IMPLICATIONS IN THE COHOMOLOGY OF H-SPACES

BY

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0. Introduction

0.1. Summary of results. In his investigations of the cohomology of H-spaces, W. Browder introduced the notion of implications in Hopf algebras (see e.g. [3, p. 357]). In this study we show that in some important cases an implication in the cohomology of H-spaces (if it is not a p^{th} power) takes the form of a (nonstable) secondary cohomology operation. Some properties of this operation were investigated in [10]. The main calculations are carried out in Section 2 where we study the operation ϕ, defined on $(\ker \xi)^{2m}$ ($\xi x = x^p$) and associated with the inequality $e(\beta p^m) > 2m$ (e is the excess).

The most significant results of this study could be summarized as follows:

THEOREM A (Corollary 1.7). *Let (X, μ) be an H-space and $B \subset H^*(X, Z_p)$ a sub-Hopf algebra closed under the action of the Steenrod algebra $\mathfrak{A}(p)$; then there exists an H-space (G, μ_0) and an H-mapping $f : (X, \mu) \to (G, \mu_0)$ so that $\operatorname{im} f^* = B$. If B is associative G may be assumed homotopy associative.*

Moreover, G can be taken to be a product of Eilenberg-McLane spaces with an exotic multiplication.

Next, consider the fiber $\tilde{X}$, $j : \tilde{X} \to X$, of the map $f : X \to G$. Since G is a product of Eilenberg-MacLane spaces

$$\operatorname{im}(j^*) \cong H^*(X, Z_p)/\!\!/B \quad \text{in } H^*(\tilde{X}, Z_p).$$

In particular, if $\bar{\mu}^*(x) \in B \otimes B$ then $j^*(x)$ is primitive and we have

THEOREM B (Corollary 2.2). *Let p be an odd prime, $B_0 \subset \ker \xi$ (ξ being the p^{th} power operation) and $x \in \ker \xi$ satisfies $\bar{\mu}^* x \in B \otimes B$ and suppose that (X, μ) is homotopy associative and $H^*(X, Z_p)/\!\!/B$ is cocommutative. There exists a secondary operation ϕ defined on $\ker \xi$ so that for every $z \in (H^*(X, Z_p)/\!\!/B)^*$ we have*

$$\langle x, z\rangle = \langle \phi(x), z^p\rangle.$$

Remark. This is an implication theorem in the sense of W. Browder (see [3, p. 357]) and it considerably strengthens his results.

As a typical application of Theorem B we have

THEOREM C (Theorem 3.1 (d)–(e)). *If X is a homotopy associative H-space, p an odd prime, and $H^*(X, Z_p)$ is primitively generated, then $H^*(X, Z_p)$ is a free algebra (i.e., a tensor product of an exterior algebra on odd-dimensional generators and a polynomial algebra on even dimensional generators).*

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If $H_*(X, Z_p)$ is commutative and primitively generated then it is a free algebra.

Remark. Some versions of this theorem were proved in [8] and [9].

THEOREM D. *If X is a homotopy associative H-space, p an odd prime and $H^*(X, Z_p)$ is a finite commutative algebra, then it is an exterior algebra on odd-dimensional generators. In particular, $H^*(X, Z)$ has no p-torsion.*

Remark. This provides an alternative proof of Browder's theorem [5, p. 319, corollary to Theorem 1] stating that $H_*(G, Z_p)$ is not commutative where $p = 3, G = F_4, E_6, E_7, E_8$ (the exceptional Lie groups) or $p = 5$ and $G = E_8$.

0.2. Conventions and notations. All spaces in this study are assumed to be arcwise connected and of the homotopy type of a CW complex of finite type (i.e., having finitely many cells in each dimension). We assume all spaces have a designated base point and, unless otherwise stated, mappings between them are assumed to be base point preserving.

All Hopf algebras are assumed to be graded, connected and of finite type. If A is a Hopf algebra, PA and QA denote the module of primitives and the module of indecomposable respectively. For all details concerning properties of Hopf algebras the reader is referred to [6].

If X is a topological space, PX denotes the space of (end points free) paths. $\mathcal{L}X$ is the subspace of PX of paths initiating at $*$. ΩX is, as usual, the loop space.

We define secondary operation by universal examples as in [1, p. 55].

1. On twisted H-structures

1.1 DEFINITION. Let X, μ be an H-space and let G be a topological group. Then ΩG admits a group structure μ_0 given by

$$\mu_0(\lambda, \nu)(t) = \lambda(t)\cdot\nu(t), \quad \lambda^{-1}(t) = [\lambda(t)]^{-1}.$$

(Note that this multiplication is homotopic to the loop-addition.)

