ON THE UNIQUENESS THEOREM

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It is the purpose of this note to give an alternate proof of the following theorem which originally is an intermediate result of [1].

THEOREM (Feit and Thompson). *Let G be a simple group of odd order all of whose proper subgroups are solvable. Let E be an elementary abelian p-subgroup of order p^3 in G. Then there is only one maximal subgroup of G which contains E.*

The largest part of the proof deals with the Fitting subgroup F of a maximal subgroup H of G. In §2 we consider the case that F is a p-group; necessary results about F are derived in a well known way mainly from the Transitivity Theorem (see (1.1) below) and the ZJ-Theorem (1.2). The case that F is not a p-group is treated in §3; here a very simple observation is crucial, namely that arguments in the proof of the Transitivity Theorem can be applied to certain subgroups of F.

In §4, knowledge about F is used to obtain information about subgroups of H not necessarily contained in F. Finally transfer arguments finish the proof of the theorem.

In the remainder of this section we introduce some notation and collect some necessary lemmas.

Notation.

S_p-subgroup = Sylow p-subgroup
$X^{\#}$ = set of non-identity elements of X
$F(X)$ = Fitting subgroup of X = maximal nilpotent normal subgroup of X
$J(P)$ = subgroup generated by all the abelian subgroups of maximal possible order of P
$\text{И}_Y(A, \pi)$ = set of A-invariant π-subgroups of Y
$\text{И}_Y^*(A, \pi)$ = set of maximal elements of $\text{И}_Y(A, \pi)$
group of type $(p, p, \cdots, p)$ = elementary abelian p-group of order p^n
$r(X) \geq n$ means that X has an elementary abelian p-subgroup of order p^n
$SCN_n(P)$ = set of abelian normal subgroups of P satisfying $C_P(A) = A$ and $r(A) \geq n$
$\{\cdots\}$ = the set $\cdots$
$\langle\cdots\rangle$ = the subgroup generated by $\cdots$

In the following sections G is assumed to be a group of odd order all of whose proper subgroups are solvable.

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$SCN_n(p)$ = set of all $A \in SCN_n(P)$ where P ranges over the S_p-subgroups of G
$\mathfrak{M}$ = set of maximal subgroups of G
$\mathfrak{M}(X)$ = set of maximal subgroups of G which contain X
$\mathfrak{U}$ = set of subgroups of G which are contained in only one $H \in \mathfrak{M}$

We divide $\mathfrak{M}$ into three classes. Take $H \in \mathfrak{M}$.

$H \in \mathfrak{M}_2$ if $\pi(F(H)) \geq 2$ and (i) or (ii) holds where
(i) $r(F(H)) \geq 3$
(ii) $F(H)$ has a subgroup U of type (p, p) such that U is contained in some element of $SCN_3(p)$;
$H \in \mathfrak{M}_1$ if $F(H)$ is a p-group for some prime p, and G has a subgroup of type (p, p, p);
$H \in \mathfrak{M}_0$ if $H \notin \mathfrak{M}_1$ and $H \notin \mathfrak{M}_2$.

The meaning of any symbol not explained here can be deduced from pages 519–520 of [2]. In the following, "group" means "finite group".

Necessary results.

1.1 Transitivity Theorem (Feit and Thompson). *Let G be as above and p a prime. If $A \in SCN_3(p)$, then $C_G(A)$ is transitive on $\mathsf{N}_G^*(A, q)$, for any $q \in p'$. Furthermore, any S_p-subgroup of $N_G(A)$ normalizes some element of $\mathsf{N}_G^*(A, q)$.*

Proof. See [2, page 292, Theorems 8.5.4 and 8.5.6].

1.2 ZJ-Theorem (Glauberman). *Let X be a solvable group of odd order, p a prime, and P a S_p-subgroup of X. Then $Z(J(P))O_{p'}(X) \lhd X$.*

Proof. See [2, page 279, Theorem 8.2.11].

1.3 (P. Hall and G. Higman). *Let X be a solvable group of odd order, p a prime, and P a S_p-subgroup of $O_{p',p}(X)$. If $x \in X$ satisfies $[x, [x, P]] = 1$, then $x \in O_{p',p}(X)$.*

Proof. See [2, page 235, Theorem 6.5.3].

