A NOTE ON COBORDISM

BY

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1. Introduction

In his paper "Cobordism and Stiefel-Whitney numbers" [6] Stong proves the following result.

THEOREM (Stong). *Let M be a closed differentiable manifold of dimension $5 \cdot 2^s$. Suppose that the Stiefel-Whitney classes $w_1, w_2, w_{2^2}, \cdots, w_{2^s}$ of M are zero. Then whenever $s \geq 4$ the manifold M is cobordant to zero.*

He remarks that easy examples can be constructed to show that the above theorem is false for $s = 0$ and $s = 1$. He also comments that he does not know what the situation is like for $s = 2$ and $s = 3$. In fact the Dold manifold $P(1, 2)$ is an example to show that Stong's result is false for $s = 0$. The manifold $N^{10} = P(1, 2) \times P(1, 2)$ is a manifold with the property that all its Stiefel-Whitney numbers divisible by w_1 and w_2 are zero. Hence by a theorem of Milnor [4] N is cobordant to a manifold M^{10} for which w_1 and w_2 are actually zero. This manifold M serves as an example to show that Stong's result is not valid for $s = 1$.

The object of this paper is to prove the following:

THEOREM. *If M^{40} is a closed differentiable manifold of dimension 40 with $w_1 = w_2 = w_4 = w_8 = 0$ and further satisfying $w_{31} = 0$ then M is cobordant to zero.*

The method of proof lies in a finer analysis of the indeterminacy group occurring in Adams' formula [2]

$$Sq^{16}V = \sum_{0 \leq i \leq j \leq 3, i \neq j-1} a_{i,j}\, \Phi_{i,j}(V) \mod \sum_{0 \leq i \leq j \leq 3, i \neq j-1} a_{i,j}\, Q_{i,j}(X)$$

valid (independent of X) for any $V \in H^n(X; \mathbf{Z}_2)$ satisfying $Sq^1V = Sq^2V = Sq^4V = Sq^8V = 0$. We will state Adams' result more precisely in §2.

Throughout this paper by a manifold we mean a compact differentiable manifold without boundary. The cohomology groups considered are with $\mathbf{Z}_2$-coefficients. We denote the Steenrod Algebra mod 2 by $\mathcal{A}$.

2. Adams' Result

J. F. Adams [2] has defined secondary cohomology operations $\Phi_{i,j}$ for each pair of integers $0 \leq i \leq j$ and $i \neq j - 1$ having the following properties:

(2.1) Let X be a space. The operation $\Phi_{i,j}$ is defined on cohomology classes $u \in H^*(X)$ such that $Sq^{2^r}u = 0$ for $0 \leq r \leq j$.

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If $u \in H^m(X)$ $(m > 0)$ then $\Phi_{i,j}(u)$ is an element in $H^{m+2^i+2^j-1}(X)$ modulo an indeterminacy subgroup $Q_{i,j}(X)$. Moreover if $i < j$,

$$Q_{i,j}(X) = Sq^{2^i} H^{m+2^j-1}(X) + \textstyle\sum_{0 \le l < j} b_l H^{m+2^l-1}(X)$$

where $b_l \in \mathcal{A}$ and $\deg b_l = 2^i + 2^j - 2^l$.

Take an integer $k \geq 3$ and suppose that $u \in H^m(X)$ $(m > 0)$ is a class such that $Sq^{2^r} u = 0$ for $0 \leq r \leq k$. Adams' main result is

(2.2) THEOREM (Adams). *There is a relation*

$$Sq^{2^{k+1}} u = \sum a_{i,j} \Phi_{i,j}(u)$$

valid (independent of X) modulo $\sum a_{i,j} Q_{i,j}(X)$ where the summation is extended over all i, j such that $0 \leq i \leq j \leq k$ and $i \neq j - 1$. Here $a_{i,j}$ denotes a certain element in $\mathcal{A}$ of degree $2^{k+1} - (2^i + 2^j - 1)$.