Let $\omega : X \times X, X \vee X \to \Omega G, *$ be a given map. The ω-twisted H-structure on $X \times \Omega G$ is the multiplication.

$$\tilde{\mu} : (X \times \Omega Y) \times (X \times \Omega Y) \to X \times \Omega Y$$

given by the composition

$$X \times \Omega G \times X \times \Omega G$$

$$\xrightarrow{1 \times T \times 1} X \times X \times \Omega G \times \Omega G$$

$$\xrightarrow{\Delta_{X \times X} \times 1 \times 1} X \times X \times X \times X \times \Omega G \times \Omega G$$

$$\xrightarrow{\mu \times \omega \times \mu_0} X \times \Omega G \times \Omega G \xrightarrow{1 \times \mu_0} X \times \Omega G.$$

If $y \in PH^*(\Omega G, Z_p)$ then

$$\tilde{\mu}^*(1 \otimes y) = (1 \otimes T^* \otimes 1)\omega^* y \otimes 1 \otimes 1.$$

Let G and G' be abelian groups. Suppose G and G' are given H-structures $\mu_0 : G \times G \to G$, $\mu_0' : G' \times G' \to G'$ which do not necessarily coincide with the group multiplication. Let $g : G \to G'$ be a (μ_0, μ_0') H-mapping. There is no loss of generality in assuming that g is a multiplicative fibration: $g \circ \mu_0 = \mu_0' \circ (g \times g)$. Let (X, μ) be an H-space and $f : (X, \mu) \to (G, \mu_0)$ a multiplicative fibration.

If $g \circ f \sim *$ then there exists

$$l : X \to \mathfrak{L}G', \quad l(x)[0] = *, \quad l(x)[1] = g \circ f(x).$$

Let $E \subset G \times \mathfrak{L}G'$, $\tau : E \to G$, $\mu_1 : E \times E \to E$ and $\tilde{f} : X \to E$ be given by

$$E = \{x, \varphi \mid g(x) = \varphi(1)\}, \quad \tau = p_1 \mid E, \quad \mu_1 = (\mu_0 \times \mathfrak{L}\mu_0') \circ (1 \times T \times 1) \mid E$$

and $\tilde{f}(x) = f(x), l(x)$. Then $\tau \circ \tilde{f} = f$ and $\tau\mu_1 = \mu_0 \tau \times \tau$. We have the following commutative diagram:

$$\begin{tikzcd}
& E \arrow[d, "\tau"] & \\
X \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & G \arrow[r, "g"] & G'
\end{tikzcd}\ .$$

The obstruction for $\tilde{f}$ to be an H-mapping can be obtained from the following diagram:

$$\begin{tikzcd}
& E \times E \arrow[d, "\tau \times \tau"] \arrow[dr, "\mu_1"] & \Omega G^1, * \arrow[d, "i"] \\
X \times X \arrow[ur, "\tilde{f} \times \tilde{f}"] \arrow[r, "f \times f"] \arrow[d, "\mu"] & G \times G \arrow[dr, dashed, "\mu_0"] & E \arrow[d, "\tau"] \\
X \arrow[urr, "\tilde{f}"] \arrow[rr, "f"] & & G
\end{tikzcd}\ .$$

Since $\tau\mu_1 \tilde{f} \times \tilde{f} = \tau\tilde{f}\mu$ there exists $\omega : X \times X, X \vee X \to \Omega G', *$ with the property

$$\mu_1(\tilde{f}(x), \tilde{f}(y)) = \eta([\tilde{f} \circ \mu(x, y)], \omega(x, y))$$

where $\eta : E \times \Omega G \to E$ is given by $\eta[(x, \varphi), \lambda] = x, \varphi \cdot \lambda$ ($\cdot$ is the group multiplication).

1.1.1. Lemma. *$\omega : X \times X, X \vee X \to \Omega G', *$ can be given by*

$$\omega(x, y) = [l\mu(x, y)]^{-1} \cdot \mathfrak{L}\mu_0'(l(x), l(y)).$$

Proof.

$$\begin{aligned}\eta(\tilde{f}\mu(x, y), \omega(x, y)) &= f\circ\mu(x, y), l\mu(x, y)\cdot\omega(x, y)\\ &= \mu_0(f(x), f(y)), \mathfrak{L}\mu_0'(l(x), l(y))\\ &= \mu_1(\tilde{f}(x), \tilde{f}(y)).\end{aligned}$$

Let $f_1 : X \times \Omega G \to E$ be given by

$$f_1(x, \lambda) = \eta(\tilde{f}(x), \lambda).$$

Then we have

1.1.3. Lemma. *If $X \times \Omega G'$ is given the ω-twisted multiplication $\bar{\mu}$ induced by ω then $f_1 : (X \times \Omega G', \bar{\mu}) \to (E, \mu_1)$ is an H-mapping.*

Proof.

$$\begin{aligned}f_1\bar{\mu}[(x, \lambda), (\bar{x}, \bar{\lambda})] &= f_1[\mu(x, \bar{x}), \omega(x, \bar{x})\cdot\lambda\cdot\bar{\lambda}]\\ &= f\mu(x, \bar{x}), l\mu(x, \bar{x})\cdot\omega(x, \bar{x})\cdot\lambda\cdot\bar{\lambda}\\ &= \mu_0(f(x), f(\bar{x})), \mathfrak{L}\mu_0'(l(x), l(\bar{x}))\cdot\lambda\cdot\bar{\lambda},\\ \mu_1(f_1(x, \lambda), f_1(\bar{x}, \bar{\lambda})) &= \mu_1[(f(x), l(x)\cdot\lambda), (f(\bar{x}), l(\bar{x})\cdot\bar{\lambda})]\\ &= \mu_0(f(x), f(\bar{x})), \mu_0'(l(x)\cdot\lambda, l(\bar{x})\cdot\bar{\lambda}),\end{aligned}$$

and the lemma follows from the fact that

$$\mu_0'(l(x), l(\bar{x}))\cdot\lambda\cdot\bar{\lambda} \sim \mu_0'(l(x)\cdot\lambda, l(\bar{x})\cdot\bar{\lambda})$$

via a homotopy leaving the end points fixed.

