ON RELATORS AND DIAGRAMS FOR GROUPS WITH ONE DEFINING RELATION

BY

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Introduction

For a group given by generators and defining relators, Van Kampen [3] described a suggestive representation for each relator (see [2, page 7], for definitions). A connected and simply connected plane complex was used, with a generator assigned to each oriented 1-cell, so that defining relators (or their inverses) corresponded to boundaries of 2-cells and the relator corresponded to the boundary of the complex.

The plane configuration which will serve to represent relators in arbitrary presented groups is slightly more general. It is a finite, connected planar graph, together with an embedding of the graph in the Euclidean plane. These graphs were used implicitly by the author in [4] to give a new proof of the solution of the word problem for sixth groups. More extensive results on the word problem were obtained by Lyndon in [1] with the aid of these planar graphs. In Lyndon's terminology, these planar graphs are maps; when generators are assigned to their oriented edges in a suitable manner, they are referred to as diagrams.

In such a map, a face may have a boundary containing fewer vertices than edges. If there is such a face in a diagram for a relator in some presented group, then that face corresponds to a defining relator (usually assumed to be a cyclically reduced word) and some proper subword of that defining relator is a relator.

The known results in sixth groups [4, page 558] imply that no proper subword of any defining relator is a relator. As usual this statement refers to a particular presentation for the group and each defining relator is a cyclically reduced word. Our main result is that the same conclusion holds for each group with one defining relator. For the proof we find it convenient to replace a planar graph by an abstract structure, called a surface. This leads to abstract versions of maps and diagrams. The proofs of a key preliminary result (Theorem 1) and of the main result (Theorem 2) are based on a scheme used by Magnus to prove the Freiheitssatz (see [2]). But the basic tool in these proofs is a diagram. We close with a diagram-theoretic modification of the Magnus proof of the Freiheitssatz.

1. Surfaces and spheres

A *surface* is determined by a finite, non-empty set S, with an even number $2e$ of elements, and by two permutations f, g on S such that g is a product of e

Received February 2, 1970.

disjoint transpositions. We think of the surface as an ordered triple but we write it simply as Sfg. Let fg denote the composition of f and g with f applied first. The *oriented edges, vertices, edges,* and *faces* are the elements of S and the orbits of fg, g, f, respectively. The numbers of vertices, edges, and faces are denoted by $|fg|$, $|g|$, $|f|$, respectively. We write $f \doteq f_1 \cdots f_n$ if $f_1, \cdots, f_n$ are the restrictions of f to the orbits of f. Here $n = |f|$.

The edges and vertices which *belong* to a face F are just the ones which have a non-empty intersection with F. Two faces are *vertex disjoint* (or *edge disjoint*) if no vertex (or no edge) belongs to both of them.

Remark 1. If two faces F, G are vertex disjoint, then they are edge disjoint

Proof. If an edge E belongs to F and to G, then we must have $E = \{x, y\}$ with x in F, y in G. Then xf^{-1} and y are in the same vertex V because $(xf^{-1})fg = xg = y$. So V belongs to F and to G, a contradiction.

A face F consisting of $n \geq 1$ oriented edges is said to be *simple* if n distinct vertices belong to F.

Remark 2. No simple face F, consisting of $n \geq 3$ oriented edges, contains an edge.

Proof. If an edge $\{x, y\}$ is contained in such an F, then $xg = y$ and two cases arise. If $xf = y$, then xf^{-1} and y are distinct and (since $(xf^{-1})(fg) = y$) are in the same vertex; if $xf \neq y$, then yf^{-1} and x are distinct and are in the same vertex. Either way, at most $n - 1$ distinct vertices can belong to F and so F is not simple, a contradiction.

A surface Sfg is the *union* of surfaces $S_if_ig_i$, $1 \leq i \leq m$, if S is the union of the S_i and if, for each i, f_i and g_i are the restrictions of f and g, respectively, to S_i. This is a disjoint union of surfaces if the S_i are pairwise disjoint. The surface is *connected* if no proper subset of S is closed under f and g. It is then clear that each surface is uniquely a disjoint union of a finite number of connected surfaces. Finally, a *sphere* is a connected surface Sfg with Euler characteristic $|fg| - |g| + |f|$ equal to 2.

LEMMA 1. *Let Sfg be a sphere with $|f| \geq 2$. Let a, b be distinct oriented edges belonging to different faces. Assume a, b are in the same vertex. Suppose*

$$f \doteq (a_1 \cdots a_r)(b_1 \cdots b_s)v \quad \textit{and} \quad fg \doteq (c_1 \cdots c_t d_1 \cdots d_u)w$$

for some permutations v, w and some oriented edges a_i, b_i, c_i, d_i where $a = a_r = c_t$, $b = b_s = d_u$. Then Spg is a sphere where $p = (a_1 \cdots a_r b_1 \cdots b_s)v$.

Proof. Observe that

$$p = (a_1 \cdots a_r b_1 \cdots b_s)(b_s \cdots b_1)(a_r \cdots a_1)f = (a_r b_s)f = (c_t d_u)f.$$

So $pg = (c_t d_u)(c_1 \cdots c_t d_1 \cdots d_u)w$ and thus

$$pg \doteq (c_1 \cdots c_t)(d_1 \cdots d_u)w.$$

Therefore Spg has Euler characteristic 2. It is connected because any subset T (of S) which is closed under p and g must contain all or none of the a_i, b_i; so T is closed under f and g, and $T = S$ or T is empty.