1.4 (Feit and Thompson). *Let p be an odd prime and P a p-group. If $r(P) \geq 3$, then $SCN_3(P)$ is not empty.*

Proof. See [2, page 202, Theorem 5.4.15].

1.5 (Burnside). *Let H be a subgroup of G, p a prime, and P a p-subgroup of H. Assume the following:*
(i) *If $u^g \in H$ for some $u \in P$ and $g \in G$, then $u^g u^{-1} \in H'$ (the latter holds when u^g and u are already conjugate in H);*
(ii) $P \not\subseteq H'$;
(iii) $(p, |G : H|) = 1$.
Then G is not simple.

Proof. See the proof of [3, page 203, Theorem 14.3.1].

1.6. *Let X be a solvable group. Then*
(i) $C_X(F(X)) \subseteq F(X)$;
(ii) $C_X(P) \subseteq O_{p',p}(X)$, *for any S_p-subgroup of $O_{p',p}(X)$.*

Proof. This is easily verified.

1.7. *Let the group K act on the nilpotent group Y, and assume $(|K|, |Y|) = 1$. Set $Y_1 = C_Y(K)$. If $C_Y(Y_1) \subseteq Y_1$, then $Y_1 = Y$.*

Proof. By Frattini argument applied to $KY_1 \lhd N_{KY}(Y_1)$. (We may assume that K is a p-group.)

1.8. *Let X be a solvable group, p a prime, and P a p-subgroup of X. Then $O_{p'}(C_X(P)) \subseteq O_{p'}(X)$.*

Proof. Without loss, $O_{p'}(X) = 1$. Set $Y = PO_p(X)$, $K = O_{p'}(C_X(\mathrm{P}))$, and $Y_1 = C_Y(K)$. Obviously, $PC_Y(P) \subseteq Y_1$. This implies $C_Y(Y_1) \subseteq Y_1$. Now (1.7) yields $Y = C_Y(K)$; by (1.6), $K = 1$.

1.9. *Let P be a p-group, p odd. Then we have the following.*
(i) If cl $(\mathrm{P}) \leq 2$, *then* $\{g \in P \mid g^p = 1\}$ *is a subgroup of* P (i.e. $\Omega_1(P)^p = 1$).
(ii) *If P is non-cyclic, then $\Omega_1(Z_2(P))$ is non-cyclic.*
(iii) *If N is a non-cyclic normal subgroup of P, then N has a subgroup of type (p, p) which is normal in P.*
(iv) *If $r(P) \geq 3$, then any normal subgroup of type (p, p) of P is contained in some element of $SCN_3(P)$.*
(v) *Let X be a p'-group acting on P; if X centralizes $\Omega_1(P)$, then X centralizes P.*
(vi) *Let K be a solvable group of automorphisms of P; if $r(P) \leq 2$ and K has odd order, then K' is a p-group (in particular, K has a normal S_p-subgroup).*

Proof. For (i), (ii), and (v) see [2, page 183, Lemma 5.3.9.i, page 199, Theorem 5.4.10.i, and page 184, Theorem 5.3.10 respectively].

As for (iii), observe that $\Omega_1(Z_2(N))$ is a non-cyclic normal subgroup of exponent p of P.

Now assume $r(P) \geq 3$, and let U be a normal subgroup of type (p, p) of P. By (1.4), there exists some $A \in SCN_3(P)$. Set $B = UC_A(U)$. Then $B \lhd P$, $r(B) \geq 3$, and $B' = 1$. Enlarge B to a maximal abelian normal subgroup of P, which then is an element of $SCN_3(P)$. This proves (iv).

In order to prove (vi), we may assume that P is not cyclic. Set $V = \Omega_1(Z_2(P))$ and $\bar{V} = V/V'$. By (i) and (ii), $V^p = 1$ and $|V| \geq p^2$. On the other hand, $r(P) \leq 2$ implies $|V| \leq p^3$ and $\Omega_1(C_P(V)) \subseteq V$. Then $|\bar{V}| = p^2$; and since a p'-subgroup of $C_K(\bar{V})$ also centralizes V, it follows from (v) and (1.7) that $C_K(\bar{V})$ is a p-group. Now the assertion follows from the structure of $GL(2, p)$, the automorphism group of $\bar{V}$, see [4, Kapitel II, §8].

