order ideals when the scalars are real, cf. §1.) This study draws much of its motivation from the properties of irreducible positive operators discovered in [9], in particular, properties of the spectrum. The notion of maximal T-invariant ideal is the key device to utilize these properties for more general positive operators, even in arbitrary Banach lattices (cf. [6] and [10, Appendix, §3]).

§1 of the present paper gives a survey of the tools to be used in the sequel. §2 is concerned with basic techniques for the study of closed ideals invariant under T (briefly, T-ideals) and gives a representation theorem for maximal T-ideals (Thm. 1). §3 exhibits the bijective correspondence, for ergodic Markov operators on $C(X)$ (§3, Def. 3), between the family of maximal T-ideals and the extreme points of the set of positive, normalized, T-invariant measures on X (Thm. 2). A dual result (Thm. 3) characterizes, in a similar fashion, the set of all (non-trivial) minimal closed order ideals in $L^1(\mu)$ that are invariant under a given ergodic and stochastic operator T.

It is perhaps not entirely unnecessary to point out that the spaces $C(X)$ include all spaces $L^\infty(\mu)$ and, after the adjunction of a unit, spaces of all continuous functions on a locally compact space that vanish at infinity. Also, studying ideals invariant under T is the same as studying ideals invariant under the semigroup of operators generated by T. More generally, many of the techniques used in this paper can be employed to study the family of ideals invariant under a given semigroup of positive operators.

1. Preliminaries

Let X be a compact (Hausdorff) space. We denote by $C(X)$ the Banach algebra (under the standard norm) of continuous complex-valued functions on X. The real subalgebra $C_R(X)$ of real-valued functions is a Banach lattice; if $|f|$ denotes the function $t \to |f(t)|$, $f \to |f|$ is a mapping of $C(X)$ into $C_R(X)$, and $C(X)$ can be considered to be the complexification of $C_R(X)$.

It is well known that the closed ideals J in $C(X)$ are sets of the form $J = \{f : f(t) = 0 \text{ for all } t \in S_J\}$, where S_J is a compact subset of X uniquely determined by J and called the **support** of J. Such an ideal can therefore be equally characterized as a closed vector subspace J of $C(X)$ such that

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$f \in J$ implies $g \in J$ whenever $|g| \leqq |f|$. In particular, in $C_R(X)$ the closed algebraic ideals are identical with the closed order ideals.

We denote by $M(X)$ the Banach space of all complex Radon measures on X; that is, $M(X)$ is the strong dual of the Banach space $C(X)$. $M(X)$ is the complexification of the (real) Banach space $M_R(X)$ of real Radon measures on X; each element of $M(X)$ is a linear combination (with complex coefficients) of measures of the form

$$f + ig \rightarrow \phi(f) + i\phi(g)$$

where $f, g \in C_R(X)$ and $\phi \in M_R(X)$. By a **positive** Radon measure we understand a Radon measure $\phi \in M_R(X)$ such that $\phi(f) \geqq 0$ whenever $f \geqq 0$.

The strong dual of $M(X)$ is a Banach space that can be identified with a Banach space $C(Z)$, where Z is an extremally disconnected compact space, and there exists a continuous, surjective map $q : Z \rightarrow X$ such that $f \rightarrow f \circ q$ is an isometric isomorphism of the Banach algebra $C(X)$ into the Banach algebra $C(Z)$. Moreover, for the topology of uniform convergence on the order intervals in $M(X)$, $C(Z)$ is a complete locally convex vector lattice in which $C(X)$ is dense. (For details, see [10, Chap. V §§ 7, 8].)

We shall employ the following notation. Elements of X will be denoted by $s, t, \cdots$; elements of $C(X)$ by $f, g, \cdots$; elements of $M_R(X)$ by $\phi, \psi, \cdots$. The point measure (unit mass) at $s \in X$ is denoted by ε_s. If $\phi \in M_R(X)$, the **support** of ϕ is denoted by S_ϕ. Each $\phi \geqq 0$ generates an ideal

$$I_\phi = \{f : \phi(|f|) = 0\}$$

whose support is identical with the support S_ϕ of ϕ. If I is a closed ideal in $C(X)$, the quotient algebra $C(X)/I$ will be canonically identified with $C(S_I)$. The annihilator I^0 of I in $M_R(X)$ is a weakly closed band.