3. Right action of the Steenrod Algebra on $H^*(M)$

Let M^n be a connected manifold of dimension n. Adams [1] has defined a right action of $\mathcal{A}$ on $H^*(M)$ and this right action has been later exploited by Brown and Peterson [3]. We need the identities obtained by Brown and Peterson relating the usual left action of $\mathcal{A}$ on $H^*(M)$ with the above right action of $\mathcal{A}$. We recall how this right action is defined.

Given any $\alpha \in \mathcal{A}^i$ and any $x \in H^k(M)$ define $(x)\alpha \in H^{k+i}(M)$ by the property

$$(x)\alpha \cup y = x \cup \alpha(y), \quad \forall y \in H^{n-k-i}(M).$$

Because of Poincare duality $(x)\alpha$ is well-defined.

Let $c : \mathcal{A} \to \mathcal{A}$ denote the canonical conjugation and $\nabla : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ denote the usual diagonal map in $\mathcal{A}$. (Refer, Chap. II of [5]). For any $\alpha \in \mathcal{A}^j$ with $j \geq 1$ we can write $\nabla(\alpha)$ as $\alpha \otimes 1 + 1 \otimes \alpha + \sum \alpha_i' \otimes \alpha_i''$ for some α_i' and α_i'' in $\mathcal{A}$ with $\deg \alpha_i' > 0$, $\deg \alpha_i'' > 0$. For the right action of $\mathcal{A}$ on $H^*(M)$ we have the following identities:

(3.1) $$(x)\, Sq^0 = x, \quad \forall x \in H^k(M)$$

(3.2) $$(1)\, c(Sq^i) = \bar{w}_i(M)$$

where $\bar{w}_i(M)$ is the i-th dual Stiefel-Whitney class of M.

(3.3) For any $x \in H^k(M)$ and $y \in H^l(M)$ with $k \geq 0$, $l \geq 0$ arbitrary and for any $\alpha \in \mathcal{A}^j$ with $j \geq 1$ we have

$$(x \cup y)\alpha = (x)\alpha \cup y + x \cup c(\alpha)(y) + \textstyle\sum_i (x)\alpha_i' \cup c(\alpha_i'')(y)$$

where $\nabla(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha + \sum \alpha_i' \otimes \alpha_i''$ with $\deg \alpha_i' > 0$; $\deg \alpha_i'' > 0$.

The formulae (3.2) and (3.3) are due to Brown and Peterson. Actually

(3.3) is the analogue for the right action of $\mathcal{A}$ on $H^*(M)$ of the well-known Cartan-formula for the left action of $\mathcal{A}$ on $H^*(M)$, namely

$$(3.4) \qquad \alpha(x \cup y) = \alpha(x) \cup y + x \cup \alpha(y) + \textstyle\sum_i \alpha_i'(x) \cup \alpha_i''(y)$$

The identity of Wu which states that $Sq^i x = v_i \cup x$ for any $x \in H^{n-i}(M)$ where v_i is the i-th Wu class of M can be stated in terms of right action as

$$(3.5) \qquad (1)\, Sq^i = v_i .$$

4. Manifolds M^{40}

Throughout the rest of this paper $M = M^{40}$ denotes a 40-dimensional connected manifold such that $w_1(M) = w_2(M) = w_4(M) = w_8(M) = 0$. We denote the j-th Stiefel-Whitney class of M by w_j; the j-th dual Stiefel-Whitney class of M by $\bar{w}_j$ and the j-th Wu class of M by v_j.

LEMMA 4.1. *The only possible non-zero Stiefel-Whitney classes in positive dimensions of M are* w_{16}, w_{24}, w_{28}, w_{30}, w_{31} *and* w_{32}.

Proof. Immediate from Propositions 2 to 4 and Theorem 2 of [6].

Thus the only *possible non-zero* Stiefel-Whitney number is $w_{16}\cdot w_{24}[M]$. Sections 4 and 5 of this paper mainly analyse this Stiefel-Whitney number.