1.2. Suppose in addition that (X, μ) and (G, μ_0) are homotopy associative; i.e., there exist

$$a : X \times X \times X \to P(X) \quad \text{and} \quad a_0 : G \times G \times G \to P(G),$$

$$a : \mu(\mu \times 1) \sim \mu(1 \times \mu) \quad \text{and} \quad a_0 : \mu_0(\mu_0 \times 1) \sim \mu_0(1 \times \mu_0).$$

Let $E_0 \subset X \times \mathfrak{L}G$ be the fiber of f; i.e.,

$$E_0 = \{x, \varphi \mid f(x) = \varphi(1)\},$$

$\tau_0 : E_0 \to X$ is the projection, $\mu_2 : E_0 \times E_0 \to E_0$ is the induced multiplication. We investigate the obstruction for E_0 to be homotopy associative. More precisely, we are looking for the obstructions to the existence of a mapping

$$a' : E_0 \times E_0 \times E_0 \to PE_0,$$

so that

(i) $$a' : \mu_2(\mu_2 \times 1) \sim \mu_2(1 \times \mu_2),$$

(ii) $$P\tau_0'\, a' = a\circ(\tau_0 \times \tau_0 \times \tau_0).$$

1.3. Lemma. *The obstruction for the existence of a' satisfying* (i) *and* (ii) *is* *represented by the function* $\alpha_f : X \times X \times X \to \Omega G$ *given by*

$$\alpha_f(x, y, z) = \{P(f)a(x, y, z)\}^{-1}a_0[f(x), f(y), f(z)].$$

(Denote the homotopy class of α_f by $\alpha(f)$.)

Proof.

$$\begin{aligned}
&\mu_2(\mu_2 \times 1)[(x, \varphi_1), (y, \varphi_2), (z, \varphi_3)] \\
&\quad = \mu(\mu \times 1)(x, y, z), \mathfrak{L}\mu_0(\mathfrak{L}\mu_0 \times 1)(\varphi_1, \varphi_2, \varphi_3) \\
&\quad \approx \mu(1 \times \mu)(x, y, z), \mathfrak{L}\mu_0(1 \times \mathfrak{L}\mu_0)(\varphi_1, \varphi_2, \varphi_3) - a_0(f(x), f(y), f(z)) \\
&\qquad + P(f)a(x, y, z)
\end{aligned}$$

as can be seen from Figure 1. It follows that

$$\mu_2(\mu_2 \times 1) \approx \eta(\mu_2(1 \times \mu_2), [\alpha_f(x, y, z)]^{-1}).$$

If (G', μ_0') is homotopy associative with homotopy

$$a_0' : G' \times G' \times G' \to P(G')$$

then α_g, $\alpha(g)$, $\alpha_{g \circ f}$ and $\alpha(g \circ f)$ are defined analogously and we have

$$\alpha_{g \circ f} = [\Omega(g) \circ \alpha_f] \cdot [\alpha_g \circ (f \times f \times f)].$$

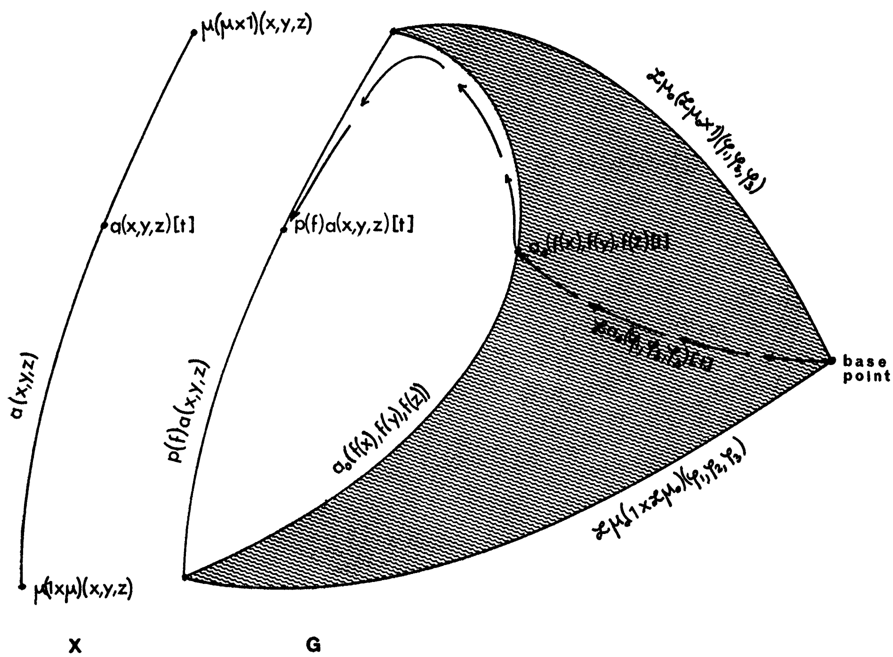

FIGURE 1

Equivalently $\alpha(g \circ f) = \Omega(g)_{\#}\, \alpha(f) + (f \times f \times f)^{\#} \alpha(g)$ where $(\quad)_{\#}$ and $(\quad)^{\#}$ are the mappings induced on homotopy classes.