Throughout the paper, T will denote a **positive** linear operator in $C(X)$, that is, a (necessarily continuous) linear map of $C(X)$ into itself such that $f \geqq 0$ implies $Tf \geqq 0$. It is clear that the restriction T_0 of such an operator to $C_R(X)$ is a positive linear operator and that, conversely, each positive operator T_0 on $C_R(X)$ has a unique linear extension T on $C(X)$ defined by

$$T(f + ig) = T_0 f + iT_0 g \qquad (f, g \in C_R(X)).$$

The **spectral radius** of T is the smallest real number $\rho \geqq 0$ such that $|\lambda| \leqq \rho$ whenever $\lambda \in \sigma(T)$, and is denoted by $r(T)$. The **peripheral spectrum** of T is the set

$$\{\lambda \in \sigma(T) : |\lambda| = r(T)\}.$$

For $\lambda \notin \sigma(T)$, we write $R(\lambda) = (\lambda - T)^{-1}$. Let us note that $r(T) \in \sigma(T)$ for each positive operator T. In fact, for each complex λ, $|\lambda| > r(T)$, and each $f \in C(X)$ we have

$$|R(\lambda)f| = |\textstyle\sum_0^\infty \lambda^{-(n+1)}T^n f| \leqq \textstyle\sum_0^\infty |\lambda|^{-(n+1)}T^n|f| = R(|\lambda|)|f|.$$

By the principle of uniform boundedness, the assumption $r(T) \notin \sigma(T)$ would imply that $\lambda \rightarrow \| R(\lambda) \|$ is uniformly bounded in the region $| \lambda | > r(T)$, which is obviously impossible.

We shall now list three important results on positive operators in $C(X)$ that will be used repeatedly in the sequel. For the convenience of the reader proofs will be given.

EIGENVALUE THEOREM [5]. *Let T be a positive operator on $C(X)$ with spectral radius r. Then r is an eigenvalue of the adjoint T' for which there exists a measure $\phi \geqq 0$ satisfying $r\phi = T'\phi$.*

Proof. We can restrict attention to the real spaces $C_R(X)$ and $M_R(X)$. Since $r \in \sigma(T) = \sigma(T')$ (see above), there exists a measure $\psi \geqq 0$ for which $R'(\lambda)\psi = (\lambda - T')^{-1}\psi$ is not bounded as $\lambda \downarrow r$. Setting

$$\psi_\lambda = R'(\lambda)\psi / \| R'(\lambda)\psi \|,$$

the directed family $(\psi_\lambda)_{\lambda > r}$ has a weak adherent point $\phi \in M_R(X)$. Since $R'(\lambda)$ $(\lambda > r)$ is a positive operator on $M_R(X)$, it follows that $\psi_\lambda \geqq 0$ and hence that $\phi \geqq 0$. Denoting by e the constantly-one function on X, we have $1 = \| \psi_\lambda \| = \psi_\lambda(e)$; thus $\| \phi \| = \phi(e) = 1$. On the other hand,

$$\lim_{\lambda \to r} \| \lambda\psi_\lambda - T'\psi_\lambda \| = 0,$$

hence $r\phi = T'\phi$ since T' is weakly continuous.

The following result is a special case of a theorem due to H. Lotz [6]. Recall that the **approximate point spectrum** of T consists of those elements $\lambda \in \sigma(T)$ for which there exists a normalized sequence (f_n) in $C(X)$ such that $\lim_n \| \lambda f_n - Tf_n \| = 0$.

IMBEDDING THEOREM. *There exists a compact space Y such that $C(X)$ is isometrically isomorphic with a subalgebra of $C(Y)$ containing the unit $\hat{e}$ of $C(Y)$, and such that each bounded linear operator U on this subalgebra has an extension U_1 to $C(Y)$ possessing the following properties:*

(i) $\| U_1 \| = \| U \|$

(ii) $\sigma(U_1) = \sigma(U)$ *and the approximate point spectrum of U is identical with the point spectrum of U_1*

(iii) *U_1 is positive if* (and only if) *U is positive.*

Proof. Put $E = C(X)$, and denote by E_0 the vector space of all bounded sequences (f_n) $(f_n \in E,\ n \in N)$. With multiplication defined coordinatewise and endowed with the norm $\| (f_n) \| = \sup_n \| f_n \|$, E_0 is a Banach algebra of type $C(W)$ [10, p. 247]. In fact, if W_0 denotes the topological direct sum of a sequence of copies of X then E_0 is the space of all bounded, continuous functions on the locally compact space W_0. Hence E_0 is isomorphic with $C(W)$, W denoting the Stone-Čech compactification of W_0. The subset I of all null sequences in E_0 then corresponds to the ideal in $C(W)$ whose elements vanish on $W\backslash W_0$. Defining $Y = W\backslash W_0$, $C(Y)$ thus is isomorphic with the B-algebra of E-valued bounded sequences modulo null sequences, and if $\hat{f}$ is an equivalence class containing the sequence (f_n) we have

$$\| \hat{f} \| = \limsup \| f_n \|.$$

It is clear that the mapping $f \to \hat{f}$, where $\hat{f}$ is the class containing the sequence $(f, f, f, \cdots)$, is an isometric isomorphism of $C(X)$ into $C(Y)$; in particular, the image of e is the unit $\hat{e}$ of $C(Y)$.