LEMMA 2. *Let $F = \{a, b\}$ be a simple face of a sphere Sfg with $a \neq b$ and $|f| \geq 2$. Suppose $f \doteq (ab)p$ for some permutation p. Then $g \doteq (ac)(bd)v$ for some permutation v and for some oriented edges c, d such that a, b, c, d are distinct. Furthermore, Tpq is a sphere where $T = S - F$ and $q \doteq (cd)v$.*

Proof. Let $c = ag$ and $d = bg$. Then $c \neq a$, $d \neq b$ because g has no fixed points; $c \neq d$ because $a \neq b$ and g is one-to-one. Also $d \neq a$ since $d = a$ implies $\{a, b\}$ is closed under f and g so that $S = \{a, b\}$, contrary to $|f| \geq 2$. Finally, $c \neq b$ since $c = b$ implies $d = bg = cg = a$, which again yields a contradiction. Thus a, b, c, d are distinct and we have $g \doteq (ac)(bd)v$ for some permutation v.

Since F is simple, it follows that a and b are in different vertices and $fg \doteq (ad_1 \cdots d_s)(bc_1 \cdots c_r)w$ for some permutation w and some oriented edges d_i, c_i. We note that $d_1 = afg = bg = d$ and $c_1 = bfg = ag = c$.

We form a surface Sfk where $k = (ab)(cd)v$. Since v and $(ac)(bd)$ commute, we have $k = (ac)(bd)v(ac)(bd)(ab)(cd) = g(ad)(bc)$. So

$$fk = fg(ad)(bc) = w(ad_1 \cdots d_s)(bc_1 \cdots c_r)(ad)(bc).$$

Then $fk \doteq w(a)(b)(d_1 \cdots d_s)(c_1 \cdots c_r)$.

Therefore Sfk has Euler characteristic 4. It is a disjoint union of Tpq and a sphere $\langle\{a, b\}, (ab), (ab)\rangle$. The latter expression is an ordered triple consisting of a set followed by two permutations on that set. It follows that Tpq has Euler characteristic 2.

To see that Tpq is connected, we consider any subset U (of T) which is closed under p amd q. U must contain either both c and d or neither c nor d. Hence either $U \cup \{a, b\}$ or U is closed under f and g. It follows that either $U \cup \{a, b\} = S$ or U is empty i.e. either $U = T$ or U is empty. Thus Tpq is a sphere and we are done.

LEMMA 3. *Let $F = \{a, b, e_1, \cdots, e_t\}$ be a simple face of a sphere Sfg where $a, b, e_1, \cdots, e_t$ are distinct oriented edges, $|f| \geq 2$, and $t \geq 1$. Suppose $f \doteq (abe_1 \cdots e_t)p$ for some permutation p. Then $g \doteq (ac)(bd)v$ for some permutation v and for some oriented edges c, d such that $a, b, c, d, e_1, \cdots, e_t$ are distinct. Furthermore, Skg is a sphere where $k \doteq (ab)(e_1 \cdots e_t)p$ and one of its faces $\{a, b\}$ is simple.*

Proof. Let $c = ag$ and $d = bg$. Since $\{a, c\}$ and $\{b, d\}$ are edges, neither of them is contained in F, by Remark 2. Hence each of the oriented edges c, d is different from each oriented edge in F. Finally, $c \neq d$ because $a \neq b$ and g is one-to-one. Thus $a, b, c, d, e_1, \cdots, e_t$ are distinct and we have $g \doteq (ac)(bd)v$ for some permutation v.

Since F is simple, we find e_t and b in different vertices. But e_t and c are in the same vertex because $e_t fg = ag = c$. So b and c are in different vertices. Thus $fg \doteq (a_1 \cdots a_r b)(cd_1 \cdots d_s)w$ for some permutation w and some oriented edges a_i, d_i where $r, s \geq 1$. We note that $d_s fg = c$. So $d_s f = cg = a$. Hence $d_s = e_t$. Also $a_1 = bfg = e_1 g$. Then

$$k = (ab)(e_1 \cdots e_t)(e_t \cdots e_1 ba)(abe_1 \cdots e_t)p = (be_t)f.$$

So $kg = (be_t)fg = (be_t)(a_1 \cdots a_r b)(cd_1 \cdots d_s)w$. Thus

$$kg \doteq (a_1 \cdots a_r bcd_1 \cdots d_s)w.$$

Therefore Skg has Euler characteristic 2. To see that it is connected, we consider any subset U (of S) which is closed unker k and g. U must contain either all or none of the e_i. Hence either $U \cup \{a, b\}$ or U is closed under f and g. Then either U is empty or $U = S$. Thus Skg is indeed a sphere.

To see that $\{a, b\}$ is a simple face of Skg, we observe that e_t, a, b are in distinct orbits of fg because F is a simple face. Thus a, b, c are in distinct orbits of fg because $e_t fg = c$. Therefore a is not equal to any a_i or d_i. Then a, b are in distinct orbits of kg since $kg \doteq (a_1 \cdots a_r bcd_1 \cdots d_s)w$.