LEMMA 4.2. *For an M of the above type we have*

$$Sq^8 w_{16} = w_{24}, \qquad Sq^{12} w_{16} = Sq^4 w_{24} = w_{28},$$

$$Sq^{14} w_{16} = Sq^6 w_{24} = Sq^2 w_{28} = w_{30},$$

$$Sq^{15} w_{16} = Sq^7 w_{24} = Sq^3 w_{28} = Sq^1 w_{30} = w_{31},$$

$$Sq^8 w_{24} = Sq^4 w_{28} = Sq^2 w_{30} = Sq^1 w_{31} = 0.$$

Proof. Immediate from the Wu formula

$$(4.3) \qquad Sq^i w_j = \sum_{t=0}^{i} \binom{j-i+t-1}{t} w_{i-t}\cdot w_{j+t} \quad \text{for} \quad i < j.$$

LEMMA 4.4. *The dual Stiefel-Whitney classes of M are given by*

$$\bar{w}_i = 0$$

for

$1 \leq i \leq 15$; $\quad 17 \leq i \leq 23$; $\quad 25 \leq i \leq 27$; $\quad i = 29$ *and* $i \geq 33$.

$$\bar{w}_{16} = w_{16};\ \bar{w}_{24} = w_{24};\ \bar{w}_{28} = w_{28};\ \bar{w}_{30} = w_{30};\ \bar{w}_{31} = w_{31}$$

and

$$\bar{w}_{32} = w_{32} + w_{16} \cup w_{16}.$$

Proof. Immediate from the Whitney duality formula.

LEMMA 4.5. *For any $\alpha \in \mathcal{A}^i$ with $i \leq 15$ we have*

$$\alpha(w_j) = \lambda_{\alpha,j} w_{j+i}$$

for some $\lambda_{\alpha,j} \in \mathbf{Z}_2$.

Proof. Immediate consequence of (repeated application of) the Wu formula (4.3) and the fact that $w_\mu = 0$ for $1 \leq \mu \leq 15$.

LEMMA 4.6. *For any $\alpha \in \mathcal{A}^i$ with $1 \leq i \leq 15$ we have*

$$(1)\alpha = 0$$

where $1 \in H^0(M)$ is the unit element.

Proof. It suffices to show that

$$(1)Sq^{d_1} \cdots Sq^{d_r} = 0$$

whenever $d_1 + \cdots + d_r \leq 15$ and $d_1 \geq 1$. But

$$(1)Sq^{d_1} \cdots Sq^{d_r} = (v_{d_1})Sq^{d_2} \cdots Sq^{d_r}$$

by (3.5). Since $w_\mu = 0$ for $1 \leq \mu \leq 15$ we get from the inductive formula

$$w_\mu = v_\mu + Sq^1 v_{\mu-1} + \cdots + Sq^{[\mu/2]} v_{\mu-[\mu/2]}$$

the relation $v_\mu = 0$ for $1 \leq \mu \leq 15$. Hence $(1)Sq^{d_1} \cdots Sq^{d_2} = 0$ whenever $1 \leq d_1$ and $d_1 + \cdots + d_r \leq 15$.

LEMMA 4.7. *For any $\alpha \in \mathcal{A}^i$ with $i \leq 15$ we have*

$$(w_j)\alpha = \mu_{\alpha,j} w_{j+i}$$

for some $\mu_{\alpha,j} \in \mathbf{Z}_2$.

Proof. If $\deg \alpha = 0$ there is nothing to prove since $(w_j)Sq^0 = w_j$. Where $1 \leq \deg \alpha \leq 15$ an application of formula (3.3) together with Lemma 4.6 yields

$$(w_j)\alpha = (1 \cup w_j)\alpha = c(\alpha)(w_j).$$

But $c(\alpha)(w_j) = \lambda_{c(\alpha),j} w_{j+i}$ by Lemma 4.5. Hence $\mu_{\alpha,j} = \lambda_{c(\alpha),j}$ satisfies the requirements of Lemma 4.7.