For any bounded linear operator U on $C(X)$, we define U_0 on E_0 by $U_0(f_n) = (Uf_n)$. Since $U_0(I) \subset I$, U_0 induces a bounded operator U_1 on $E_1 = C(Y)$. Clearly, $\| U \| = \| U_1 \|$ and U_1 is positive if U is (the converse is also true). It is clear from the construction of $C(Y)$ that each approximate eigenvalue of U is an eigenvalue of U_1, and that each approximate eigenvalue of U_1 is an approximate eigenvalue of U (hence actually an eigenvalue of U_1). On the other hand, if $\lambda \in \sigma(U)$ is not an approximate eigenvalue of U, then $\lambda - U$ is an isomorphism of E onto a closed proper subspace H. Hence if $g \notin H$, we have

$$\inf_n \|(\lambda - U)f_n - g \| \geqq \delta > 0$$

for each sequence $(f_n) \in E_0$ which shows that $\|(\lambda - U_1)\hat{f} - \hat{g} \| \geqq \delta$ for each $\hat{f} \in C(Y)$, and hence that $\lambda \in \sigma(U_1)$. The proof that $\lambda \in \sigma(U_1)$ implies $\lambda \in \sigma(U)$ is similar.

A positive operator T on $C(X)$ is called a **Markov** operator if $Te = e$. The unimodular eigenfunctions of these operators have a special property which we record in the following lemma [9, p. 307].

LEMMA ON UNIMODULAR EIGENFUNCTIONS. *Let T be a Markov operator. If α, β are unimodular eigenvalues of T and g, h are corresponding unimodular eigenfunctions, then $\alpha\beta^* gh^* = T(gh^*)$.*[2]

Proof. In fact, for each $f \in C(X)$ we have $Tf(s) = \int f(t)\, d\mu_s(t)$ where μ_s is a positive Radon measure of mass 1 (precisely, $\mu_s = T' \varepsilon_s$). Now if $\alpha g(s) = \int g(t)\, d\mu_s(t)$ then

$$1 = \int \frac{g(t)}{\alpha g(s)}\, d\mu_s(t) .$$

Since $\mu_s \geqq 0$, $\mu_s(e) = 1$ and the integrand is of absolute value 1, it follows that it must be identically $= 1$ on the support of μ_s. The same argument applies to β and h, so we have $g(t)h^*(t) = \alpha\beta^* g(s)h^*(s)$ for all t in the support of μ_s. Therefore

$$\alpha\beta^* g(s)h^*(s) = \int g(t)h^*(t)\, d\mu_s(t),$$

and the lemma is proved.

[2] The asterisk denotes complex conjugates.

2. Maximal ideals

We suppose in the sequel that X is a *non-empty* compact space. As usual, we begin with a definition.

DEFINITION 1. Let T be any positive operator on $C(X)$. By a T-**ideal** we understand a closed proper ideal in $C(X)$ which is invariant under T. A T-ideal is **maximal** if it is not properly contained in any other T-ideal.

PROPOSITION 1. *Every positive operator T on $C(X)$ possesses at least one maximal T-ideal, and each T-ideal is contained in some maximal T-ideal.*

Proof. The family of T-ideals is not empty, since it clearly contains (0). Let us show that this family is inductively ordered under set inclusion $\subset$. In fact, if (I_α) is a non-empty, totally ordered subfamily then the family (S_α) of the respective supports has non-empty intersection S, since X is compact. The ideal with support S is clearly invariant under T and hence the least upper bound of the given subfamily. The assertions follow now from an application of Zorn's lemma.

Examples. 1. If $X = [0, 1]$ and T is the Volterra operator given by $Tf(s) = \int_0^s f(t)\,dt$, then each T-ideal is of the form

$$I_a = \{f : f(s) = 0 \quad \text{for all} \quad s, 0 \leqq s \leqq a\},$$

where $a \in [0, 1]$. The only maximal ideal is I_0 $(a = 0)$.

2. Let X be the one-dimensional torus, represented by the complex unit circle Γ. Define T on $C(\Gamma)$ by $Tf(z) = f(\alpha z)$, $\alpha \in \Gamma$.

The T-ideals are exactly those ideals in $C(\Gamma)$ whose support is invariant under $z \to \alpha z$. There are essentially two different cases:

(a) α is a root of unity, and hence a primitive nth root of unity for some $n \in N$. There exist infinitely many T-ideals, and the maximal T-ideals are exactly those whose support forms a configuration of n points which is invariant under the rotation through $2\pi/n$.

(b) α is not a root of unity. Then (0) is the only T-invariant ideal, hence maximal. In fact, since the set $(\alpha^n)_{n \in N}$ is dense in Γ, each invariant ideal J must satisfy $S_J = \Gamma$, thus $J = (0)$.

3. Let X be finite and P a permutation matrix (viewed as a positive operator on $C(X)$). If $(c_1)(c_2) \cdots (c_k)$ is the decomposition of the corresponding permutation into independent cycles, the P-ideals are those whose support is the set of all elements occurring in certain cycles $(c_{i_1}), \cdots, (c_{i_m})$. Any one cycle determines a maximal P-ideal, and this exhausts the supply of maximal P-ideals.