2. Verbal surfaces

Let a, b, c, d be elements (of any sort). We call $abcd$ an *array*. Similarly, $a_1 \cdots a_m$ denotes an array of $m \geq 1$ elements. We need not require that $a_1, \cdots, a_m$ be distinct. If A, B are, respectively, the arrays $a_1 \cdots a_m$ and $b_1 \cdots b_n$, then AB is defined to be the array $a_1 \cdots a_m b_1 \cdots b_n$.

If A, B are arrays, then (A, B) denotes the set of all arrays C such that either $C = AB$, or $C = BA$, or $C = XBY$ and $A = XY$ for some arrays X, Y. We think of the latter case as inserting B into A. This can arise only if $A = a_1 \cdots a_m$ where $m \geq 2$. For arrays $A_1, \cdots, A_n$, $n \geq 1$, we define the set $(A_1, \cdots, A_n)$ inductively. (A_1) has one element A_1. $(A_1, \cdots, A_n)$ is the set of all arrays V such that V is in (U, A_n) for some U in $(A_1, \cdots, A_{n-1})$.

Now let $p = (c_1 \cdots c_r)$ be a cyclic permutation on the set consisting of $r \geq 1$ distinct elements $c_1, \cdots, c_r$. We say the array $c_1 \cdots c_r$ *represents* p. If we denote this array by C, then exactly r arrays (C and its cyclic permutations) can represent p.

Let Sfg be a surface with $f \doteq f_1 \cdots f_n$ and $g \doteq g_1 \cdots g_e$ for some cyclic permutations f_i, g_j. We call Sfg *verbal* if

$$(A_1, \cdots, A_n) \cap (I_1, \cdots, I_e)$$

is not empty for some arrays A_i, I_j representing f_i, g_j, respectively.

Remark 3. If W is in $(A_1, \cdots, A_r, X)$ and if X is in (Y, Z) where W, X, Y, Z, and the A_i are arrays, then W is in

$$(A_1, \cdots, A_r, Y, Z).$$

Remark 4. If C is a cyclic permutation of AB where A, B, C are arrays, then C is in (X, Y) where either $Y = A$ and X is a cyclic permutation of B or $Y = B$ and X is a cyclic permutation of A.

Remark 5. If W is in $(A_1, \cdots, A_r, C)$ and C is a cyclic permutation of AB, where W, A, B, C and the A_i are arrays, then W is in

$$(A_1, \cdots, A_r, X, Y)$$

where X, Y satisfy the conclusions of Remark 4.

Remark 6. Each connected verbal surface Sfg is a sphere.

Proof. We need some results in [4]. The terminology there is related to our present set-up as follows. Let W be an array in

$$(A_1, \cdots, A_n) \cap (I_1, \cdots, I_e)$$

where $A_1, \cdots, A_n, I_1, \cdots, I_e$ represent the cycles of f and g, respectively, and $n = |f| \geq 1$, $e = |g| \geq 1$. Then, as in [4], we have

$$1 \to W \quad (\text{insert } A_1, \cdots, A_n)$$

$$W \to 1 \quad (\text{delete } I_e, \cdots, I_1).$$

So (S, f, g, θ) is a structure where θ is a cyclic permutation on the set S and θ is represented by the array W. This structure is minimal (see definition in [4, page 561]) because Sfg is connected. The structure is cancelled (see definition in [4, page 560]) because g has no fixed points.

By Theorem 6.1 in [4], (S, f, g) is a spherical complex, in the terminology of [4]. This is equivalent to saying Sfg is a sphere because the notions of connectedness here and in [4] are equivalent (see definition in [4, page 561]).

Remark 7. Each verbal surface Sfg is a disjoint union of a finite number of spheres.

Proof. Let W be in $(A_1, \cdots, A_n) \cap (I_1, \cdots, I_e)$ for some arrays A_i, I_j representing, respectively, the cycles of f and g where $n = |f| \geq 1$ and $e = |g| \geq 1$. Let T be any non-empty subset (of S) which is minimal with respect to the property that T is closed under f and g. Then T is a union of faces $F_1, \cdots, F_r$ and also a union of edges $E_1, \cdots, E_s$ with $r, s \geq 1$. If the corresponding cycles of f and g are $f_1, \cdots, f_r$ and $g_1, \cdots, g_s$, respectively, then there exists some subsequence $B_1, \cdots, B_r$ of $A_1, \cdots, A_n$ and some subsequence $J_1, \cdots, J_s$ of $I_1, \cdots, I_e$ such that B_i, J_j represent f_i, g_j for all i, j, $1 \leq i \leq r$, $1 \leq j \leq s$.

We form an array V (from W) by deleting all oriented edges in $S - T$. Then V is in $(B_1, \cdots, B_r) \cap (J_1, \cdots, J_s)$. So Tpq is a connected verbal surface, where $p = f_1 \cdots f_r$ and $q = g_1 \cdots g_s$. The proof is completed by observing that S is a disjoint union of a finite number of sets such as T and that Tpq is a sphere by *Remark* 6.

Remark 8. Each sphere Sfg, with $n = |f| \geqq 1$, is a verbal surface.