1.4. PROPOSITION. *With the notations of* 1.1.1 *we have*

$$\alpha_{g \circ f}(x, y, z) \approx [\omega(x, y)] \cdot [\omega(\mu(x, y), z)] \cdot [\omega(x, \mu(y, z))]^{-1} \cdot [\omega(y, z)]^{-1}.$$

Proof. Consider Figure 2. It follows that

$$\alpha_{g \circ f} \approx l\mu\,(\mu \times 1) - \mathcal{L}\mu_0'(\mathcal{L}\mu_0' \times 1)(l \times l \times l) + \mathcal{L}\mu_0'(1 \times \mathcal{L}\mu_0')(l \times l \times l) - l\mu(1 \times \mu).$$

Now

$$\begin{aligned} l\mu(\mu \times 1)(x, y, z) &= \mathcal{L}\mu_0'(l\mu(x, y), l(z)) \cdot \omega(\mu(x, y), z) \\ &= \mathcal{L}\mu_0'(\mathcal{L}\mu_0'(l(x), l(y)) \cdot \omega(x, y), l(z)) \cdot \omega(\mu(x, y), z) \\ &\approx [\mathcal{L}\mu_0'(\mathcal{L}\mu_0' \times 1) \circ l \times l \times l(x, y, z)] \cdot \omega(x, y) \cdot \omega(\mu(x, y), z). \end{aligned}$$

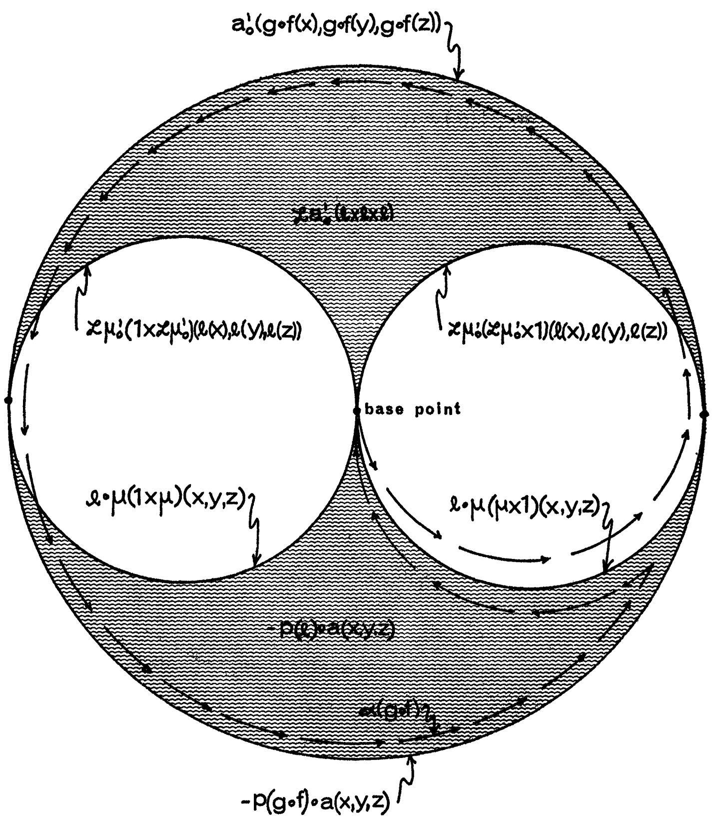

FIGURE 2

Similarly

$l\mu(1 \times \mu)(x, y, z)$

$$\approx \mathcal{L}\mu_0'(1 \times \mathcal{L}\mu_0')(l \times l \times l)(x, y, z)\cdot\omega(y, z)\cdot\omega(x, \mu(y, z)).$$

Hence, $\alpha_{g\circ f} \approx \omega(x, y)\cdot\omega(\mu(x, y), z) - \omega(y, z)\cdot\omega(x, \mu(y, z))$ and 1.4 follows.