The preceding examples suggest the following definition.[3]

DEFINITION 2. The positive operator T is called **irreducible** if there exist no T-ideals distinct from (0).

[3] In [9] and [10], the term irreducible is used in a slightly different sense.

If J is a T-ideal in $C(X)$, T induces a positive operator T_J on $C(X)/J \cong C(S_J)$.

PROPOSITION 2. *Let J be a fixed T-ideal and denote by q the canonical map $C(X) \to C(X)/J$. Then $I \to q(I)$ is a bijective map of the set of all T-ideals containing J onto the set of all T_J-ideals. In particular, a T-ideal $I \supset J$ is maximal if and only if $q(I)$ is a maximal T_J-ideal.*

Proof. Clearly, if $I \supset J$ is a T-ideal then $q(I)$ is a T_J-ideal, since q is a homomorphism and since $T_J \circ q = q \circ T$. (Note that $q(I)$, which can be identified with the Banach algebra I/J, is closed in $C(X)/J$.) Likewise, if K is a T_J-ideal, then $q^{-1}(K)$ is an ideal invariant under T. The mapping is bijective, since clearly $q(q^{-1}(K)) = K$ (for q is surjective), and since $q^{-1}(q(I)) = I$ (for $q^{-1}(0) = J \subset I$). The remainder is now obvious.

COROLLARY. *Let J be a T-ideal in $C(X)$. Then J is maximal if and only if T_J is irreducible on $C(S_J)$.*

Our first theorem concerns the representation of maximal T-ideals. Recall that e denotes the constant-one function on X.

THEOREM 1. *Let T be any positive operator in $C(X)$. Every maximal T-ideal is of the form $I_\phi = \{f : \phi(|f|) = 0\}$ for a suitable normalized eigenvector $\phi \geqq 0$ of the adjoint operator T': $\rho\phi = T'\phi$. Here $\rho = 0$ occurs if and only if ϕ is a point measure ε_s such that $Te(s) = 0$.*

Proof. Let I be a maximal T-ideal. By Proposition 2, Corollary, T_I is irreducible on $C(S_I)$. Denote by ρ the spectral radius of T_I. It follows from the eigenvalue theorem (§1) that there exists a positive measure $\hat{\phi}$ on S_I satisfying $\rho\hat{\phi} = T_I'\hat{\phi}$. Clearly $I_{\hat{\phi}}$ is a T_I-ideal; hence $I_{\hat{\phi}} = (0)$, since T_I is irreducible. On the other hand, if q is the canonical map $C(X) \to C(S_I)$, then $\phi = \hat{\phi} \circ q$ is a positive measure on X for which $I = I_\phi$ and such that $\rho\phi = T'\phi$.

If $\phi = \varepsilon_s$ where $Te(s) = 0$ then, clearly, $T'\phi = 0$. Conversely, suppose that I_ϕ is maximal and $T'\phi = 0$. $\phi(Te) = 0$ shows that the support of ϕ is contained in $S_0 = \{s \in X : Te(s) = 0\}$. On the other hand, each closed ideal whose support is contained in S_0 is a T-ideal (immediate verification). Thus I_ϕ defines a maximal ideal of the algebra $C(S_0)$ (Prop. 2), and hence ϕ is necessarily of the form ε_s for some $s \in S_0$, Q.E.D.

PROPOSITION 3. *If I, J are distinct maximal T-ideals, then $S_I \cap S_J = \emptyset$. In particular, if $I = I_\phi$ and $J = I_\psi$ then the measures ϕ, ψ are orthogonal.*[4]

Proof. If $S = S_I \cap S_J$ were non-empty, S would be the support of a T-ideal K containing both I and J. Since I, J are maximal and distinct, it follows that S is empty.

[4] Orthogonal as elements of the Banach lattice $M_R(X)$.

PROPOSITION 4. *Suppose that $\sigma(T)$ is nowhere dense in the real interval $[0, r(T)]$. If ϕ_1, ϕ_2 are positive eigenvectors of T' belonging to distinct eigenvalues $\rho_1 > \rho_2$ ($\geqq 0$), then $I_{\phi_1} \neq I_{\phi_2}$. In particular, if I_{ϕ_1} and I_{ϕ_2} are both maximal T-ideals then $\phi_1 \perp \phi_2$.*