Proof. In the terminology of [4], (S, f, g) is a spherical complex with n boundaries. By Theorem 6.3 in [4], there exists a minimal, cancelled structure (S, f, g, θ). Therefore, there is an array W, representing θ, and there are arrays $A_1, \cdots, A_n, I_1, \cdots, I_e$ representing the cycles of f and g, respectively, with $|g| = e$, such that

$$1 \to W \quad (\text{insert } A_1, \cdots, A_n)$$

$$W \to 1 \quad (\text{delete } I_e, \cdots, I_1).$$

Hence W is in $(A_1, \cdots, A_n) \cap (I_1, \cdots, I_e)$ and so Sfg is verbal.

LEMMA 4. *Let Sfg be a sphere with $|f| \geq 1$. Let a, b be distinct oriented edges in the same face and in the same vertex. Suppose*

$$f \doteq (a_1 \cdots a_r b_1 \cdots b_s)v \quad \textit{and} \quad fg \doteq (c_1 \cdots c_t d_1 \cdots d_u)w$$

for some permutations v, w and some oriented edges a_i, b_i, c_i, d_i where $a = a_r = c_t$, $b = b_s = d_u$ and $r, s, t, u \geq 1$. Let

$$p \doteq (a_1 \cdots a_r)(b_1 \cdots b_s)v.$$

Then Spg is a disjoint union of two spheres where $\{a_1, \cdots, a_r\}$ is a face of one of these spheres, and $\{b_1, \cdots, b_s\}$ is a face of the other.

Proof. Observe that

$$p = (a_1 \cdots a_r)(b_1 \cdots b_s)(b_s \cdots b_1 a_r \cdots a_1)f = (a_r b_s)f.$$

So $pg = (a_1 b_s)(c_1 \cdots c_t d_1 \cdots d_s)w$ and thus

$$pg \doteq (c_1 \cdots c_r)(d_1 \cdots d_s)w.$$

Therefore Spg has Euler characteristic 4.

To see that Spg is verbal, we suppose that arrays $A_1, \cdots, A_n$ represent the cycles of f. One of these arrays is a cyclic permutation C of AB where $A = a_1 \cdots a_r$, $B = b_1 \cdots b_s$. So A and B represent two cycles of p. By Remark 4, C is in (X, Y) where X, Y are arrays representing these same two cycles of p. We form a sequence $D_1, \cdots, D_{n+1}$ of arrays from $A_1, \cdots, A_n$ by replacing C (in the latter sequence) by two successive terms X, Y. Then any array in $(A_1, \cdots, A_n)$ will also be in $(D_1, \cdots, D_{n+1})$ and, furthermore, $D_1, \cdots, D_{n+1}$ represent the cycles of p. It follows that Spg is verbal because Sfg is verbal.

By Remark 7, Spg is a disjoint union of two spheres. The oriented edges a, b must be in different spheres because f, p disagree only on a and on b. This completes the proof.

3. Maps and diagrams

A *labelled sphere*, over a free group F with a given set of free generators, is determined by a sphere Sfg and a function L which assigns a label xL to each

x in S, where xL is a generator (of F) or its inverse. It is required that if $xg = y$ then xL and yL are inverses of each other.

A map M is determined by a sphere Sfg and one face H of that sphere. If, in addition, Sfg and some L determine a labelled sphere, over some F, then we say that M and L determine a *diagram* over F. In this context, the faces of the map (or of the diagram) are all the orbits of f, except H; the vertices, edges, and oriented edges of the map (or of the diagram) are those of Sfg.

We now establish notation which will be used repeatedly. We follow essentially the pattern in [2, pages 254 and 256]. G denotes a group given by two or more generators and by one defining relation $R = 1$ where R is a non-empty cyclically reduced word involving all the generators. Since subscripts on generators will serve another purpose, we denote the generators by $b, c, \cdots, t$. By this, we mean, for instance, b, t or b, c, t or b, c, d, t if there are exactly 2, 3, or 4 generators, respectively. We now assume that the generator b has a zero exponent sum in the word R. Then N denotes the smallest normal subgroup (of G) containing all the generators (of G) except b. We use the powers of b (b^i, for any integer i) as a Schreier system of coset representatives for N (in G) to get a Reidemeister-Schreier rewriting process.

We use the symbols $c_i, \cdots, t_i$ to denote the elements $b^icb^{-i}, \cdots, b^itb^{-i}$, respectively, where i is any integer. The rewriting process changes a word X into a word X'. Here X is a word in the generators $b, c, \cdots, t$ and X defines an element of N. X' is a word in the symbols $c_i, \cdots, t_i$. The rewriting process changes X in the following way. If p denotes a particular symbol in X and p occurs among $c, c^{-1}, \cdots, t, t^{-1}$, then p is replaced by p_k (e.g. c is replaced by c_k, t^{-1} is replaced by t_k^{-1}) where k is the b-exponent sum of the initial segment of X preceding p. The process is completed by discarding any b symbols in X.