1.5. PROPOSITION. *Let (G, μ_0) be a homotopy associative H-space, H- a group. A twisting function $\omega : G \times G, G \vee G \to H, *$ induces a homotopy associative H-structure on $G \times H$ if and only if*

$$\omega(x, \bar{x})\cdot\omega(\mu(x, \bar{x}), \bar{\bar{x}}) \sim \omega(\bar{x}, \bar{\bar{x}})\cdot\omega(x, \mu(\bar{x}, \bar{\bar{x}})).$$

If (G, μ_0), (G', μ_0') are H-spaces, H, H' are groups,

$$g : G \to G' \quad \text{and} \quad h : H \to H'$$

are an H-mapping and a homomorphism respectively,

$$\omega : (G \times G, G \vee G) \to (H, *), \quad \omega' : (G' \times G', G' \vee G') \to (H', *)$$

are twistings, then $g \times h$ is an H-mapping (with respect to the twisted multiplications) if and only if $h \circ w \sim \omega' \circ h \times h$. If $(G \times H, \tilde{\mu}_0, \tilde{a}_0)$ and $(G' \times H', \tilde{\mu}_0', \tilde{a}_0')$ are homotopy associative (twisted) H-spaces, $g \times h$- an H-mapping, then the homotopy class $\alpha[g \times h]$ of

$$\alpha_{g\times h} : (G \times H)^3 \to \Omega(G' \times H')$$

is in

$$\operatorname{im}(P_1 \times P_1 \times P_1)^\# : [G^3, \Omega(G' \times H')] \to [(G \times H)^3, \Omega(G' \times H')].$$

The proof is straightforward.

1.6. PROPOSITION. *Let (X, μ), (G, μ_0) and $k : X \to G$ be H-spaces and H-mapping. Let $x \in H^m(X, Z_p)$, and suppose*

$$\bar{\mu}^*x \in \operatorname{im} k^* \otimes \operatorname{im} k^* \qquad (\bar{\mu}^*x = \mu^*x - 1 \otimes x - x \otimes 1).$$

Let $\tilde{k} : X \to K(Z_p, m)$ be given by $\tilde{k}^*\iota_m = x$. Then $G \times K$ can be given a twisted multiplication $\tilde{\mu}$ so that $(k \times \tilde{k}) \circ \Delta$ is an H-mapping (with respect to μ and $\tilde{\mu}$).

Proof. If $\bar{\mu}^*x = k^* \otimes k^*x_0$, $x_0 \in \bar{H}^*(G, Z_p) \otimes \bar{H}^*(G, Z_p)$, choose

$$\omega : (G \times G, G \vee G) \to (K(Z_p, m), *)$$

to be such that $\omega^*\iota_m = x_0$.

1.7. COROLLARY. *Let (X, μ) be an H-space, let $B \subset H^*(X, Z_p)$ be a sub-Hopf algebra closed under the action of $\mathfrak{A}(p)$ and generated by a finite coalgebra B'.*

There exists $G = \prod_{j\in J} K(Z_p, n_j)$, J-finite, with an H-structure μ_0 and an H-mapping, $f : (X, \mu) \to (G, \mu_0)$ so that $f^*H^*(G, Z_p) = B$. Moreover, if B is co-associative G may be assumed homotopy associative.

Proof. Let $\{x_1, \cdots, x_n\} = B'$. $\dim x_i \leq \dim x_{i+1}$. Suppose G_i, $\mu_0(i)$, f_i are constructed so that

$$G_i = \textstyle\prod_{j \in J_i} K(Z_p, n_j), \qquad f_i : (X, \mu) \to (G_i, \mu_0(i))$$

are H-spaces and H-mappings respectively and im f_i^* is the algebra $B'(i)$ generated by $x_1 \cdots x_i$. If $\bar{f}_{i+1} : X \to K(Z_p, \dim x_{i+1})$ is given by $\bar{f}_{i+1}^* \iota = x_{i+1}$ then by 1.6,

$$G_{i+1} = G_i \times K(Z_p, \dim x_{i+1})$$

can be given an H-structure $\mu_0(i+1)$ so that $f_{i+1} = (f_i \times \bar{f}_{i+1}) \circ \Delta$ is an H-mapping. As G_i can be constructed so that $H^*(G_i, Z_p)$ actually contains a coalgebra isomorphic to $B'(i)$ (the coalgebra generated by $\{x_1, \cdots, x_i\}$), the twisting can be chosen to be homotopy associative provided B' is coassociative.

2. Implications and secondary operations

Consider the following group homomorphism g_0 and its induced fibration:

$$\begin{tikzcd}
 & \Omega K(Z_p, 2mp) = K(Z_p, 2mp-1) \arrow[dl, "h_0"'] \arrow[dr] & \\
E_0 \arrow[rr, "\tilde{g}_0"] \arrow[d, "\tau_0"] & & \mathcal{L} K(Z_1, 2mp) \arrow[d] \\
K(Z_p, 2m) \arrow[rr, "g_0"] & & K(Z_p, 2mp)
\end{tikzcd}$$

$$g_0^* \iota_{2mp} = \mathcal{P}^m \iota_{2m} = \xi \iota_{2m} = \iota_{2m}^{\mathcal{P}}.$$

2.1.1. LEMMA. *There exists a class $v \in H^{2mp}(E_0, Z_p)$ satisfying*

$$h_0^* v = \beta \iota_{2mp-1}$$

$$\bar{\mu}_{E_0}^* v = \sum_{a=1}^{p-1} \frac{1}{p} \binom{p}{a} (\tau_0^* \iota_{2m})^a \otimes (\tau_0^* \iota_{2m})^{p-a}.$$