Proof. Suppose that $I_{\phi_1} = I_{\phi_2} = I$. Then ϕ_1, ϕ_2 define strictly positive measures on S_I such that $\rho_i \phi_i = T_I' \phi_i$ $(i = 1, 2)$. If ρ denotes the spectral radius of T_I, then, clearly, we have $0 \leqq \rho_2 < \rho_1 \leqq \rho \leqq r(T)$. Since $\sigma(T)$ is nowhere dense in $[0, r(T)]$ there exists λ, $\rho_2 < \lambda < \rho_1$, for which $R(\lambda) = (\lambda - T)^{-1}$ exists. On the other hand, the C. Neumann's series

$$S\psi = \sum_0^\infty \lambda^{-(n+1)}(T')^n\psi$$

converges for every element $\psi \in M_R(S_I)$ such that $|\psi| \leqq c\phi_2$ for some $c > 0$; in fact, since T' is positive, we have

$$|(T')^n\psi| \leqq (T')^n|\psi| \leqq c\rho_2^n\phi_2$$

and certainly $S\psi = R'(\lambda)\psi$ for these ψ. It follows that $R'(\lambda)$ is a positive operator on the closed order ideal $B_2 \subset M_R(S_I)$ generated by ϕ_2. Since ϕ_2 is strictly positive on (the positive cone of) $C(S_I)$, it follows by taking polars that the positive cone of B_2 is weak* dense in the positive cone of $M_R(S)$ and hence that $R'(\lambda)$, being weak* continuous, is a positive operator on $M_R(I)$. But this contradicts the resolvent equation

$$R'(\mu) = R'(\lambda) - (\mu - \lambda)R'(\mu)R'(\lambda)$$

when applied to $\mu > r(T)$; in fact we would have $R'(\mu) \leqq R'(\lambda)$, and hence $\| R'(\mu) \| \leqq \| R'(\lambda) \|$, for all $\mu > r(T)$ which is impossible because of $r(T) \in \sigma(T)$ (§1).

Consequently, we must have $I_{\phi_1} \neq I_{\phi_2}$.

PROPOSITION 5. *Let T be any positive operator in $C(X)$, and let T_1 be the positive operator induced by T on $C(Y)$* (see the imbedding theorem, §1). *Then the correspondence*

$$I_1 \to I = C(X) \cap I_1$$

maps the set of all maximal T_1-ideals onto the set of all maximal T-ideals.

Proof. If I is a maximal T-ideal, then (identifying $C(X)$ with a subalgebra of $C(Y)$) the ideal in $C(Y)$ generated by I is T_1-invariant and contained in a maximal T_1-ideal I_1 (Proposition 1). Clearly $I = C(X) \cap I_1$, since I is maximal. On the other hand, if I_1 is a maximal T_1-ideal, then $I = C(X) \cap I_1$ is a T-ideal in $C(X)$ which is proper, since $\hat{e} \notin I_1$ but $e \in C(X)$. It remains to show that I is maximal. To this end we show that for every T-ideal $J \supset I$, the closed ideal K in $C(Y)$ generated by $J + I_1$ is proper; since K is clearly T_1-invariant, the maximality of I_1 then implies $J = I$.

Recall the construction of Y (§1): $Y = W \backslash W_0$ where W is the Stone-

Čech compactification of the topological sum W_0 of an infinite sequence of copies of X. Suppose that K is not proper; then it follows that $J + I_1$ contains a strictly positive function $\hat{f}$. The equivalence class $\hat{f}$ contains a sequence $(f + g_n)$ where $f \geqq 0$, $g_n \geqq 0$ are in $C(X)$ and such that $f(t) + g_n(t) \geqq 3\delta$ for some $\delta > 0$ and *all* $n \in N$; in addition, $f \in J$ and $(g_n)^\wedge \in I_1$. (There exists $\delta > 0$ with $3\delta < \inf\{\bar{f}(t) : t \in Y\}$, for any representative $\bar{f} \in C(W)$ of $\hat{f}$. Now the set $\{t \in W : \bar{f}(t) \leqq 3\delta\}$ is compact and contained in W_0.) It follows that $g_n(t) \geqq 2\delta$ for all $n \in N$ and $t \in U$, where U is a suitable open neighborhood of the support $S_J \subset X$ of J. An application of Urysohn's theorem yields the existence of $h \in C(X)$ such that $0 \leqq h \leqq g_n$ for all n, and such that $h(t) \geqq \delta$ whenever $t \in S_J$. Hence the stationary sequence (h) defines a continuous function on W whose restriction to Y, $\hat{h}$, belongs to I_1. Thus

$$(f + h)^\wedge \in (J + I_1) \cap C(X) = J + I = J;$$

but this is contradictory, since $f + h$ does not vanish on S_J, and hence K is a proper ideal in $C(Y)$.

3. Ergodic Markov operators

We have seen (§2, Theorem 1) that each maximal T-ideal I can be obtained, by means of the correspondence $\phi \rightarrow I_\phi$, from an eigenvector $\phi \geqq 0$ of T'. Supposing ϕ to be normalized (i.e., $\| \phi \| = 1$), is ϕ uniquely determined by I? By transition to the quotient $C(X)/I$, the problem is seen to be equivalent to this: Does the adjoint T' of an irreducible operator T (Definition 2) have a unique (normalized) positive eigenvector? Further, what particular properties distinguish the eigenvectors $\phi \geqq 0$ of T' that determine maximal T-ideals I_ϕ, from other positive eigenvectors of T'? These questions do not appear to have simple or easy answers for the most general positive operators T on $C(X)$, but we can obtain a complete answer for a rather wide class of operators that are important in applications. These are the ergodic Markov operators.