COROLLARY 4.8.

$$(w_{24})\mathcal{A}^i = 0 \quad \textit{for } i \neq 0, 4, 6, 7, 8$$

and

$$(w_{31})\mathcal{A}^i = 0 \quad \textit{for } i \neq 0, 1.$$

Proof. Immediate consequence of Lemma 4.7 and Lemma 4.1.

LEMMA 4.9. $(w_{24})\mathcal{A}^8 = 0$.

Proof. The elements Sq^8; Sq^7Sq^1; Sq^6Sq^2; $Sq^5Sq^2Sq^1$ form a basis for $\mathcal{A}^8$ over $\mathbf{Z}_2$. Hence it suffices to verify that $(w_{24})\alpha = 0$ when α is one of the above four basis elements.

By Corollary (4.8) we have $(w_{24})Sq^5 = 0$ and hence $(w_{24}Sq^5Sq^2Sq^1 = 0)$.
Take any $x \in H^8(M)$. We have

$$Sq^8(w_{24} \cup x) = v_8 \cup (w_{24} \cup x) = 0.$$

Cartan's formula and Lemmas 4.1 and 4.2 give

(i) $$w_{24} \cup Sq^8x + w_{28} \cup Sq^4x + w_{30} \cup Sq^2x + w_{31} \cup Sq^1x = 0.$$

Similarly we have $Sq^4(w_{28} \cup x) = v_4 \cup (w_{28} \cup x) = 0$ yielding

(ii) $$w_{28} \cup Sq^4x + w_{30} \cup Sq^2x + w_{31} \cup Sq^1x = 0.$$

Adding (i) and (ii) we get $w_{24} \cup Sq^8x = 0$. Thus $(w_{24})Sq^8 \cup x = 0$ and this $\forall\, x \in H^8(M)$. Poincare duality for M^{40} now yields

$$(w_{24})Sq^8 = 0.$$

Also we have $Sq^7(w_{24} \cup Sq^1x) = v_7 \cup w_{24} \cup Sq^1x = 0$ yielding

(iii) $$w_{24} \cup Sq^7Sq^1x + w_{28} \cup Sq^3Sq^1x + w_{31} \cup Sq^1x = 0.$$

Similarly $Sq^3(w_{28} \cup Sq^1x) = 0$ yields

(iv) $$w_{28} \cup Sq^3Sq^1x + w_{31} \cup Sq^1x = 0.$$

Adding (iii) and (iv) we get $w_{24} \cup Sq^7Sq^1x = 0$. This means $(w_{24})Sq^7Sq^1 = 0$.
The proof for $(w_{24})Sq^6Sq^2 = 0$ is similar and hence omitted.

5. Thom class of the normal bundle of M^{40} in S^{40+d}

We want to bring into force the relationship between Stiefel-Whitney classes and the Steenrod squares via the Thom isomorphism. For this purpose we imbed M^{40} differentiably in S^{40+d} for some d. Let $\nu = \nu^d$ denote the normal bundle of M^{40} in S^{40+d}. Let E denote a closed tubular neighborhood of M in S^{40+d} with $\dot{E}$ as the boundary. E can be identified with the total space of the disk bundle associated to ν.

Let $p : E \to M$ denote the projection and $\Phi : H^i(M) \to H^{i+d}(E, \dot{E})$ the Thom isomorphism. Then $H^d(E, \dot{E}) \simeq \mathbf{Z}_2$ with $\Phi(1) = U$ as the generator and

$$\Phi : H^i(M) \to H^{i+d}(E, \dot{E})$$

is given by $\Phi(x) = p^*(x) \cup U$ where $p^* : H^*(M) \to H^*(E)$ is induced by the homotopy equivalence $p : E \to M$. As is well known, $\bar{w}_i = \Phi^{-1}(Sq^iU)$. Let $T(\nu)$ denote the Thom space of ν and $\eta : (E, \dot{E}) \to (T(\nu), \infty)$ the canonical projection. If $k : T(\nu) \to (T(\nu), \infty)$ denotes the inclusion, the composite isomorphism