1.10. *Let X be a solvable group of odd order, p a prime, and P a S_p-subgroup of X. If P' is cyclic, then $P \subseteq O_{p',p}(X)$.*

Proof. We may assume that $Q = O_{p',p}(X)$ is an elementary abelian p-group. Then $Q \subseteq P$ and $|P' \cap Q| \leq p$. This implies $[P, [P, Q]] = 1$. Now (1.3) yields $P \subseteq Q$.

2. Maximal subgroups in $\mathfrak{M}_1$

In this section we prove a uniqueness theorem for certain subgroups of type (p, p) and discuss maximal subgroups of G whose Fitting subgroup is a p-group.

2.1. *Let p be a prime, $H \in \mathfrak{M}$, and U a subgroup of type (p, p) of H satisfying*

$$C_G(x) \subseteq H, \quad \textit{for all } x \in U^{\#}.$$

Then $U \in \mathfrak{U}$, i.e. $\mathfrak{M}(U) = H$.

Proof. Let P be a S_p-subgroup of H containing U. We first show that $N_G(P) \subseteq H$; then P is also a S_p-subgroup of G.

Since U is non-cyclic and abelian, every U-invariant p'-subgroup of G is contained in H (otherwise U would act fixed-point-free on some non-trivial section of G). Thus, $O_{p'}(H) = \langle \mathsf{N}_G(P, p') \rangle$. Hence, $N_G(P)$ normalizes $O_{p'}(H)$. This implies $N_G(P) \subseteq H$, provided $O_{p'}(H) \neq 1$.

If $O_{p'}(H) = 1$, then the ZJ-Theorem (1.2) yields $Z(J(P)) \lhd H$, and again we get $N_G(P) \subseteq H$.

Now assume $H \neq M \in \mathfrak{M}(U)$. Choose M in such a way that $|M \cap H|_p$ is maximal, and let R be a S_p-subgroup of $M \cap H$ containing U. Without loss, $R \subseteq P$. Choice of M and $N_G(P) \subseteq H$ yield $N_G(R) \subseteq H$. As a consequence, R is a S_p-subgroup of M. We have $O_{p'}(M) \subseteq H$. Set $Z = Z(J(R))$. By the ZJ-Theorem, $M = O_{p'}(M)N_M(Z)$. Thus, $N_G(Z) \not\subseteq H$; now choice of M and $N_P(Z) \supset R$ imply $R = P$. If $O_{p'}(H) = 1$, then $N_G(Z) = H$. Hence, $O_{p'}(H) \neq 1$.

Set $S = P \cap O_{p',p}(M)$. Then $M = O_{p'}(M)N_M(S)$. This implies $S \neq P$. Then it follows from (1.9.vi) and (1.6) that $SCN_3(P)$ is non-empty. Take $A \in SCN_3(P)$. By (1.3), $A \subseteq S$.

Take any prime $q \in p'$. By the Transitivity Theorem (1.1), P normalizes some maximal A-invariant q-subgroup Q of G. Being U-invariant, Q is contained in H; then $P \subseteq N_H(Q)$ implies $Q \subseteq O_{p'}(H)$, and Q must be a S_q-subgroup of $O_{p'}(H)$.

From the Transitivity Theorem we know that $C_G(A)$ is transitive on $\mathsf{N}^*_G(A, q)$. However, $C_G(A) = AO_{p'}(C_G(A))$ implies $C_G(A) \subseteq H$ (because $C_G(A)$ is U-invariant). Consequently, every A-invariant q-subgroup of G is contained in $O_{p'}(H)$. Since this holds for any prime $q \in p'$, we get $O_{p'}(H) = \langle \mathsf{N}_G(A, p') \rangle$. This together with $A \subseteq S \subseteq H$ yields $O_{p'}(H) = \langle \mathsf{N}_G(S, p') \rangle$. Then $N_G(S)$ normalizes $O_{p'}(H) \neq 1$. Thus, $N_G(S) \subseteq H$, a contradiction.

2.2. *Suppose $H \in \mathfrak{M}_1$. Let U be a subgroup of type (p, p) of H, where p is the only prime divisor of $|F(H)|$.*

If $C_H(U)$ has a subgroup of type (p, p, p), then $U \in \mathfrak{U}$.