DEFINITION 3. A bounded operator U on a Banach space E is called **ergodic** if for each $x \in E$, the convex closure $K(x)$ of the orbit $(x, Ux, U^2x, \cdots)$ contains a fixed vector x_0 of U.[5]

It is well known and follows from standard arguments of ergodic theory [3, pp. 8–11], that whenever $\{U^n : n \in N\}$ is equicontinuous, then $x_0 \in K(x)$ is unique, and $x \rightarrow x_0$ is a projection P onto the space of fixed vectors of U. More precisely, $\lim_n \| M_n x - Px \| = 0$ where M_n is the n-th average,

$$M_n = n^{-1}(I + U + \cdots U^{n-1}).$$

If $U = T$ is a positive operator and P exists, then P is a positive projection and $PC_R(X)$ is a sublattice of $C_R(X)$. We shall also have use for a stronger notion of ergodicity.

[5] Usually it is the semigroup (U^n) that is called ergodic.

DEFINITION 3a. U is called **uniformly ergodic** if (M_n) is a Cauchy sequence for the uniform operator topology.

PROPOSITION 6. *If $T \geqq 0$ has spectral radius $r(T) = 1$ and*

$$\lim_{\lambda \downarrow 1}(\lambda - 1)R(\lambda)$$

exists for the strong (or even weak) operator topology, then T is ergodic.

Proof. If the limit exists for one of the topologies mentioned, it is clearly a positive linear operator P, hence bounded.[6] Also, for any $f \in C(X)$ we have $Pf \in K(f)$, and it is readily seen from the definition of the resolvent $R(\lambda)$ that $PT = TP = P$; hence Pf is a fixed vector of T.

Examples. 1. Let T be weakly compact, and $r(T) = 1$. Then T^2 is compact (theorem of Dunford-Pettis); hence $\lambda = 1$ is a pole of the resolvent. If this pole is of order 1, then T is uniformly ergodic; this is, for example, true whenever (T^n) is bounded.

2. Let T be the operator defined in §2, Example 2(a). Here P is the projection defined by

$$Pf(z) = n^{-1}(f(z) + f(\alpha z) + \cdots + f(\alpha^{n-1}z)).$$

As each periodic operator, T is uniformly ergodic. (T is called periodic if there exists an integer $n \geqq 1$ such that $T^n = I$.)

3. If T is the operator defined in §2, Example 2(b), T is ergodic and

$$Pf = \lim M_n f = \left(\int_\Gamma f(s)\, ds\right) e.$$

Thus P is of rank 1.

PROPOSITION 7. *Suppose T is positive and satisfies $\| T \| = r(T) = 1$. Then the set of real T-invariant measures is a (non-trivial) weakly closed vector sublattice of $M_R(X)$, and each of these measures has its support in*

$$M = \{s \in X : Te(s) = 1\}.$$

Proof. It follows from the eigenvalue theorem (§1) that the set F of measures in question is non-trivial, and evidently F is weakly closed. The condition $\| T \| = 1$ is equivalent to ($Te \leqq e$ and $M \neq \emptyset$). Now if $\phi \in F$ (that is, $\phi = T'\phi$), then $|\phi| \leqq T'|\phi|$; thus

$$|\phi|(e) \leqq |\phi|(Te) \leqq |\phi|(e).$$

Therefore, $|\phi|(e - Te) = 0$ which shows that $S_\phi \subset M$ and $T'|\phi| = |\phi|$.

Since F is weakly closed in the adjoint L-space $M_R(X)$, the set of measures $\phi \geqq 0$ in F is a convex cone with compact base

$$\Phi = \{\phi \in F : \phi \geqq 0, \phi(e) = 1\}.$$

[6] In fact, $T \geqq 0$ implies that the unit ball $[-e, e]$ of $C_R(X)$ is mapped by T into the (bounded) interval $[-Te, Te]$.

By the theorem of Krein-Milman, Φ is the weakly closed convex hull of its subset Λ of extreme points. The much stronger theorem of Choquet-Bishop-de Leeuw (see, e.g., [1]) asserts that each $\phi \in \Phi$ is the barycenter of a probability measure m on Φ that is concentrated on Λ. Under the assumptions of Proposition 7, Φ is even a simplex and hence m unique [1]. Let us recall that a positive operator T on $C(X)$ is called a **Markov** operator if $Te = e$. The following theorem is the principal result of this section.