Let $P_i = (b^iRb^{-i})'$ for each integer i. Then each P_i is a cyclically reduced word (see problem 2, page 98, in [2]). The Reidemeister-Schreier method leads to a presentation for N:

$$N = \langle \cdots, c_{-1}, c_0, c_1, \cdots, t_{-1}, t_0, t_1, \cdots; \cdots, P_{-1}, P_0, P_1, \cdots\rangle$$

Since R involves $c, \cdots, t$, P_0 involves some $c_i, \cdots$, some t_j. Therefore P_r involves $c_{i+r}, \cdots, t_{j+r}$. It follows that each generator in the presentation for N appears in at least one defining relator in this presentation. In contrast to [2, p. 257], we choose to define N_i as the group having one defining relator P_i and having, as generators, the generators involved in P_i, for each integer i.

Finally, let $H = \{a_1, \cdots, a_r\}$ be a face of a labelled sphere Sfg over a free group F. Suppose $f \doteq (a_1 \cdots a_r)p$ for some permutation p. Let L be the label function. We say that a non-empty word W (in the generators of F) *corresponds* to H or H *corresponds* to W if the word $a_1L \cdots a_rL$ is, as it stands, W or some cyclic permutation of W.

THEOREM 1. *Let M be a diagram over the free group with free generators $c_i, \cdots, t_i$ for i ranging over the integers. Suppose M is determined by a face H*

on a labelled sphere Sfg with label function L and $|f| \geq 2$. *Assume each face of M corresponds to a relator in* N_i, *for some integer i depending on the face. Assume a word W not freely equal to* 1 *corresponds to H. The word W and the relators corresponding to the faces of M need not be freely reduced or cyclically reduced. Then there exists a diagram M′, over the same free group, determined by the face H′ on a labelled sphere Tpq with label function L′ and with* $|p| \geq 2$ *such that*:

(1) $T \subseteq S$, $L' = L$ *on* T, *and* $H' \subseteq H$.

(2) *Each face of M′ corresponds to a cyclically reduced relator in some* N_i, *where i depends on the face.*

(3) *If two different faces (of M′) correspond to relators in the same* N_i, *then the two faces are vertex disjoint.*

(4) *Each face of M′ is simple.*

(5) *The cyclically reduced form of W corresponds to H′.*

Note. The following two lemmas will be needed in the proof of Theorem 1 which requires an induction argument. Therefore, during the proof of Theorem 1, we assume that the conclusions of Theorem 1 are true when the induction hypothesis is valid. It is under these circumstances that Lemma 6 will be used. This note is intended to quiet the fear of using circular reasoning when invoking Lemma 6 (and thereby invoking its predecessor Lemma 5). This fear is raised because these lemmas assume that the conclusions of Theorem 1 hold for some diagram.

LEMMA 5. *Let M′ be a diagram over the free group with free generators* $c_i, \cdots, t_i$ *for i ranging over the integers. Let x denote one of these generators. Suppose M′ is determined by a face H′ on a labelled sphere Tpq. Assume the conclusions of Theorem* 1 *are satisfied by M′. Let I be the set of all integers i such that some relator in* N_i *corresponds to some face of M′. If x is involved in* P_j *for a unique integer j in I, then the label for some oriented edge b in H′ is x or* x^{-1}.

Proof. For such an x and such a j, we conclude that each non-empty cyclically reduced relator in N_j involves x (by the Freiheitssatz). Let F be a face (of M') corresponding to a non-empty cyclically reduced relator in N_j. Call this relator Y. Then Y involves x. Suppose an oriented edge a, in F, has the label x^{-1} (or x). Let $b = aq$. Then x (or x^{-1}) is the label for b. Now no P_i, for i in I and $i \neq j$, involves x. Also any face (of M'), different from F and corresponding to a relator in N_j, has no vertex in common with F, and hence no edge in common with F (by Remark 1). So b is not in any face (of M') different from F. Since F is simple, b is not in F if F contains at least three oriented edges (by Remark 2). Finally, if F contains exactly two oriented edges, then b is not in F because, otherwise, F would be equal to $\{a, b\}$ and the non cyclically reduced word xx^{-1} would correspond to F. Therefore, b is in H' and we are done.

LEMMA 6. *If the assumptions of LEMMA 5 hold and if H' corresponds to a word W in the generators of N_k, for some integer k, then $I = \{k\}$.*

Proof. Let r, s be, respectively, the minimum and maximum integers in I. Let m, n be, respectively, the minimum and maximum subscripts on a, c involved in P_0. If $x = c_{r+m}$ and $y = c_{s+n}$, then the unique P_i (as i ranges over I) which involves x is P_r. Similarly, if P_i involves y and i is in I, then $i = s$. Thus x and y are involved in W by Lemma 5.

If $k \neq r$ then x is not involved in P_k; hence a is not involved in W, a contradiction. Therefore $k = r$. Similarly $k = s$ (using y).

Proof of Theorem 1. Use induction on the number $|f| + |g|$ which is $\geqq 3$. If $|f| + |g| = 3$ then since $|g| = 1$, S must have precisely two elements. Then we can use $M' = M$. In this case, each of the two faces of Sfg will correspond to a cyclically reduced word of length 1. Now suppose $|f| + |g| > 3$.

We now consider 5 cases:

0. M satisfies (2), (3), (4), (5).
1. M does not satisfy (3).
2. M does not satisfy (4).
3. M satisfies (4), but not (2).
4. M satisfies (2), (3), (4), but not (5).