Proof. Let P be a S_p-subgroup of H and E a subgroup of type (p, p, p) such that $U \subseteq E \subseteq P$. By (1.4), we find some $A \in SCN_3(P)$.

Assume $\mathfrak{M}(A) = H$. Then $C_G(x) \subseteq H$ for all $x \in A^{\#}$. Take a subgroup U_1 of type (p, p) of A which is normalized by E, and set $U_2 = C_E(U_1)$. Then $|U_2| \geq p^2$. By (2.1), $\mathfrak{M}(U_1) = H$. Hence, $C_G(x) \subseteq H$ for all $x \in U_2$. By (2.1), $\mathfrak{M}(U_2) = H$. Then a third application of (2.1) yields $\mathfrak{M}(U) = H$. Thus, if suffices to prove $\mathfrak{M}(A) = H$.

By (1.3), $A \subseteq O_p(H) = F(H)$. Then $Z(F(H)) \subseteq A$ and therefore $C_G(A) \subseteq H$. As A is an S_p-subgroup of $C_H(A)$, it follows from (1.6) and (1.7) that $C_H(A) = A$. Thus, $C_G(A) = A$.

By the ZJ-Theorem, $Z(J(P)) \lhd H$; in particular, $N_G(P) \subseteq H$, and P is a S_p-subgroup of G. Then $A \in SCN_3(p)$.

Take any $q \in p'$. It follows from the Transitivity Theorem and $C_G(A) = A$ that $\mathsf{M}_G(A, q)$ has only one maximal element, say Q. Then $N_G(A)$ normalizes Q, and Q is also the unique element of $\mathsf{M}_G^*(F(H), q)$, because of $A \subseteq F(H) \subseteq N_G(A)$. Now H has to normalize Q, and this implies $Q = 1$.

Thus, $\mathsf{M}_G(A, p') = 1$.

Now assume $H \neq M \in \mathfrak{M}(A)$. Choose M in such a way that $|M \cap H|_p$ is maximal, and let R be a S_p-subgroup of $M \cap H$. Choice of M implies that R is a S_p-subgroup of M. We already know that $O_{p'}(M) = 1$. From the ZJ-Theorem we get $Z(J(R)) \lhd M$. Hence, $N_G(R) \subseteq M$, and R is a S_p-subgroup of G. But then $Z(J(R))$ is also normal in H, a contradiction.

3. Maximal subgroups in $\mathfrak{M}_2$

Throughout this section let $H \in \mathfrak{M}_2$, and set $F = F(H)$ and $\pi = \pi(F)$. For any nilpotent group X and any prime set σ we write X_σ for $O_\sigma(X)$.

3.1. *Let E be a subgroup of F satisfying $Z(F) \subseteq E$. Take $p \in \pi$ and assume $E \subseteq M \subset G$. Then $E_{p'} \subseteq O_{p'}(M)$.*

Proof. For $P = E_p$ we have $1 \neq Z(F)_p \subseteq P$ and therefore $C_G(P) \subseteq H$. Hence, $E_{p'} \subseteq O_{p'}(C_M(P))$. By (1.8), this implies $E_{p'} \subseteq O_{p'}(M)$.

3.2. *Let E be a subgroup of F satisfying $Z(F) \subseteq E$. If $\mathsf{M}(E, \pi') = 1$, then $E \in \mathfrak{U}$.*

Proof. Take $M \in \mathfrak{M}(E)$. We have to show that $M = H$.

Set $D = F(M)$ and $\sigma = \pi(D)$. By assumption, $\sigma \subseteq \pi$.

For $p \in \pi$ we have $E_{p'} \subseteq O_{p'}(M)$, by (3.1). Thus, $E_{\sigma'} \subseteq O_{\sigma'}(M) = 1$, i.e. $\sigma = \pi$. As a second consequence, $D_p \subseteq C_G(E_{p'}) \subseteq H$.

From $|\pi| \geq 2$ we can now conclude $D \subseteq H$. Obviously, $D_{p'} \subseteq C_G(E_p)$.

As an easy consequence, $O_{p'}(M) \subseteq C_G(E_p) \subseteq H$. Then $O_{p'}(M) \subseteq O_{p'}(C_H(D_p))$. By (1.8), this implies $O_{p'}(M) \subseteq O_{p'}(H)$.