THEOREM 2. *Suppose T is an ergodic Markov operator on $C(X)$, and denote by Φ the* (weakly compact) *set of all positive, normalized T-invariant measures on X. Then $\phi \rightarrow I_\phi$ is a bijection of the set Λ of extreme points of Φ onto the family of all maximal T-ideals. Moreover, every T-ideal of the form I_ϕ $(\phi \in \Phi)$ is the intersection of all maximal T-ideals containing it.*

Proof. First, a lemma.

LEMMA. *If $\phi \in \Lambda$ and ψ is a T-invariant measure contained in the weakly closed band $\bar{B}_\phi \subset M_R(X)$ generated by ϕ, then $\psi = c\phi$ $(c \in R)$.*

In fact, suppose first that $0 \leqq \psi \leqq \phi$, $\psi \neq 0$. Then we have $\phi = \psi + (\phi - \psi)$ which is consistent with the extreme point property of ϕ only when $\psi = c\phi$, $0 < c \leqq 1$. If, more generally, $|\psi|$ is majorized by some multiple of ϕ then ψ^+ and ψ^- are non-negative multiples of ϕ by the preceding; hence $\psi = c\phi$ $(c \in R)$. Now let $P = \lim_n M_n$ (strong operator topology) denote the positive projection onto the space of fixed vectors of T. The adjoint P' maps $M_R(X)$ onto the space of invariant measures (cf. Proposition 7). Obviously, $P'\phi = \phi$, hence, by the preceding, P' maps B_ϕ (the band in $M_R(X)$ generated by ϕ) onto the one-dimensional space $F_\phi = \{c\phi : c \in R\}$. Since P' is weakly continuous and F_ϕ complete for the weak topology, it follows that $P'(\bar{B}_\phi) = F_\phi$. This proves the lemma, because $P'\psi = \psi$.

Let us show that $\phi \rightarrow I_\phi$ is injective on Λ, with range in the set of maximal T-ideals. Suppose $\phi \in \Lambda$ and denote by T_ϕ the operator induced on $C(S_\phi) \cong C(X)/I_\phi$. The dual of $C_R(S_\phi)$ is $\bar{B}_\phi$, and by the lemma above we know that ϕ is the only normalized, positive T-invariant measure on S_ϕ. Since ϕ is strictly positive on S_ϕ it follows from Theorem 1 that T_ϕ is irreducible, hence from the corollary of Proposition 2 that I_ϕ is maximal. Now if $\psi \in \Lambda$ is $\neq \phi$, then either $S_\phi = S_\psi$ or $S_\phi \cap S_\psi = \emptyset$, since I_ψ also is maximal. If we had $S_\phi = S_\psi$ we would have $\psi \in \bar{B}_\phi$, hence $\phi = \psi$ by the lemma above. Thus $I_\phi \neq I_\psi$.

We show that each maximal T-ideal is an I_λ, $\lambda \in \Lambda$. By the remarks preceding Theorem 2 there exists, for each $\phi \in \Phi$, a (unique) probability measure m on Φ which is concentrated on Λ, that is, for which $m(S) = 0$ whenever $S \subset \Phi$ is a Borel set not intersecting Λ. We shall prove that

$$I_\phi = \bigcap_{\lambda \in S_m \cap \Lambda} I_\lambda$$

where $S_m \subset \Phi$ is the support of the Radon measure m on Φ. In fact, let $f \geqq 0$ and $\lambda(f) = 0$ for all $\lambda \in \Lambda \cap S_m$; if

$$S = \{\tau \in \Phi : \tau(f) > 0\},$$

then $S \cap (\Lambda \cap S_m) = \emptyset$ so $S \cap S_m$ is a Borel set not intersecting Λ, whence $m(S \cap S_m) = 0$. This implies, clearly, that $\phi(f) = \int f\,dm = 0$; thus $f \in I_\phi$. Conversely, if $f \in I_\phi$ then $\tau(f) = 0$ for all $\tau \in S_m$; thus $\lambda(f) = 0$ for all $\lambda \in S_m \cap \Lambda$ which proves the above formula. Now because each I_λ $(\lambda \in \Lambda)$ is maximal and $I_{\lambda_1} \neq I_{\lambda_2}$ for $\lambda_1 \neq \lambda_2$ by the first part of the proof, I_ϕ $(\phi \in \Phi)$ is a maximal T-ideal if and only if $\phi = \lambda$ for some $\lambda \in \Lambda$. More generally, we have shown that each I_ϕ $(\phi \in \Phi)$ is the intersection of all maximal T-ideals containing it, and Theorem 2 is proved.

Remark. The assumption that T be ergodic (that is, in our terminology, that $\lim M_n = P$ exists for the strong operator topology) cannot be dropped from Theorem 2. In fact, Raimi [7] exhibits a Markov operator T on $C(N^*)$, where $N^* = \beta N \backslash N$, such that each maximal T-ideal can be represented by at least two T-invariant measures (which are both extreme points of Φ). In particular, the subsequent corollary of Thm. 2 is false unless T is assumed to be ergodic.