$$H^i(M) \xrightarrow{\Phi} H^{i+d}(E, \dot{E}) \xrightarrow{(\eta^*)^{-1}} H^{i+d}(T(\nu), \infty) \xrightarrow{k^*} H^{i+d}(T(\nu))$$

will be denoted by Ψ. Sometimes we will refer to Ψ also as the Thom-isomorphism

Let us denote the class $\Psi(1) \in H^d(T(\nu))$ by V. Then $\Psi^{-1}(Sq^i V) = \bar{w}_i$. From Lemma 4.4 we have $\bar{w}_i = 0$ for $1 \leq i \leq 15$. It follows that $Sq^i V = 0$ for $1 \leq i \leq 15$. Thus we are in a position to apply Adams' result to V. We get

$$(5.1) \quad Sq^{16} V = \sum\nolimits_{0 \leq i \leq j \leq 3, 0 \neq j-1} a_{i,j}\, \Phi_{i,j}(V) \bmod \sum\nolimits_{0 \leq i \leq j \leq 3, i \neq j-1} a_{i,j}\, Q_{i,j}(T(\nu))$$

where $Q_{i,j}$ is the subgroup of indeterminacy corresponding to the secondary operation $\Phi_{i,j}$.

Set $P_{i,j} = \eta^* k^{*^{-1}}(Q_{i,j}(T(\nu)))$ and $\theta_{i,j} = \eta^* k^{*^{-1}} \Phi_{i,j}(V)$. Then equation (5.1) yields

$$(5.2) \qquad Sq^{16} U = \sum\nolimits_{0 \leq i \leq j \leq 3, i \neq j-1} a_{i,j}\, \theta_{i,j} \bmod \sum\nolimits_{0 \leq i \leq j \leq 3, i \neq j-1} a_{i,j}\, P_{i,j}.$$

We are interested in the Stiefel-Whitney number $w_{24} \cdot w_{16}[M]$. From Lemma 4.4 we have $\bar{w}_{16} = w_{16}$ and $\bar{w}_{24} = w_{24}$. We have

$$Sq^{16} U = p^*(\bar{w}_{16}) \cup U = p^*(w_{16}) \cup U.$$

Hence

$$p^*(w_{24}) \cup Sq^{16} U = p^*(w_{24}) \cup p^*(w_{16}) \cup U = \Phi(w_{24} \cdot w_{16}).$$

Since Φ is an isomorphism it follows that $w_{24} \cdot w_{16}$ is zero whenever $p^*(w_{24}) \cup Sq^{16} U$ is zero. Motivated by this we analyse the class $p^*(w_{24}) \cup Sq^{16} U$ further.

PROPOSITION 5.3. *We have* $p^*(w_{24}) \cup Sq^{16} U = \lambda p^*(w_{31}) \cup \theta_{1,3}$ *for some* $\lambda \in \mathbf{Z}_2$.

For the proof of this proposition we need the following

LEMMA 5.4. *For any* $x \in H^i(M)$ *and* $\alpha \in \mathcal{A}^j$ *with* $1 \leq j \leq 15$ *we have*

$$\alpha\{p^*(x) \cup U\} = p^*(\alpha(x)) \cup U.$$

Proof. Let $\nabla(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha + \sum \alpha_i' \otimes \alpha_i''$ with $\deg \alpha_i' > 0$, $\deg \alpha_i'' > 0$. Then by Cartan's formula we have

$$\alpha\{p^*(x) \cup U\} = \alpha(p^*(x)) \cup U + p^*(x) \cup \alpha(U) + \sum \alpha_i'(p^*(x)) \cup \alpha_i''(U).$$