Case 0. We use $M' = M$.

Case 1. If property (3) does not hold for Sfg, we suppose that relators X, Y in some N_i correspond to two different faces containing oriented edges a, b, respectively, and that a, b are in the same vertex, as in Lemma 1. By Lemma 1 there is a sphere Spg with the face H which, together, determine a diagram M'', if we keep the labels from M. Then there is a face (of M'') corresponding to a relator $Z = UV$ in N_i where U, V are, respectively, cyclic permutations of X and Y. We note that W still corresponds to the face H of the labelled sphere Spg and $|p| < |f|$. We find a suitable diagram M' by applying the induction assumption to M''.

Case 2. Assume M does not satisfy (4) so some particular face of M corresponding to a relator in N_i is not simple. We suppose that the assumptions and all notation in the statement of Lemma 4 hold for Sfg. We mean this to include the representation of

$$f \doteq (a_1 \cdots a_r b_1 \cdots b_s)v,$$

the definition of the permutation p, and the conditions on the oriented edges a, b in the above-mentioned particular face. Thus Spg is a disjoint union of two spheres (taken in some order). We make these labelled spheres by keeping the labels from M.

Suppose H and $A = \{a_1, \cdots, a_r\}$ are faces of the first sphere and $B =$

$\{b_1, \cdots, b_s\}$ is a face of the second sphere. Let a word U be chosen to correspond to the face (of M) containing a, b in such a way that U is of the form XY where X, Y correspond to the faces A, B, respectively, of Spg. This implies that U is a relator in N_i and that X and Y are words in the generators of N_i.

We apply the induction assumption to the diagram M_1 determined by the face B on the second sphere. We thereby get a diagram M_1' determined by a face H_1' on some third sphere. Here H_1' corresponds to the cyclically reduced form Z of the word Y. By applying Lemma 6 to M_1' and H_1', we conclude that each face of M_1' corresponds to a cyclically reduced relator in N_i (because Z is a word in the generators of N_i).

We observe that all but one (namely B) of the faces of the third labelled sphere correspond to relators in N_i. It follows from Theorem 6.4 in [4] that the face B also corresponds to a relator in N_i, i.e. Z is a relator in N_i. Therefore Y is a relator in N_i and so is X.

Now we apply the induction assumption to the diagram M'' determined by the face H on the first sphere to get a suitable diagram M'.

Case 3. Assume M satisfies (4) but not (2) so each face of M is simple and some face H'' of M corresponds to a noncyclically reduced word. We consider two cases. When H'' contains more than two oriented edges, we form the sphere Skg as in Lemma 3 (assuming that the oriented edges a, b in the statement of Lemma 3 have labels which are inverse generators). We keep the labels of M to make Skg a labelled sphere. Going from Sfg to Skg we lose one vertex and gain one simple face $\{a, b\}$.

We apply Lemma 2 to the face $\{a, b\}$ on the sphere Skg to get a sphere Tpq with face H. We use the restrictions of L to T as a label function for Tpq. Since $1 + |q| = |g|$ and $1 + |p| = |f|$, we can use the induction assumption on the diagram determined by the face H on the labelled sphere Tpq to get a suitable diagram M'.

In the second case, $H'' = \{a, b\}$ is a simple face and aL, bL are inverse generators. We again apply Lemma 2 to a face H'' on the sphere Sfg to get a labelled sphere Tpg, as above. We find a suitable M' as before.

Case 4. Assume M satisfies (2), (3), (4) but not (5) so W is not cyclically reduced. Then the cyclic word W contains some subword consisting of a generator and its inverse. Since W is not freely equal to 1, the length of W is at least 3. Suppose $f \doteq (abc_1 \cdots c_r)p$ for some oriented edges a, b, c_i, $1 \leq i \leq r$ and some permutation p such that

$$H = \{a, b, c_1, \cdots, c_r\}$$

and aL, bL are a generator and its inverse.

Subcase A. Assume b and c_r are in different vertices. Say

$$fg \doteq (d_1 \cdots d_s)(e_1 \cdots e_t)w$$

for some oriented edges d_i, e_i and some permutation w where $d_s = b$, $e_t = c_r$ and $s, t \geq 1$. Let $k \doteq (ab)(c_1 \cdots c_r)p$. We claim that Skg is a sphere. In fact, $k = (bc_r)f$ and so

$$kg \doteq (d_1 \cdots d_s e_1 \cdots e_t)w.$$

Thus Skg had Euler characteristic 2. To see that Skg is a verbal surface, we may use the argument involving Spg in the proof of Lemma 4. Therefore, by Remark 7, Skg is a sphere. We can now apply Lemma 2 to the face $\{a, b\}$ on the sphere Skg and find a suitable diagram M' as in Case 3.