On the other hand, if T is a Markov operator on $C(X)$, not necessarily irreducible but such that the set Φ of positive, normalized T-invariant measures is a singleton, $\Phi = \{\phi\}$, then T is ergodic. Indeed, if $\psi \geqq 0$ is any normalized measure on X, each weak* limit point of the sequence

$$M_n' \psi = n^{-1}(I + T' + \cdots (T')^{n-1})\psi$$

is normalized and T-invariant, hence $= \phi$ by hypothesis; more generally, for any $\psi \geqq 0$ the weak* limit $\lim M_n' \psi = P'\psi$ exists and $P'\psi = \psi(e)\phi$. Therefore, P' is a weak* continuous projection which, in turn, implies that $P = \lim M_n$ exists for the weak operator topology. Thus T is ergodic (Def. 3).

COROLLARY 1. *An ergodic Markov operator is irreducible if and only if there exists a unique (normalized) positive T-invariant measure ϕ, and ϕ is strictly positive.*

We can generalize Theorem 2 somewhat by replacing the assumption $Te = e$ by the weaker hypothesis $\| T \| = r(T) = 1$. Φ is defined as before (cf. Proposition 7).

COROLLARY 2. *Suppose $T \geqq 0$ is ergodic and $\| T \| = r(T) = 1$. By virtue of $\phi \longrightarrow I_\phi$, the set Λ of extreme points of Φ is in one-to-one correspondence with the set of all maximal T-ideals whose support is contained in*

$$M = \{s \in X : Te(s) = 1\}.$$

Proof. By Proposition 7, there exists at least one T-ideal having its support in M. Now if J denotes the intersection of all these ideals, J is a T-

ideal. In view of Proposition 2, it suffices for the proof to apply Theorem 2 to the operator T_J on $C(X)/J$.

Theorem 2 has a counterpart for stochastic operators on abstract L-spaces. Recall that a positive (linear) operator on a space $L^1(\mu)$ is called **stochastic** if $\| Tf \| = \| f \|$ for each $f \geqq 0$ and that, in $L^1(\mu)$, a *band* is the same as a closed, solid vector subspace (cf. [10, Chap V, §§7, 8]).

THEOREM 3. *Let T be a stochastic and ergodic operator on a space $L^1(\mu)$, and define $\Phi = \{f : f \geqq 0, Tf = f, \| f \| = 1\}$. There exist minimal T-invariant bands $\neq (0)$ if and only if the set Φ has extreme points, and the latter are in one-to-one correspondence with the former by virtue of $f \rightarrow B_f$, where B_f denotes the band in $L^1(\mu)$ generated by f.*

Proof. We need a lemma.

LEMMA. *If T is stochastic on $L^1(\mu)$ and there exist no non-trivial T-invariant bands, then the space of fixed vectors of T is at most one-dimensional.*[7]

In fact, $f = Tf$ implies $|f| \leqq T|f|$; since the norm is additive on the positive cone of $L^1(\mu)$, we obtain

$$\| T|f| - |f| \| = \| T|f| \| - \| f \| = 0.$$

Hence, $|f| = T|f|$, that is, the fixed vectors of T form a vector sublattice of $L^1(\mu)$. Thus if $f = Tf$, then f^+ and f^- are fixed under T; since the bands B_{f^+} and B_{f^-} are T-invariant and lattice disjoint, our assumption implies that either $f^+ = 0$, or else $f^- = 0$. Thus the fixed space of T is a totally ordered vector lattice and hence, since it is Archimedean, at most one-dimensional. This proves the lemma.

Now let f be an extreme point of Φ and denote, as before, by P the projection which is the pointwise limit of the averages

$$M_n = n^{-1}(I + T + \cdots + T^{n-1}).$$

Clearly the band B_f is T-invariant. Suppose that D is a T-invariant band such that $(0) \neq D \subset B_f$. If g is any element satisfying $|g| \leqq cf$ $(c > 0)$, the extreme point property of f implies that Pg is a scalar multiple of f; this follows exactly as in the lemma contained in the proof of Thm. 2. Again, we conclude that $P(B_f)$ is the one-dimensional subspace of $L^1(\mu)$ generated by f. Thus, since $D \neq (0)$, it follows that $D \cap \Phi = \{f\}$. Hence $f \in D$ and, therefore, $D = B_f$. This shows B_f to be minimal.

Conversely, let B be a minimal T-invariant band $\neq (0)$. Clearly, the restriction of T to B satisfies the hypothesis of the lemma above, which implies that $B \cap \Phi$ is a singleton, $\{f\}$ say. (Since T is ergodic and $B \neq (0)$, $\Phi \cap B$ cannot be empty.) This implies in turn that f is an extreme point of Φ, for B is solid. Finally, it is clear that $B = B_f$, Q.E.D.

[7] We consider the *real* space $L^1(\mu)$, and suppose it to be $\neq (0)$.

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