Proof. Since $g_0^* \beta \iota_{2mp} = \beta \mathcal{P}^m \iota_{2m} = 0$, $\sigma^* \beta \iota_{2mp-1} \in \operatorname{im} h_0^*$. If E_0' is the fiber of

$$K(Z_p, 2m+1) \xrightarrow{g_0'} K(Z_p, 2mp+1)$$

$(g_0'^* \iota_{2mp+1} = \mathcal{P}^m \iota_{2m+1})$ then $\Omega E_0' = E_0$, $\Omega g_0' = g_0$ and as $g_0'^*$ is a mono in $\dim \leq 2mp + 2$,

$$\beta \iota_{2mp-1} \notin \operatorname{im} h_0^* \sigma^*$$

$(\sigma^* : QH^*(E_0', Z_p) \to PH^*(E_0, Z_p))$, since im $\sigma^* = PH^*(E_0, Z_p)$ in dim $2mp$ it follows that $\beta \iota_{2mp-1} \notin h_0^*(PH^*(E_0, Z_p))$. This is possible only if there exists

$\nu \in H^*(E_0, Z_p)$ with

$$\bar{\mu}^*_{E_0} \nu = \sum_{a=1}^{p-1} \frac{1}{p} \binom{p}{a} (\tau_0^* \iota_{2m})^a \otimes (\tau_0^* \iota_{2m})^{p-a} \quad \text{and} \quad h_0^* \nu = \beta \iota_{2mp-1}.$$

Let ϕ be the secondary operation associated with the universal example $\langle E_0, \tau_0^* \iota_{2m}, \nu \rangle$. The main calculation of this section is the following:

2.1. PROPOSITION. *Let (X, μ) be a homotopy associative H-space. Let $B \subset H^*(X, Z_p)$ be a sub-Hopf algebra over $\mathfrak{A}(p)$ generated by a finite coalgebra. $B \subset \ker \xi$. Put*

$$q : H^*(X, Z_p) \to H^*(X, Z_p)/\!\!/B$$

and let $\mathfrak{M}$ be the coproduct in $H^*(X, Z_p)/\!\!/B$ induced by μ.

If $x \in (\ker \xi)^{2m}$, $\bar{\mu}^* x \in B \otimes B$, then

$$\begin{aligned} \mathfrak{M} q\phi(x) &= q \otimes q\bar{\mu}^* \phi(x) \\ &= q \otimes q \sum_{a=1}^{p-1} \frac{1}{p} \binom{p}{a} x^a \otimes x^{p-a} + \beta \ker (\overline{\mathfrak{M}} \otimes 1 - 1 \otimes \overline{\mathfrak{M}}). \end{aligned}$$

As a corollary we have

2.2. COROLLARY. *Let (X, μ), B and x be as in 2.1. If p is odd and $H^*(X, Z_p)/\!\!/B$ is cocommutative then*

$$q^p \bar{\mu}^{*p-1} \phi(x) = q(x) \otimes \cdots \otimes q(x)$$

where $\bar{\mu}^{*k}$ is given by $\bar{\mu}^{*1} = \bar{\mu}^*$, $\bar{\mu}^{*k} = (\bar{\mu}^{*k-1} \otimes 1)\bar{\mu}^*$. Hence, for every $z \in [H^*(X, Z_p)/\!\!/B]^*$, $\langle x, z \rangle = \langle \phi(x), z^p \rangle$.

Proof of 2.2. If $\bar{v} \in \phi(x)$ then by 2.1,

$$q \otimes q\bar{\mu}^* v = q \otimes q \sum_{a=1}^{p-1} \frac{1}{p} \binom{p}{a} x^a \otimes x^{p-a} + \beta z$$

where $z \in \ker (\mathfrak{M} \otimes 1 - 1 \otimes \mathfrak{M})$. Hence, z represents a class in

$$\mathrm{Ext}^{2,2mp-1}_{[H^*(X,Z_p)//B]^*} (Z_p, Z_p)$$

but since $[H^*(X, Z_p)/\!\!/B]^*$ is commutative the class of z can be represented by a class in

$$PH^*(X, Z_p)/\!\!/B \otimes PH^*(X, Z_p)/\!\!/B.$$

Changing $\bar{v}$ if necessary by an element in im β, we can assume that

$$\beta z \in [PH^*(X, Z_p)/\!\!/B] \otimes H^*(X, Z_p)/\!\!/B$$

and as $p > 2$, $(\bar{\mu}^{*p-2} \otimes 1)(\beta z) = 0$ and 2.2 follows.

The remainder of this section is devoted to the proof of 2.1. Following Corollary 1.7 we first construct homotopy associative H-spaces G, μ_0 and G', μ_0'

$$G = \textstyle\prod_j K(Z_p, n_j), \; G' = \prod_j K(Z_p, pn_j)$$

and H-mappings $f : X \to G$ and $g = \prod_j g_j : G \to G'$ so that im $f^* = B$ and