Now every assumption actually used so far remains valid if H is replaced by M, M by H, and E by $D = F(M)$. For this reason, we also have $O_{p'}(H) \subseteq O_{p'}(M)$. Thus, $O_{p'}(M) = O_{p'}(H) \neq 1$. This implies $M = H$.

3.3. *Let E be a subgroup of F satisfying $C_F(E) \subseteq E$. Assume $q \in \pi'$ and $E \subseteq M \subset G$. Then $\langle И_M(E, q)\rangle$ is a q-group.*

Proof. For $K \in И_H(E, \pi')$ we have $[K, E] \subseteq K \cap F = 1$. This implies $K \subseteq C_H(F) \subseteq F$, see (1.7) and (1.6), and therefore $K = 1$. Thus, $И_H(E, \pi') = 1$.

Now let $Q \in И_M(E, q)$. Take two different primes $p, r \in \pi$, and set $P = F_p$ and $R = F_r$. By (3.1), $R \subseteq O_{p'}(M)$; hence, $[R, Q] \subseteq O_{p'}(M)$. Now $C_Q(R) \in И_H(E, \pi') = 1$ implies $[R, Q] = Q$. Thus, $Q \subseteq O_{p'}(M)$.

Since $p \in \pi$ was arbitrary, we get $Q \subseteq O_{\pi'}(M)$. Set $N = O_{\pi'}(M)$. We have $C_N(P) \in И_H(E, \pi') = 1$. It follows easily from Sylow's theorem that every P-invariant q-subgroup of N is contained in a P-invariant S_q-subgroup of N, and that all P-invariant S_q-subgroups of N are conjugate under $C_N(P)$. Thus, $C_N(P) = 1$ forces $\langle И_N(P, q)\rangle$ to be a q-group. Now the assertion is clear.

3.4. *Let V be an elementary abelian p-subgroup of F, for some $p \in \pi$. Assume either $|V| = p^3$, or $|V| = p^2$ and $V \subseteq A$ for some $A \in SCN_3(p)$. Set $E = C_F(V)$. Then $И_G(E, \pi') = 1$.*

Proof. Take any $q \in \pi'$. Suppose $R, Q \in И_G^*(E, q)$, $R \neq Q$.

If $R \cap Q \neq 1$, then $\langle N_R(R \cap Q), N_Q(R \cap Q)\rangle$ is a q-group, by (3.3). However, this is impossible when R and Q are chosen in such a way that $|R \cap Q|$ is maximal. Thus, $R \cap Q = 1$ (for any two different elements of $И_G^*(E, q)$).

Then it follows from (3.3) that, for every $z \in Z(E)$, $C_R(z) = 1$ or $C_Q(z) = 1$. This implies that $Z(E)$ is metacyclic; in particular, $|V| = p^2$. Clearly, $\Omega_1(Z(F)_p) \subseteq V$; consequently, $A \subseteq C_G(V) \subseteq H$ and therefore $A \subseteq N_H(E)$.

As V is not cyclic, we find an element $z \in V^\#$ satisfying $C_Q(z) \neq 1$. Set $S = \langle И_{C_G(z)}^*(E, q)\rangle$; by (3.3), S is a q-group. A normalizes S. Since any two maximal elements of $И_G(E, q)$ have trivial intersection and A normalizes E, it follows that $S \subseteq Q$ and $A \subseteq N_G(Q)$.

In the same way we see that R is A-invariant.

Now the Transitivity Theorem (1.1) yields an element $c \in C_G(A)$ such that $\langle Q^c, R\rangle$ is a q-group. However, $c \in C_G(A) \subseteq C_G(V) \subseteq N_H(E)$ implies $Q^c \in И_G^*(E, q)$. Thus, $Q^c = R$ and therefore $C_R(z) = C_Q(z)^c \neq 1$, a contradiction.

This proves that $И_G^*(E, q)$ has only one element, say Q. Then $E_1 = N_F(E)$ normalizes Q, and consequently Q is also the unique element of $И_G^*(E_1, q)$. As F is a nilpotent normal subgroup of H, it follows easily that H normalizes Q; but then $Q \subseteq F_q = 1$. The proof of (3.4) is finished.