But since $Sq^j U = 0$ for $1 \leq j \leq 15$ it follows that $\beta(U) = 0$ for any $\beta \in \mathcal{A}^j$ with $1 \leq j \leq 15$. Hence

$$\alpha\{p^*(x) \cup U\} = \alpha(p^*(x)) \cup U = p^*(\alpha(x)) \cup U.$$

Proof of Proposition 5.3. From (5.2) we get

$$(5.5) \quad \begin{aligned} &p^*(w_{24}) \cup Sq^{16} U \\ &\quad = \sum\nolimits_{0 \leq j \leq j \leq 3, i \neq j-1} p^*(w_{24}) \cup a_{i,j}\theta_{i,j} \quad \bmod \mathrm{p}^*(w_{24}) \cup \sum a_{i,j} P_{i,j}. \end{aligned}$$

To prove Proposition (5.3) we have only to prove the following three statements:

(a) Each of the groups $p^*(w_{24}) \cup a_{i,j} P_{i,j}$ is zero.
(b) For $(i, j) \neq (1, 3)$ the element $p^*(w_{24}) \cup a_{i,j}\theta_{i,j}$ is zero.
(c) $p^*(w_{24}) \cup a_{1,3}\theta_{1,3} = \lambda p^*(w_{31}) \cup \theta_{1,3}$ for some $\lambda \in \mathbf{Z}_2$.

Denote the groups $H^{d+2^i+2^j-1}(E, \dot{E})$ and $H^{2^i+2^j-1}(M)$ by $L_{i,j}$ and $B_{i,j}$ respectively. Since $\theta_{i,j} \in L_{i,j}$ to prove (a) and (b) it suffices to prove statements (a′) and (b′) mentioned below:

(a′) For $(i, j) \neq (1, 3)$ the group $p^*(w_{24}) \cup a_{i,j} L_{i,j}$ is the zero subgroup of $H^{40+d}(E, \dot{E})$.
(b′) $p^*(w_{24}) \cup a_{1,3} P_{1,3} = 0$.

First consider $p^*(w_{24}) \cup a_{i,j} L_{i,j}$ with $(i, j) \neq (1, 3)$. Let $e_{i,j} \in L_{i,j}$ be an arbitrary element. We can write $e_{i,j}$ as $p^*(x_{i,j}) \cup U$ for some $x_{i,j} \in B_{i,j}$. Hence

$$\begin{aligned} p^*(w_{24}) \cup a_{i,j} e_{i,j} &= p^*(w_{24}) \cup a_{i,j}\{p^*(x_{i,j}) \cup U\} \\ &= p^*(w_{24}) \cup p^*\{a_{i,j}(x_{i,j})\} \cup U \end{aligned}$$

by Lemma 5.4, because $\deg a_{i,j} = 16 - (2^i + 2^j - 1)$ and for $0 \leq i \leq j \leq 3$ we have $1 \leq \deg a_{i,j} \leq 15$. Thus

$$p^*(w_{24}) \cup a_{i,j} e_{i,j} = p^*(w_{24} \cup a_{i,j}(x_{i,j})) \cup U.$$

Now $\deg a_{i,j} + \deg x_{i,j} = 16$ and since M is of dimension 40 by the definition of right action of $\mathfrak{A}$ on $H^*(M)$ we have

$$w_{24} \cup a_{i,j}(x_{i,j}) = (w_{24})a_{i,j} \cup x_{i,j}.$$

The $a_{i,j}$'s occurring in the sum (5.5) with $(i, j) \neq (1, 3)$ are $a_{0,0}$; $a_{0,2}$; $a_{0,3}$; $a_{1,1}$; $a_{2,2}$ and $a_{3,3}$ and their respective degrees are 15, 12, 8, 13, 9 and 1. By Corollary 4.8 and Lemma 4.9 we have $(w_{24})\mathfrak{A}^{\mu} = 0$ for $\mu = 15, 12, 8, 13, 9$ and 1. Hence $(w_{24})a_{i,j} \cup x_{i,j} = 0$ and it follows that $p^*(w_{24}) \cup a_{i,j} e_{i,j} = 0$ for every $e_{i,j} \in L_{i,j}$ with $(i, j) \neq (1, 3)$. This proves statement (a′).