Subcase B. Assume b and c_r are in the same vertex. Say

$$fg \doteq (d_1 \cdots d_s e_1 \cdots e_t)w$$

for some oriented edges d_i, e_i and some permutation w where $d_s = b$, $e_t = c_r$ and $s, t \geq 1$. Let $k \doteq (ab)(c_1 \cdots c_r)p$. We claim that Skg has Euler characteristic 4. In fact, $k = (bc_r)f$ and so

$$kg \doteq (d_1 \cdots d_s)(e_1 \cdots e_t)w.$$

Skg is a verbal surface, as in Subcase A. Therefore, by Remark 7, Skg is a disjoint union of 2 spheres. One of these spheres (call it Tpq) has a face $H'' = \{c_1, \cdots, c_r\}$ and the other sphere has a face $\{a, b\}$. These two faces are on different connected pieces of Skg because Sfg is connected and f, k differ only on b and on c_r. By restricting L to T, we make Tpq into a labelled sphere. We can now apply the induction assumption to the diagram M'' determined by H'' and Tpq to get a suitable diagram M'. This completes the proof of Theorem 1.

In order to prove our main result, we need a final reference to [4] and a final lemma.

Remark 9. For each relator W in a group given by generators and non-empty cyclically reduced defining relators such that each generator appears in at least one defining relator, there exists a labelled sphere with two or more faces such that W corresponds to one face and each of the other faces corresponds to a defining relator or its inverse.

Proof. Use Theorem 6.2 in [4] and note that each spherical complex is a sphere.

LEMMA 7. *Let W be a word in the generators of N_k for some integer k. If W is a relator in N, then W is a relator in N_k. (This implies that the smallest subgroup of N, containing the generators involved in P_k, is isomorphic to N_k, for each integer k.)*

Proof. We can assume that W is non-empty and cyclically reduced. By Remark 9, W corresponds to one face H' of some labelled sphere Tpq (with at least two faces) such that defining relators of N correspond to the other faces. Thus H' and Tpq determine a diagram M'. By Theorem 1, we can assume that Tpq satisfies the conclusions of Theorem 1. By Lemma 6, each face of Tpq (except H') corresponds to P_k or to P_k^{-1}. It follows from Theorem 6.4 in [4] that W is a relator in N_k.

4. Main result

THEOREM 2. *Let G be a group given by one or more generators and one defining relator R, which is a non-empty cyclically reduced word involving all the generators. Then no proper subword of R is a relator.*

Proof. We use induction on the length of R. When the length is 1, the theorem is true. To avoid a trivial situation, we can assume that there are at least 2 generators. Let U be a proper subword of R. Assume U is a relator. Since R can be replaced by any of its cyclic permutations when describing G, we can assume that R is a product UV which is cyclically reduced, as it stands.

Case 1. Suppose some generator, say b, has a zero exponent sum in R. We now use the notation in Section 3. So we have a normal subgroup N, a presentation for N using generators $c_i, \cdots, t_i$ as i ranges over the integers, and a rewriting process $X \to X'$. In particular, $U \to U'$ and U' is a relator in N.

Since U is non-empty and freely reduced as a word in the generators of G, the same is true for U' as a word in the generators of N (see problem 2, page 98, in [2]). Furthermore, R' and $U'V'$ are identical words in the generators of N (by property (vi), page 92, in [2]). Thus U' is a proper subword of $R' = P_0$.

Now consider the word W' which is the cyclically reduced form of the word U'. By Remark 9, there exists a diagram M determined by a face H on a labelled sphere Sfg such that H corresponds to W' and each face of M corresponds to P_i or P_i^{-1} for some integer i depending on the face.

We apply Theorem 1 to M to get a diagram M', determined by the face H on a labelled sphere Tpq such that W' corresponds to H. By Lemma 7, W' is a relator in N_0. Hence, so is U'. This gives a contradiction because the induction assumption can be applied to the group N_0 with one defining relator $P_0 = R' = U'V'$.

Case 2. Suppose each of the generators $b, c, \cdots, t$ has a non-zero exponent sum in R. Let b, t have exponent sums m, n respectively. Instead of $G = b, c, \cdots, t; R(b, c, \cdots, t)$ we now consider

$$E = \langle x, c, \cdots, t; R(x^n, c, \cdots, t)\rangle.$$

G can be mapped homomorphically into E by the mapping $b \to x^n$, $c \to c, \cdots,$ $t \to t$. Since we are assuming that U, written functionally as

$$U(b, c, \cdots, t),$$

is a relator in G, we find that the word

$$U(x^n, c, \cdots, t)$$

is a relator in E and it is a proper subword of

$$R(x^n, c, \cdots, t).$$

We apply Tietze transformations to E to get a presentation in which the single defining relator $R(x^n, c, \cdots, yx^{-m})$ has a zero exponent sum in the generator x.

$$E = \langle x, c, \cdots, t, y; R(x^n, c, \cdots, t), y = tx^m \rangle,$$

$$E = \langle x, c, \cdots, t, y; R(x^n, c, \cdots, t), t = yx^{-m} \rangle,$$

$$E = \langle x, c, \cdots, y; R(x^n, c, \cdots, yx^{-m}) \rangle.$$

Again the word

$$U(x^n, c, \cdots, yx^{-m})$$

is a relator in E and it is a proper subword of $R(x^n, c, \cdots, yx^{-m})$. Since the latter word has a zero exponent sum in x, we can use x (just as we used b in Section 3) to go to the normal subgroup N (in E) generated by $c, \cdots, y$.

$$N = \langle \cdots, c_{-1}, c_0, c_1, \cdots, y_{-1}, y_0, y_1, \cdots; \cdots, P'_{-1}, P'_0, P'_1, \cdots \rangle.$$

Here we have a rewriting process $X \to X'$ which comes from the Schreier representatives x^k, k an integer, for N in E. So

$$P'_i = (x^i R(x^n, c, \cdots, yx^{-m}) x^{-i})'.$$

Since the x-symbols in $R(x^n, c, \cdots, yx^{-m})$ will contribute no symbols to P'_0, the latter word has smaller length than $R(b, c, \cdots, t)$.

