ON EIGENFUNCTION EXPANSIONS FOR ELLIPTIC OPERATORS

BY

RICHARD BEALS

Introduction

The eigenfunction expansion theorem for singular self-adjoint elliptic operators is well known. In this paper we present a proof which is more elementary in some respects than those given previously, and which has the advantage of applying to operators with merely measurable (and locally bounded) coefficients.

A general eigenfunction expansion theorem for operators in Lebesgue spaces was proved by Mautner [10] and extended by Bade and Schwartz [1]; a somewhat different result is due to Gelfand and Kostyucenko [9]. Gårding [8] and Browder [4], [5], obtained the expansion theorem for elliptic operators under various assumptions; see also Nelson [11]. In each case the technical problem is to show that some function $h(A)$ of the given operator A has a kernel. In the papers cited this problem is solved by using some variant of the Dunford-Pettis theorem or another Banach space differentiation theorem, together with the fact, or assumption, that the range of $h(A)$ consists of locally bounded functions. When A is an elliptic operator, $h(A)$ is taken to be $(A - \lambda)^{-q}$ for λ in the resolvent of A and q sufficiently large. Then the regularity theory for elliptic operators and the Sobolev imbedding theorem give the desired conclusion. When q has to be taken greater than 1, the regularity theory needed requires a certain amount of differentiability of the coefficients of A.

The point of the present proof is that for A elliptic and q large enough, $(A - \lambda)^{-q}$ is "locally" an operator of Hilbert-Schmidt type. The existence of a (square-integrable) kernel for operators of this type is well-known and more elementary than the Dunford-Pettis theorem and the Sobolev imbedding theorem.

The proof of the assertion about $(A - \lambda)^{-q}$ depends on some of the simple observations about compact operators and Sobolev spaces which were applied in a much more delicate way in [2], [3] to obtain the asymptotic distribution of eigenvalues for elliptic operators without smooth coefficients.

1. Some compact operators

If H and K are Hilbert spaces and $S : H \to K$ a linear operator, we denote the domain and range of S by $D(S)$ and $R(S)$ respectively. For bounded S, the characteristic numbers $\mu_j(S)$, $j = 1, 2, \cdots$, are defined by

$$(1) \qquad \mu_j(S) = \inf_{\operatorname{codim}(H_1)<j} \sup_{u \in H_1, \|u\|=1} \| Su \|.$$

If S is compact, $\{\mu_j^2(S)\}$ is the sequence of eigenvalues of S^*S [7, Theorem

Received October 26, 1966.

X.4.3]. We need the properties [7, Corollary X.9.3 and Lemma X.9.6]:

$$\mu_j(S^*) = \mu_j(S), \tag{2}$$

$$\mu_j(ST) \leq \| T \| \mu_j(S), \tag{3}$$

$$\mu_{j+k-1}(S + T) \leq \mu_j(S) + \mu_k(T), \tag{4}$$

$$\mu_{j+k-1}(ST) \leq \mu_j(S)\mu_k(T). \tag{5}$$

It follows readily from (3) that if S is a bounded operator in H and W a partial isometry, then $\mu_j(SW) = \mu_j(S)$, all j.

LEMMA 1.1. *Let H, H_1, H_2 be separable Hilbert spaces and let*

$$S : H_1 \to H \quad \textit{and} \quad T : H_2 \to H$$

be bounded operators. If S is compact and $R(T) \subseteq R(S)$, then there is a constant c such that $\mu_j(T) \leq c\mu_j(S)$, all j.