$g_0^* \iota_{pn_j} = \iota_{n_j}^p$. Let $f_0 : X \to K(Z_p, 2m)$ be given by $f_0^* \iota_{2m} = x$. By 1.6,

$$G_0 = G \times K(Z_p, 2m) \quad \text{and} \quad G_0' = G' \times K(Z_p, 2mp)$$

can be given homotopy associative H-structures $\bar{\mu}_0$ and $\bar{\mu}_0'$ so that $(f \times f_0) \circ \Delta$ and $g \times g_0$ are H-mappings. Assume f, g are multiplicative fibrations and let $j : \tilde{X} \to X$ be the inclusion of the fiber of g. We have the following commutative diagram:

$$\begin{array}{ccccc}
\tilde{X} & \xrightarrow{\tilde{f} = f_0 \circ j} & K(Z_p, 2m) = K & \xrightarrow{g_0} & K' = K(Z_p, 2mp) \\
\Big\downarrow j & & \Big\downarrow j_0 & & \Big\downarrow j_0' \\
X & \xrightarrow{(f \times f_0)\Delta = f'} & G_0 = G \times K & \xrightarrow{g \times g_0 = g_0'} & G_0' = G' \times K'.
\end{array}$$

We assume $(f \times f_0)\Delta$, $g \times g_0$, $\tilde{f}$ and g_0 are multiplicative. Note that $\tilde{X}$ is not necessarily homotopy associative. By [7, Theorem 4.5], or by [9, Theorem 4.9],

$$\operatorname{coker} j^* = H^*(X, Z_p) /\!\!/ \operatorname{im} f^* = H^*(X, Z_p) /\!\!/ B.$$

Now, $\alpha(g_0' \circ j_0) = \alpha(j_0' \circ g_0) = 0 = \alpha(j_0)$; hence, by 1.2,

$$(j_0 \times j_0 \times j_0)^{\#} \alpha(g \times g_0) = 0.$$

Since $\Omega g_0' \sim *$ it follows that $\alpha(g_0' \circ f')$ considered as a vector of cohomology classes in $H^*(X \wedge X \wedge X, Z_p)$ is in the ideal generated by

$$(\operatorname{im} f^*)^- \otimes (\operatorname{im} f^*)^- \otimes (\operatorname{im} f^*)^- = \bar{B} \otimes \bar{B} \otimes \bar{B}.$$

Let $\tau : E \to G$ be the fibration induced by g from

$$\Omega G' \to \mathfrak{L} G' \to G'$$

and put $h : \Omega G' \to E$. Consider once again the universal example $(E_0, y = \tau_0^* \iota_{2m}, v)$ for ϕ. $h_0^* v = \beta \iota_{2mp-1}$. We have

$$(g \times g_0) \circ (f \times f_0) \circ \Delta \sim *, \quad g_0 \circ \tilde{f} \sim *, \quad \Omega g \sim *, \quad \Omega g_0 \sim *.$$

As in 1.1 we have H-mappings $\bar{\tilde{f}}$ and $\bar{f}$,

$$\bar{\tilde{f}} : X_2 = \tilde{X} \times K(Z_p, 2mp - 1) \to \mathrm{E}_0,$$

$$\bar{f} : X_1 = X \times \Omega G' \times K(Z_p, 2mp - 1) \to E \times E_0,$$

with respect to some twisted multiplications μ_2 and μ_1 in X_2 and X_1 respectively.

Now,

$$l : (g \times g_0) \circ (f \times f_0) \circ \Delta \sim *$$

and

$$\tilde{l} : g_0 \circ \tilde{f} \sim *$$

can be chosen so that $\tilde{l} = \mathfrak{L}(P_2) \circ l \circ j$ and hence we have a commutative diagram of H-mappings:

$$\begin{tikzcd} X \times K(Z_p, 2mp-1) \arrow[r, "\bar{\bar{f}}"] \arrow[d, "j \times 1"] & E_0 \arrow[d, "i_0"] \\ X_1 = X \times \Omega G' \times K(Z_p, 2mp-1) \arrow[r, "\bar{f}"] & E \times E_0 . \end{tikzcd}$$

By 1.3 and 1.4

$$[(1 \otimes \bar{\mu}_1^*) - (\bar{\mu}_1^* \otimes 1)]\bar{\mu}_1^*(1 \otimes \iota_{2mp-1}) = (P_1^* \otimes P_1^* \otimes P_1^*)(\Omega P_2)_{\#}\, \alpha(g_0' \circ f')$$

(where $P_2 : G' \times K(Z_p, 2mp) \to K(Z_p, 2mp)$ and $P_1 : X_1 \to X$ are the projections) and hence $[(1 \otimes \bar{\mu}_1^*) - (\bar{\mu}_1^* \otimes 1)]\bar{\mu}_1^*(1 \otimes \iota_{2mp-1})$ is in the image of the ideal generated by $B \otimes B \otimes B$. It follows that

$$\bar{\mu}_2^*(1 \otimes \iota_{2mp-1}) \in \ker\, (\bar{\mu}_2^* \otimes 1 - 1 \otimes \bar{\mu}_2^*).$$

Now,

$$\bar{\bar{f}}^*v = (j^* \otimes 1) \circ \bar{f}^*(1 \otimes v) = j^*u \otimes 1 + 1 \otimes \beta\iota_{2mp-1}$$

where $u \in \phi(x)$ and

$$(P_1^* \otimes P_1^*)j^* \otimes j^* \sum_{a=1}^{p-1} \frac{1}{p} \binom{p}{a} x^a \otimes x^{p-a} = \bar{\mu}_2^*\bar{\bar{f}}^*v = (P_1^* \otimes P_1^*)(j^* \otimes j^*)\bar{\mu}_1^*u + \beta\bar{\mu}_2^*(1 \otimes \iota_{2mp-1})$$

and putting $(P_1' \otimes P_1')^*z = \bar{\mu}_2^*(1 \otimes \iota_{2mp-1})$, ($P_1' : X_2 \to X$ — the projection), $z \in \ker\, (\bar{\mu}_2^* \otimes 1 - 1 \otimes \bar{\mu}_2^*)$, by replacing j^* by q and $\bar{\mu}_2^*$ by $\mathfrak{M}$, 2.1 follows.