Since the word $(U(x^n, c, \cdots, yx^{-m}))'$ is a word in the generators of N_0 and this word is a relator in N, we conclude that this word is a relator in N_0, by Lemma 7. Furthermore, it is a proper subword of P_0. By applying the induction assumption to the group N_0 which has P_0 as its single defining relator, we get a contradiction.

5. A new proof of the Freiheitssatz

THEOREM 3. (The Freiheitssatz). *Let G be a group given by one or more generators and one defining relation $R = 1$, where R is a non-empty cyclically reduced word involving all the generators. Let W be a non-empty cyclically reduced word in the generators. If $W = 1$ in G, then W involves all the generators.*

Proof. We use induction on the length of R. When the length is 1, the theorem is true. To avoid a trivial situation, we can assume that there are at least 2 generators. We shall show that a typical generator (to be called t) is involved in W.

Case 1. The generators of G are b, t and t has a non-zero exponent sum in R. For a brief proof, see [2, page 253].

Case 2. The generators of G are b, t and t has a zero exponent sum in R. Form the normal subgroup N (of G) generated by t, the rewriting process $X \to X'$, and the presentation for N as in Section 3. Here t takes the place of b in section 3. Then W' is a cyclically reduced word in the generators of N and W' is a relator.

By the induction assumption, we are allowed to apply the Freiheitssatz to each group N_j which has a single defining relator P_j whose length is smaller than the length of R. If we regard the assumptions of Lemma 5 as applying to the subgroup N of our present group G, then the proof of Lemma 5 still holds since the application of the Freiheitssatz will be permissible. It follows that Theorem 1 is also valid when applied to the present N and G.

By Remark 9, there exists a diagram M determined by a face H on a labelled sphere Sfg such that H corresponds to W' and each face of M corresponds to some P_i or P_i^{-1}. We apply Theorem 1 to M to get a diagram M', determined by a face H' on a labelled sphere Tpq, such that W' corresponds to H'.

Let m, n be the minimum and maximum subscripts on b's appearing in $R' = P_0$. Then $m < n$, otherwise $R = t^m b^k t^{-m}$ for some integer k, a contradiction. Let I be the set of integers i such that P_i or P_i^{-1} corresponds to some face of M'. Suppose r, s are, respectively, the minimum and maximum integers in I, so that $r \leq s$. Since $r + m < s + n$, either $r + m$ or $s + n$ is different from zero. Assume $r + m \neq 0$. (The other case is similar.)

By applying Lemma 5 to M' and $x = b_{r+m}$, we conclude that W' involves x. Suppose $W' = Cb_{r+m}^e D$ for some words C, D in the generators of N, with $e = \pm 1$. Then $W = Ab^e B$ for some words A, B in the generators of G. It is understood that b^e is replaced by b_{r+m}^e during the rewriting process. Therefore, t must have an exponent sum $r + m$ in the word A. Thus A and hence W involve t.

Case 3. G has 3 or more generators and the exponent sum (in R) of some generator different from t (say b) is zero. We follow the steps in Case 2, except that the use of t there is replaced by the use of b here. We find M', I, r, s as before and let m be the minimum subscript on t's appearing in $R' = P_0$. Here $r + m$ may be zero. Once again W' involves the generator $x = t_{r+m}$. In the present case, we conclude that W involves t.

Case 4. G has 3 or more generators and the exponent sum on each generator in R, other than t, is different from zero. In this case, W will be renamed V. Let m, n be the exponent sums of b, c, respectively in the word R. We form the group E:

$$E = \langle x, c, \cdots, t; R(x^n, c, \cdots, t) \rangle.$$

We note again that G can be mapped homomorphically into E by the mapping $b \to x^n$, $c \to c$, $\cdots$, $t \to t$. Then since $V(b, c, \cdots, t)$ is a relator in G, we have that

$$Y = V(x^n, yx^{-m}, \cdots, t)$$

is a relator in E. By Tietze transformations we arrive at another presentation for E:

$$E = \langle x, y, \cdots, t; R(x^n, yx^{-m}, \cdots, t)\rangle.$$

(This is accomplished by interchanging the roles of c and t in the Tietze transformations of Case 2 of Theorem 2.)

Let N be the smallest normal subgroup (of E) containing $y, \cdots, t$. We now follow the steps in Case 2 (of Theorem 3), except that G, t, W are replaced by the present E, x, Y, respectively. We find M', I, r, s, as before and let k be the minimum subscript on t's appearing in

$$P_0 = (R(x^n, yx^{-m}, \cdots, t))'$$

Here $r + k$ may be zero. Again Y' involves t_{r+k}. Hence Y involves t. Therefore V involves t.

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