As for statement (b′) we have

$$\begin{aligned} P_{1,3} &= \eta^* k^{*^{-1}} Q_{1,3}(T(\nu)) \\ &= \eta^* k^{*^{-1}}\{Sq^2 H^{d+7}(T(\nu)) + \textstyle\sum_{0 \leq l < 3} b_l H^{d+2^l-1}(T(\nu))\} \end{aligned}$$

by (2.2).

Setting $b_3 = Sq^2$ we have

$$P_{1,3} = \eta^* k^{*^{-1}}\{\textstyle\sum_{0 \leq l \leq 3} b_l H^{d+2^l-1}(T(\nu))\}$$

with $b_l \in \mathfrak{A}$ of deg $10 - 2^l$. However

$$\eta^* k^{*^{-1}} b_l H^{d+2^l-1}(T(\nu)) = b_l H^{d+2^l-1}(E, \dot{E})$$

and

$$p^*(w_{24}) \cup a_{1,3}P_{1,3} = p^*(w_{24}) \cup a_{1,3}(\textstyle\sum_{0\le l\le 3} b_l H^{d+2^l-1}(E, \dot{E}))$$
$$= \textstyle\sum_{0\le l\le 3} p^*(w_{24}) \cup a_{1,3} b_l H^{d+2^l-1}(E, \dot{E}).$$

If $e_l \in H^{d+2^l-1}(E, \dot{E})$ is an arbitrary element we can write it as $p^*(x_l) \cup U$ with $x_l \in H^{2^l-1}(M)$. Then we have

$$p^*(w_{24}) \cup a_{1,3} b_l e_l = p^*(w_{24}) \cup a_{1,3} b_l\{p^*(x_l) \cup U\}.$$

Since $1 \le \deg b_l \le 15$ and $\deg a_{1,3} = 7$ applying Lemma 5.4 twice we have

$$p^*(w_{24}) \cup a_{1,3} b_l e_l = p^*(w_{24} \cup a_{1,3} b_l(x_l)) \cup U.$$

By Lemma 4.7 we have $(w_{24})a_{1,3} = \lambda w_{31}$ for some $\lambda \in \mathbf{Z}_2$. Hence

$$(w_{24})a_{1,3} b_l = (\lambda w_{31})b_l = \lambda(w_{31})b_l.$$

Since $\deg b_l = 10 - 2^l \ne 0$ and 1 we see from Corollary 4.8 that $(w_{24})a_{1,3} b_l = 0$. Hence

$$p^*(w_{24}) \cup a_{1,3} b_l e_l = p^*((w_{24})a_{1,3} b_l \cup x_l) \cup U = 0.$$

This completes the proof of statement (b′).

As for statement (c) we can write $\theta_{1,3}$ as $p^*(x_{1,3}) \cup U$ for some $x_{1,3} \in B_{1,3}$. Then as before

$$\begin{aligned} p^*(w_{24}) \cup a_{1,3}\theta_{1,3} &= p^*(w_{24} \cup a_{1,3}(x_{1,3})) \cup U \\ &= p^*((w_{24})a_{1,3} \cup x_{1,3}) \cup U \\ &= p^*(\lambda w_{31} \cup x_{1,3}) \cup U \\ &= \lambda p^*(w_{31}) \cup p^*(x_{1,3}) \cup U \\ &= \lambda p^*(w_{31}) \cup \theta_{1,3}. \end{aligned}$$

This completes the proof of Proposition 5.3.

6. The main theorem

As remarked earlier the main result proved here is

THEOREM 6.1. *If M^{40} is a 40-dimensional closed differentiable manifold with $w_1 = w_2 = w_4 = w_8 = w_{31} = 0$ then M is cobordant to zero.*

Proof. Follows immediately from Lemma 4.1 and Proposition 5.3.

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