3. Applications

In this chapter we bring some of the applications of 2.2.

3.1. DEFINITION. Let A be a commutative Hopf algebra. $x \in A$ is called a split generator if the inclusion of the subalgebra A_1 generated by x into A splits as a mapping of algebras. If $v : A \to A_1$ is such a splitting we have coalgebra inclusion $v^* : A_1^* \to A^*$.

Let $u_0, u_1, \cdots u_r \cdots, 0 \leq r < t$ ($t = \infty$ is not excluded), be the elements in A^* with $\langle u_r, a^{p^r} \rangle = 1$. We refer to $v^*u_0 \cdots v^*u_r \cdots$ as to the set of cogenerators associated with v and x.

3.2. THEOREM. *Let (X, μ) be a homotopy associative H-space and p an odd prime. Suppose $H_*(X, Z_p)$ is commutative.*

(a) *Let $x \in H_*(X, Z_p)$ be a split generator, $x^{p^t} = 0$, and let v^*u_0,*

$v^*u_1, \cdots v^*u_{t-1}$ be the set of cogenerators associated with some v and x. If $(v^*u_m)^p = 0$, then $\langle \phi v^*u_m, x^{p^{m+1}}\rangle = 1$, and hence, $(v^*u_{t-1})^p \neq 0$.

(b) *Let $x, v^*u_0 \cdots v^*u_{t-1}$ be as in* (a). *There exists s, $0 \leq s < t$ such that $(v^*u_s)^p$ is a non-zero primitive element of $H^*(X, Z_p)$.*

(c) *If $x \in H_{2n}(X, Z_p)$ is a split generator $x^{p^t} = 0$, then there exists a (split) generator $\bar{x} \in H_{2np^s+1}(X, Z_p)$ for some s, $0 \leq s < t$, $\bar{x}$ is a 1-implication of x^{p^s} in the sense of W. Browder (see [3, p. 357]).*

(d) *If $H^*(X, Z_p)$ is primitively generated then it is a free algebra.*

(e) *If $H_*(X, Z_p)$ is primitively generated then it is a free algebra.*

(f) *If for every $m > 0$ there exists $r_0(m) > 0$ such that for $r \geq r_0(m)$, $PH^{2mp^r}(X, Z_p) = 0 = QH^{2mp^r}(X, Z_p)$ then $H^*(X, Z_p)$ is the exterior algebra on odd-dimensional generators and $H_*(X, Z)$ has no p-torsion.* (*In particular* (f) *holds if $H^*(X, Z_p)$ is finite dimensional.*)

Proof. (a) The coalgebra generated by the v^*u_i $i < m$ satisfies the condition for B, $qu_m \neq 0$, and one can see that for every $z \in H_*(X, Z_p) z^{p^m}$ is annihilated by B; hence $x^{p^m} \in [H^*(X, Z_p)/\!/B]^*$ and

$$1 = \langle v^*u_m, x^{p^m}\rangle = \langle \phi v^*u_m, x^{p^{m+1}}\rangle$$

by 2.2.

(b) If s is the smallest integer such that $(v^*u_s)^p \neq 0$ then $(v^*u_s)^p$ is a primitive.

(c) This is the dualization of (b).

(d) If $u \in PH^*(X, Z_p)$ is even dimensional and $u^p = 0$ then there exists a split generator $x \in H_*(X, Z_p)$ with $\langle u, x\rangle = 1 = \langle \phi u, x^p\rangle$ and $x^p \neq 0$. Hence, if $H^*(X, Z_p)$ is primitively generated all even-dimensional generators have an infinite height.

(e) If $H_*(X, Z_p)$ is primitively generated, ϕ is defined on all $H^{\text{even}}(X, Z_p)$ and by (a) no even-dimensional generator can have a finite height.

(f) If $y \in H_{2m}(X, Z_p)$ is a split generator of finite height, then by (c) there exists a (split) generator in dim $2mp^s$, $s \geq 1$. Since $PH^{2mp^r}(X, Z_p) = 0$, hence $QH_{2mp^r}(X, Z_p) = 0$ for large r, so the existence of even-dimensional generator implies the existence of one of infinite height. If k is the smallest dimension in which a generator of infinite height exists then y^{p^r} is primitive for all sufficiently large t which contradicts the fact that $QH^{2mp^r}(X, Z_p) = 0$ (and hence $PH_{2mp^r}(X, Z_p) = 0$) for r large. Hence $QH_{\text{even}}(X, Z_p) = 0$.

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