Proof. We consider explicitly only the case when H, H_1, H_2 and $R(S)$ are infinite-dimensional. Let H_0 be the orthogonal complement of the null space of S, and W a partial isometry of H_1 onto H_0. Replacing S by SW and H_1 by H_0, we may assume that S is 1–1. Similarly, by using isometries to transfer the operators, we may assume that $H_1 = H_2 = H$. Replacing S by $(SS^*)^{1/2}$, which has the same range [3, Lemma 1.1], we may assume S is positive. Then there is a complete orthonormal sequence $\{u_j\} \subseteq H$ with $Su_j = \mu_j u_j$, where $\mu_j = \mu_j(S)$. With respect to the inner product $\langle u, v\rangle = (S^{-1}u, S^{-1}v)$, $R(S)$ is a Hilbert space K with norm $|u| = \langle u, u\rangle^{1/2}$. Then $T = JT_1$, where $T_1 : H \to K$ is closed, hence continuous and $J : K \to H$ is the injection mapping. Let H_j be the closed subspace of H generated by $\{u_k \mid k \geq j\}$. Then $T^* = T_1^* J^*$ and

$$\begin{aligned}\mu_j(T) = \mu_j(T^*) &\leq \sup_{u \in H_j, \|u\|=1} \| T^* u \| \\ &\leq \| T_1^* \| \sup_{u \in H_j, \|u\|=1} \| J^* u \|.\end{aligned}$$

Now $\{v_k = \mu_k u_k\}$ is a complete orthonormal sequence in K. It follows easily that $J^* u_k = \mu_k v_k$, and hence that for $u \in H_j$, $| J^* u | \leq \mu_j \| u \|$. Therefore the desired inequality holds with $c = \| T_1^* \| = \| T_1 \|$.

We shall say that S is of class $a \geq 0$ if there is a constant c such that $\mu_j(S) \leq cj^{-a}$, all j. In particular any bounded operator is of class 0. An easy consequence of (4) and (5) is

LEMMA 1.2. *If S and T are operators in H of classes a and b respectively, then $S + T$ is of class* min (a, b) *and ST is of class $a + b$.*

Since, as noted above, $\{\mu_j^2(S)\}$ is the sequence of eigenvalues of S^*S for S compact; since $\mu_j(S) \to 0$ implies S compact, we have

LEMMA 1.3. *If S is of class $a > \frac{1}{2}$, then it is of Hilbert-Schmidt type.*

2. Sobolev spaces and elliptic operators

Let Ω be an open subset of E^n. Denote by $\mathfrak{D}(\Omega)$ the space of infinitely differentiable complex-valued functions on Ω with compact support, and by $L^2(\Omega)$ the usual L^2-space with inner product (u, v). For m a non-negative integer, $H^m(\Omega)$ is the space of functions u whose distribution derivatives $D^\alpha u$ of order $|\alpha| \leq m$ are all in $L^2(\Omega)$. This is a Hilbert space with inner product $(u, v)_m = \sum (D^\alpha u, D^\alpha v)$, $|\alpha| \leq m$. If K is a compact subset of Ω, we denote by H^m_K the subspace of $H^m(\Omega)$ consisting of those u with support $\operatorname{supp}(u) \subseteq K$.

Lemma 2.1. *Suppose S is a bounded operator in $H^m(\Omega)$ with $R(S) \subseteq H^p_K$ where K is a compact subset of Ω and $p \geq m$. Then S is of class $(p - m)/n$.*

Proof. Cover a neighborhood of K by a finite number of closed cubes $K_j \subseteq \Omega$, and take functions $\varphi_j \in \mathfrak{D}(\Omega)$ with $\operatorname{supp}(\varphi_j) \subseteq K_j$, $\sum \varphi_j(x) = 1$, $x \in K$. Then $S = \sum S_j$ where $S_j u = \varphi_j Su$. By Lemma 1.2 we can therefore reduce to the case K a cube. Let $H^k_\pi(K)$ be the space of periodic distributions on K with derivatives of order $\leq k$ in $L^2(K)$. Then $R(S) \subseteq H^p_\pi(K)$. For an n-tuple $\alpha = (\alpha_1, \cdots, \alpha_n)$ of integers, let $\alpha \cdot x = \alpha_1 x_1 + \cdots + \alpha_n x_n$, $x \in E^n$. If d is the length of a side of K, the functions $u_\alpha(x) = \exp(2\pi i\alpha \cdot x)$ are a complete orthogonal system for $H^k_\pi(K)$, all k. Let $\{v_\alpha\}$ and $\{w_\alpha\}$ be the corresponding normalized sequences for $H^m_\pi(K)$ and $H^p_\pi(K)$ respectively. Then the unitary map W of $H^m_\pi(K)$ onto $H^p_\pi(K)$ taking v_α onto w_α is easily seen to be of class $(p - m)/n$ as an operator in $H^m_\pi(K)$. The desired conclusion follows from Lemma 1.1.