3.5. *Let V be as in* (3.4). *Then* $V \in \mathfrak{U}$.

Proof. By (3.4) and (3.2), $C_F(V) \in \mathfrak{U}$. Hence, $C_G(x) \subseteq H$ for all $x \in V^{\#}$. Then (2.1) yields $V \in \mathfrak{U}$.

4. The Uniqueness Theorem

Throughout this last section let p be a prime and $A \in SCN_3(p)$.

4.1. *Suppose* $U \in \mathfrak{U}$, $V \subseteq C_G(U)$, *and* $r(V) \geq 2$. *Then* $V \in \mathfrak{U}$.

Proof. If $U \subseteq H \in M$, then $U \in \mathfrak{U}$ implies $C_G(x) \subseteq H$ for all $x \in V^{\#}$. Then (2.1) yields $V \in \mathfrak{U}$.

4.2. *If* $A \in \mathfrak{U}$, *then* $E \in \mathfrak{U}$ *for every p-subgroup* E *of* G *with* $r(E) \geq 3$.

Proof. Without loss, E normalizes A. Then E normalizes a subgroup B of type (p, p) of A. From $r(E) \geq 3$ we conclude $r(C_E(B)) \geq 2$. Now repeated application of (4.1) yields $C_E(B) \in \mathfrak{U}$. In particular, $E \in \mathfrak{U}$.

4.3. *Suppose* $H \in \mathfrak{M}_1 \cup \mathfrak{M}_2$. *Then there is a prime* q *such that* $F(H)_q \in \mathfrak{U}$ *and* $E \in \mathfrak{U}$ *for every subgroup* E *of type* (q, q, q) *of* G.

Proof. If $H \in \mathfrak{M}_1$, then $r(F(H)) \geq 3$ because otherwise $F(H)$ would be a S_q-subgroup of H (for the unique prime divisor q of $|F(H)|$), see (1.9.vi). Then the assertion follows immediately from (2.2).

If $H \in \mathfrak{M}_2$, then the definition of $\mathfrak{M}_2$ yields a prime q and a subgroup V of $F(H)$ such that V is of type (q, q, q) or (q, q) and in the second case is contained in some element of $SCN_3(q)$.

By (3.5), $V \in \mathfrak{U}$ for any such V. In particular, $F(H)_q \in \mathfrak{U}$. Then H contains a S_q-subgroup of G, and therefore some $\bar{A} \in SCN_3(q)$ contains such a subgroup V (see 1.9.iii and iv). Now (4.1) yields $\bar{A} \in \mathfrak{U}$, and application of (4.2) completes the proof.

4.4. *Assume* $C_G(A) \subseteq H \in \mathfrak{M}_1 \cup \mathfrak{M}_2$. *Then* $N_G(X) \subseteq H$ *for any p-subgroup* X *of* G *containing* A.

Proof. Let q be as in (4.3). If $p = q$, then $r(A) \geq 3$ implies $A \in U$. In case $p \neq q$, enlarge $F(H)_q$ to an element Q of $\mathsf{N}_G^*(A, q)$. From $F(H)_q \in U$ we get $Q \subseteq H$. By the Transitivity Theorem (1.1), $Q \subseteq H$ and $C_G(A) \subseteq H$ implies that every A-invariant q-subgroup of G is contained in H. In particular, $K = \langle \mathsf{N}_G(X, q) \rangle \subseteq H$. Without loss, $X \subseteq H$. Then $F(H)_q \subseteq K$ and therefore $N_G(K) \subseteq H$. Obviously, $N_G(X)$ normalizes K, and the assertion follows.

In the following, let P *be a* S_p*-subgroup of* G *satisfying* $A \lhd P$. *Choose some* $H \in \mathfrak{M}$ *such that* $N_G(Z(P)) \subseteq H$ (*then* $C_G(A) \subseteq H$, *because of* $Z(P) \subseteq C_P(A) = A$). *In the next lemma we show that* $H \in \mathfrak{M}_1 \cup \mathfrak{M}_2$. *Then fix a prime* q *having the properties described in* (4.3), *and set* $Q = F(H)_q$.