Let $A = \sum a_\alpha D^\alpha$, $|\alpha| \leq m$, be a partial differential operator with coefficients a_α measurable and bounded on each compact subset of Ω. Let A_1 be the restriction of A to a subspace $D(A_1)$ with

$$\mathfrak{D}(\Omega) \subseteq D(A_1) \subseteq H^m_{\text{loc}}(\Omega),$$

where $H^m{}_{\text{loc}}(\Omega) = \{u \mid \varphi u \in H^m(\Omega), \text{all } \varphi \in \mathfrak{D}(\Omega)\}$. Assume that A_1 is closed and that the resolvent set $r(A_1)$ is not empty. Take $\lambda \in r(A_1)$ and set $S = (A_1 - \lambda)^{-1}$.

Given operators B, C let $[B, C] = BC - CB$. Given $\varphi \in \mathfrak{D}(\Omega)$, let φ also denote the operation of multiplication of a function by φ.

Lemma 2.2. *For $\varphi \in \mathfrak{D}(\Omega)$, φS is of class m/n and $[A_1, \varphi]S$ is of class $1/n$ as operators in $L^2(\Omega)$.*

Proof. $R(\varphi S) \subseteq \varphi H^m_{\text{loc}}(\Omega) \subseteq H^m_K$, where $K = \operatorname{supp}(\varphi)$. Therefore by Lemma 2.1, φS is of class m/n.

Since A_1 is closed and of order m, it is clear that $D(A_1) \supseteq H^m_K$. Thus $\varphi : D(A_1) \rightarrow D(A_1)$. Take $\psi \in \mathfrak{D}(\Omega)$ such $\psi(x) = 1$ for $x \in K = \operatorname{supp}(\varphi)$. Then

$$[A_1, \varphi]S = [A_1, \varphi]\psi S.$$

Let $K^* = \text{supp}(\psi)$. Then ψS is continuous to $H^m_{K^*}$. By Lemmas 1.1 and 2.1 the injection mapping of $H^m_{K^*}$ to $H^{m-1}_{K^*}$ is of class $1/n$ in the latter space. But $[A_1, \varphi]$ is of order $\leq m - 1$, hence continuous from $H^{m-1}_{K^*}$ to $L^2(\Omega)$. It follows that $[A_1, \varphi]S$ is of class $0 + 1/n + 0 = 1/n$.

LEMMA 2.3. *For $\varphi \in \mathfrak{D}(\Omega)$ and q a positive integer, φS^q is of class mq/n as an operator in $L^2(\Omega)$.*

Proof. We shall show by induction that $[A_1, \varphi]S^q$ is of class $[m(q-1)+1]/n$, and φS^q is of class mq/n. The case $q = 1$ is Lemma 2.2. Suppose this has been proved for q, and suppose that $[A_1, \varphi]S^{q+1}$ has been shown to be of class j/n for some $j < mq + 1$. Take $\psi \in \mathfrak{D}(\Omega)$ with $\psi(x) = 1$, $x \in \text{supp}(\varphi)$. Note that $[\psi, S] = S[A, \psi]S$. Then

$$\begin{aligned}[A, \varphi]S^{q+1} &= [A, \varphi]\psi S^{q+1}\\ &= [A, \varphi][\psi, S]S^q + [A, \varphi]S\psi S^q\\ &= ([A, \varphi]S)([A, \psi]S^{q+1}) + ([A, \varphi]S)(\psi S^q).\end{aligned}$$

By the induction assumptions the first term on the right is of class $1/n + j/n$ and the second is of class $1/n + mq/n \geq (j+1)/n$. So the sum is of class $(j+1)/n$. Thus $[A, \varphi]S^{q+1}$ is of class $(mq+1)/n$. As for φS^{q+1},

$$\begin{aligned}\varphi S^{q+l} &= \psi\varphi S^{q+1} = \psi[\varphi, S]S^q + \psi S\varphi S^q\\ &= (\psi S)([A, \varphi]S^{q+1}) + (\psi S)(\varphi S^q).\end{aligned}$$

By what was just proved, the first term on the right is of class $m/n + (mq+1)/n > m(q+1)/n$. By the induction assumption the second term is of class $m/n + mq/n = m(q+1)/n$. This completes the proof.

As an immediate consequence of Lemmas 1.1, 1.3, and 2.3 we have the key result.

COROLLARY 2.4. *Let S be as above, $q > n/2m$ and $\varphi \in \mathfrak{D}(\Omega)$. If H is a Hilbert space and $T : H \to L^2(\Omega)$ is bounded and has $R(T) \subseteq R(S^q)$, then φT is of Hilbert-Schmidt type.*

Remarks. At least when the coefficients a_α for $|\alpha| = m$ are continuous, the assumptions that A_1 has a non-empty resolvent set while $D(A_1) \subseteq H^m_{\text{loc}}(\Omega)$ imply that A is elliptic. Conversely if a_α is continuous for $|\alpha| = m$ and A is elliptic and formally self-adjoint, then under fairly general conditions A has a self-adjoint realization corresponding to the Dirichlet problem [6].

3. The eigenfunction expansion theorem

As in the previous section, Ω is an open subset of E^n and $A = \sum a_\alpha D^\alpha$ is an operator of order m with measurable, locally bounded coefficients. We assume that for some choice of $D(A_1)$ with

$$\mathfrak{D}(\Omega) \subseteq D(A_1) \subseteq H^m_{\text{loc}}(\Omega),$$

the restriction A_1 of A to $D(A_1)$ is self-adjoint in $L^2(\Omega)$. Denote the complex conjugate of a by a^*.

THEOREM. *There are a vector-valued measure ν on the real line R and a unitary mapping V of $L^2(\Omega)$ onto $L^2(R^1, d\nu)$ such that for $u \in D(A_1)$,*

(a) $VA_1 u(\lambda) = \lambda Vu(\lambda)$ *for ν-almost all $\lambda \in R^1$.*

Moreover there is a function $\theta(x, \lambda)$ which is $dx \times d\nu$-square integrable on each compact subset of $\Omega \times R^1$ and such that

(b) $Vu(\lambda) = \int \theta(x, \lambda)^* u(x)\, dx$ *for $u \in L^2(\Omega)$ and a.a. λ,*

(c) $V^*g(x) = \int \theta(x, \lambda) g(\lambda)\, d\nu(\lambda)$ *for $g \in L^2(R^1, d\nu)$ and a.a. x,*

(d) $A\theta_\lambda = \lambda\theta_\lambda$ *for a.a. λ, where $\theta_\lambda(x) = \theta(x, \lambda)$.*

(*The integrals in* (b) *and* (c) *are taken in the mean square sense, while* (d) *is taken in the sense of distributions.*)

Proof. The first part of the statement is just the standard spectral representation for a self-adjoint operator: there is a finite or countable set $\nu = \{\nu_j\}$ of finite measures on R^1 and a unitary mapping V of $L^2(\Omega)$ onto $L^2(R^1, d\nu) = \sum \oplus L^2(R^1, d\nu_\alpha)$ diagonalizing A_1 in the sense of (a) [7, Theorem XII.3.5]. Let $S = (A_1 + i)^{-1}$. Then $VS^q u(\lambda) = (\lambda + i)^{-q} Vu(\lambda)$. Therefore $V^*g \in R(S^q)$ if and only if $(\lambda + i)^q g(\lambda) \in L^2(R^1, d\nu)$. In particular, if g has compact support then $V^*g \in R(S^q)$, all q.