4.5. *We have $P \subseteq H'$ and $H \notin \mathfrak{M}_0$.*

Proof. We first show that $H' \cap P$ is cyclic, provided $H \in \mathfrak{M}_0$. In that case we have $r(F(H)) \leq 2$. Furthermore, $F(H)_p$ must be cyclic, because otherwise P has a normal subgroup of type (p, p) contained in $F(H)$, and this normal subgroup would be contained in some element of $SCN_3(P)$, see (1.9.iii/iv). Then (1.9.vi) yields $H' \cap P \subseteq C_H(F(H)) \subseteq F(H)$. This proves what is claimed above.

Next we prove that $Z(P) \subseteq H^g$ implies $g \in H$, for any $g \in G$. Assume this to be false. Let X be maximal among p-subgroups of H satisfying

$$C_G(X) \subseteq H \quad \text{and} \quad X \subseteq H^g \qquad \text{for some } g \in G - H.$$

It follows easily from Sylow's theorem and $N_G(P) \subseteq H$ that X is not a S_p-subgroup of H. If $X \subseteq H^g \neq H$, then X must be a S_p-subgroup of $H \cap H^g$. These two facts yield $N_G(X) \nsubseteq H$. Set $N = N_G(X)$. Let S be a S_p-subgroup of $N_H(X)$. Then $X \subset S$, and maximality of X forces S to be a S_p-subgroup of N. Without loss, $S \subseteq P$. For $S_0 = S \cap O_{p',p}(N)$ we have $N = N_N(S_0)O_{p'}(N)$. Clearly, $O_{p'}(N) \subseteq C_G(X) \subseteq H$. Thus, $N_G(S_0) \nsubseteq H$, and maximality of X yields $S_0 = X$.

As X is non-trivial, $N = N_G(X)$ is solvable. By (1.3), $N_A(X) \subseteq O_{p',p}(N)$. Hence, $N_A(X) \subseteq X$ and therefore $A \subseteq X$. By (4.4), this is possible only if $H \in M_0$. Then P' is cyclic, as we have seen above. By (1.10), $S \subseteq O_{p',p}(N)$. Thus, $S = X$, a contradiction. This proves our assertion about $Z(P)$.

Now assume $u^g \in H$ for some $u \in P$ and $g \in G$. Then there exists some $h \in H$ such that $u^g = u^h$:

We have $Z(P)^g \subseteq C_G(u^g)$ and $Z(P)^a \subseteq C_G(u^g)$ for some $a \in H$. Sylow's theorem yields an element $c \in C_G(u^g)$ such that $\langle Z(P)^{gc}, Z(P)^a\rangle$ is a p-group. As H contains a S_p-subgroup of G, $\langle Z(P)^{gc}, Z(P)^a\rangle \subseteq H^y$ for some $y \in G$. Then $Z(P)^a \subseteq H \cap H^y$ yields $y \in H$, and $Z(P)^{gc} \subseteq H^y \cap H^{gc} = H \cap H^{gc}$ yields $gc \in H$. For $h = gc$ we now have $h \in H$ and $u^g = u^h$.

Thus, conditions (i) and (iii) of (1.5) are satisfied, and therefore we get $P \subseteq H'$ from (1.5).

Then $P \cap H'$ is not cyclic; as we have seen above, this implies $H \notin \mathfrak{M}_0$.

4.6. *Let K be a q'-subgroup of H. If $C_Q(K)$ is not cyclic, then $C_Q(K) \in \mathfrak{U}$.*

Proof. By definition of q and Q, $Q = F(H)_q \in U$ and $E \in U$ for every subgroup E of type (q, q, q) of G. Then (4.1) yields $D \in U$ for any non-cyclic subgroup D of such a subgroup E of type (q, q, q). Thus, it suffices to show $R = Q$ or $r(RC_Q(R)) \geq 3$ for $R = C_Q(K)$. If $r(RC_Q(R)) \leq 2$, then $\Omega_1(RC_Q(R)) \subseteq R = C_Q(K)$, and (1.9.v) yields $RC_Q(R) \subseteq C_G(K)$. Thus, $C_Q(R) \subseteq R$, and (1.7) yields $R = Q$.