Now let $I_j \subseteq R^1$ be the interval $(-j, j)$ and let $\{\Omega_j\}$ be an increasing sequence of relatively compact open subsets of Ω with union Ω. Take functions $\varphi_j \in \mathfrak{D}(\Omega)$ with $\varphi_j(x) = 1$, all $x \in \Omega_j$. Let W_j be the restriction to $L^2(I_j, d\nu)$ of $\varphi_j V^*$. Then $R(W_j) \subseteq R(\varphi_j S^q)$, all q. It follows from Corollary 2.4 that W_j is an operator of Hilbert-Schmidt type. Therefore there is a kernel $\theta_j(x, \lambda) \in L^2(\Omega \times I_j, dx \times d\nu)$ such that for $g \in L^2(I_j, d\nu)$,

$$V^*g(x) = \int_{-j}^{j} \theta_j(x, \lambda) g(\lambda)\, d\nu(\lambda), \quad \text{a.e. in } \Omega_j.$$

Clearly for $k \geq j$, $\theta_k = \theta_j$ a.e. on $\Omega_j \times I_j$. Therefore there is a function θ, measurable and $dx \times d\nu$ square integrable on each compact subset of $\Omega \times R^1$, such that for $g \in L^2(R^1, d\nu)$,

$$V^*g(x) = \lim_j \int_{-j}^{j} \theta(x, \lambda) g(\lambda)\, d\nu(\lambda), \quad \text{a.e. in } \Omega.$$

This proves (c); (b) and (d) follow by standard arguments. For $u \in L^2(\Omega)$, $g \in L^2(R^1, d\nu)$ with compact support,

$$\int g(\lambda) Vu(\lambda)^*\, d\nu(\lambda) = (g, Vu) = (V^*g, u)$$

$$= \iint g(\lambda)\theta(x, \lambda) u(\lambda)^*\, dx\, d\nu(\lambda).$$

Then (b) follows from Fubini's theorem. Finally, for $u \in \mathfrak{D}(\Omega) \subseteq D(A_1)$, letting $\langle\ ,\ \rangle$ denote the distribution pairing we have

$$\begin{aligned}\langle A\theta_\lambda , u\rangle = \langle \theta_\lambda , Au\rangle = VAu(\lambda) = \lambda Vu(\lambda) \\ = \langle \lambda\theta_\lambda , u\rangle, \quad \text{a.a. } \lambda.\end{aligned}$$

Since $\mathfrak{D}(\Omega)$ is separable, this implies that as a distribution $A\theta_\lambda = \lambda\theta_\lambda$ for almost all λ.

References

1. W. G. Bade and J. Schwartz, *On abstract eigenfunction expansions*, Proc. Nat. Acad. Sci. U.S.A., vol. 42 (1956), pp. 519–525.
2. R. Beals, *On eigenvalue distributions for elliptic operators without smooth coefficients*, Bull. Amer. Math. Soc., vol. 72 (1966), pp. 701–705.
3. ———, *Classes of compact operators and eigenvalue distributions for elliptic operators*, Amer. J. Math., to appear.
4. F. E. Browder, *Eigenfunction expansions for singular elliptic operators. I, II*, Proc. Nat. Acad. Sci. U.S.A., vol. 40 (1954), pp. 454–463.
5. ———, *Eigenfunction expansions for formally self-adjoint partial differential operators. I, II*, Proc. Nat. Acad. Sci. U.S.A., vol. 42 (1956), pp. 769–771, 870–872.
6. ———, *On the spectral theory of elliptic differential operators. I*, Math. Ann., vol. 145 (1962), pp. 22–130.
7. N. Dunford and J. T. Schwartz, *Linear operators, part II*, New York, Interscience, 1963.
8. L. Gårding, *Eigenfunction expansions connected with elliptic differential operators*, Twelfth Congress of Scand. Math., Lund, 1953, pp. 44–55.
9. I. M. Gelfand and A. G. Kostyucenko, *Expansions in eigenfunctions of differential and other operators*, Dokl. Akad. Nauk S.S.S.R. (N.S.), vol. 103 (1955), pp. 349–352.
10. F. I. Mautner, *On eigenfunction expansions*, Proc. Nat. Acad. Sci. U.S.A., vol. 39 (1953), pp. 49–53.
11. E. Nelson, *Kernel functions and eigenfunction expansions*, Duke Math. J., vol. 25 (1958), pp. 15–27.

The University of Chicago
Chicago, Illinois