4.7. *If $C_Q(A) \neq 1$, then $A \in \mathfrak{U}$.*

Proof. Otherwise, $p \neq q$ and $1 \neq C_Q(A) \neq Q$, by (4.1) and definition of q

and Q. Set $R = C_Q(A)$. We find a non-identity subgroup T/R of $N_Q(R)/R$ on which A acts irreducibly. Set $B = C_A(T/R)$. Then A/B is cyclic whence $r(B) \geq 2$. Furthermore, B centralizes T (since $p \neq q$ and B centralizes R as well as T/R). If T is cyclic, then $\Omega_1(T) \subseteq C_Q(A)$ and therefore $T \subseteq C_Q(A) = R$, see (1.9.v), contradicting $T/R \neq 1$. Thus, $r(T) \geq 2$. In particular, $r(C_Q(B)) \geq 2$. By (4.6), $C_Q(B) \in \mathfrak{U}$. Since $r(B) \geq 2$, this implies $B \in \mathfrak{U}$, see (4.1). In particular, $A \in \mathfrak{U}$.

4.8. *Let E be a subgroup of type (p, p, p) of G. Then $E \in \mathfrak{U}$.*

Proof. Suppose by way of contradiction that (4.8) is false. Then $A \notin \mathfrak{U}$, by (4.2). Let $Z \neq 1$ be a minimal A-invariant subgroup of $Z(Q)$. By (4.7) and $A \notin \mathfrak{U}$, Z is not centralized by A. Thus, $Z = [Z, A]$. Since $A/C_A(Z)$ is cyclic, $C_A(Z)$ is not cyclic. Then (2.1) yields an element $a \in C_A(Z)^{\#}$ such that $C_G(a) \nsubseteq H$. Let M be a maximal subgroup of G containing $C_G(a)$. Then $M \neq H$.

Case 1. $M \in \mathfrak{M}_1 \cup \mathfrak{M}_2$. Set $R = P \cap O_{p'p}(H)$. By (1.3), $A \subseteq R$. This implies $N_G(R), \subseteq M$, see (4.4). Thus, $N_H(R)$ normalizes $Z(Q) \cap M$. It follows from $Q \in \mathfrak{U}$ and (4.1) that $Z(Q) \cap M$ is cyclic. However, $Z(Q) \cap M$ contains Z and is therefore not centralized by A. Then it is clear that P is not contained in $N_H(R)'$. This together with $H = N_H(R)O_{p'}(H)$ yields $P \nsubseteq H'$, which contradicts (4.5).

Case 2. $M \in \mathfrak{M}_0$. Then $r(F(M)) \leq 2$. Since $Z = [Z, A]$, it follows from (1.9.vi) that Z centralizes $F(M)_{q'}$. By definition of q, any maximal subgroup of G containing a subgroup of type (q, q, q) is conjugate to H. Thus, M has no subgroup of type (q, q, q). Then it is an easy consequence of (1.9.vi) and $C_M(F(M)) \subseteq F(M)$ that, with $K = F(M)_{q'}$, $C_M(K)$ has a normal S_q-subgroup. Thus, $Z \subseteq F(M)$. Set $Y = C_{F(M)}(Z)_q$. Then $C_G(Y) \subseteq M \cap H$, because of $Z(F(M))_q \subseteq Y$, $Z \subseteq Z(Q)$, and $Q \in \mathfrak{U}$. Now we have $K \subseteq O_{q'}(C_H(Y))$ and therefore $K \subseteq O_{q'}(H)$, see (1.8). Then $Q \subseteq C_H(K)$, and this together with $Q \in \mathfrak{U}$ and $K \lhd M$ yields $K = 1$. Now $M = C_M(K)$ has a normal S_q-subgroup, whence M contains a S_q-subgroup of G. However, this is not in accordance with $\mathfrak{M}(Q) = H \notin \mathfrak{M}_0$ and $M \in \mathfrak{M}_0$. The proof is complete.

Now the assertion of the theorem follows immediately from (1.4) and (4.8).

Bibliography

1. W. Feit and J. Thompson, *Solvability of groups of odd order*, Pacific J. Math., vol. 13 (1963), pp. 775–1029.
2. D. Gorenstein, *Finite groups*, Harper and Row, New York, 1968.
3. M. Hall, Jr., *The theory of groups*, Macmillan, New York, 1962.
4. B. Huppert, *Endliche Gruppen I*, Springer, Berlin, 1967.

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