THE AUTOMORPHISMS OF THE UNITARY GROUPS AND THEIR CONGRUENCE SUBGROUPS

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The automorphism theory of the unitary group and the projective unitary group has been investigated only over fields. In particular, no automorphism theory has been available for the groups of unitary matrices whose entries are taken from an integral domain. In this paper, we consider subgroups G of the projective unitary group $PU_n(V)$ such that for each isotropic line L of V, G contains at least one non-trivial projective transvection with proper line L.

We require the underlying hermitian form to have Witt index at least 3. We then give group-theoretical properties of the projective unitary transvections in G which suffice to distinguish the projective transvections from the other projective unitary transformations. From this it follows that an automorphism Λ of G must preserve all projective unitary transvections. This yields a bijection $L \rightarrow L'$ of the isotropic lines of the underlying vector space V; we extend this to a bijection of all the totally isotropic subspaces of V and then apply a theorem of Chow and Dieudonné [3, p. 82] to conclude the bijection of totally isotropic subspaces is induced identically by a unitary semi-similitude g of V. It is then easy to show the automorphism Λ is given by transformation by the projective semi-similitude $\bar{g}$ corresponding to g. Our results hold for Witt index at least 3 and characteristic not 2.

Having obtained the automorphisms of such projective unitary groups G we determine as a corollary all the automorphisms of any subgroup S of $U_n(V)$ that contains at least one non-trivial transvection on each isotropic line of V. In the final section we apply these results to the unitary groups U_n, U_n^+, T_n defined over integral domains and to their congruence subgroups. We show each such unitary congruence group contains a non-trivial transvection on every isotropic line. Applying our previous results we thus obtain an automorphism theory for the unitary congruence groups.

The techniques used in this paper are modifications of the original method of residual spaces introduced by O'Meara in [6] for the congruence subgroups of the special linear and general linear groups.

1. Preliminaries

Let V be an n-dimensional vector space over the field F, and let $\chi(F)$ denote the characteristic. Consider a hermitian form (x, y) on V, i.e., a map

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from $V \times V$ into F such that

$$(\alpha x + z, y) = \alpha(x, y) + (z, y)$$
$$(x, y) = (y, x)^* \quad \text{for all } x, y, z \text{ in } V, \text{all } \alpha \text{ in } F$$

where $\alpha \rightarrow \alpha^*$ is an automorphism of F of order two and not the identity.

Let E denote the fixed field of the * map; $F \mid E$ is a galois extension of degree two. Now let $(x, x) = H(x)$. Note $H(\alpha x) = \alpha\alpha^* H(x)$ for $\alpha \in F$, and since $(x, x) = (x, x)^*, H(V) \subseteq E$. If $\chi(F) \neq 2, F = E(\sqrt{\theta})$ for some $\theta \in E$ and $(\alpha + \beta\sqrt{\theta})^* = \alpha - \beta\sqrt{\theta}$ for all α and β in E.

In all that follows we shall assume that (x, y) is a non-degenerate hermitian form on the n-dimensional vector space V over F. Here non-degenerate means $(V, x) = 0$ implies $x = 0$.

Definition. Let W be a subspace of V and set

$$W^* = \{x \in V \mid (W, x) = 0\}.$$

W is called regular if $W \cap W^* = 0$, degenerate if $W \cap W^* \neq 0$; and W is called totally degenerate if $W \neq 0$ and $W \subseteq W^*$. We define rad W to be the subspace $W \cap W^*$ of W; so rad W = rad W^*. Since we always assume V to be regular, we have that $\dim W + \dim W^* = \dim V$ and therefore $(W^*)^* = W$ [3, p. 13]. Since rad $W \subseteq W^*$ we see

$$\dim \operatorname{rad} W \leqq n - \dim W.$$

We call a non-zero vector x of V isotropic if $H(x) = 0$, anisotropic if $H(x) \neq 0$. A subspace W is called isotropic if it contains an isotropic vector, anisotropic otherwise. *For the rest of this paper we make the further assumption V is isotropic.*

The n-dimensional unitary group, written $U_n(V)$ or simply $U(V)$, is all $\sigma \in GL_n(V)$ such that $(\sigma x, \sigma y) = (x, y)$ for all x, y in V. Such a σ is called a unitary transformation. Now it can be shown using Witt's theorem on the extension of unitary transformations that all maximal totally degenerate subspaces of V have equal dimension [3, pp. 21–23]. This common dimension is called the Witt index of V, $\nu(V)$; one has $2 \cdot \nu(V) \leqq n$.

Now for $\sigma \in U_n(V)$, we let $P = \{x \in V \mid \sigma x = x\}$ and we put $R = P^*$. P is called the fixed space of σ and R is the residual space of σ. We have $\dim R + \dim P = n$. Whenever a transformation $\sigma \in U_n(V)$ is under discussion, the letter P will always denote the fixed space of σ and the letter R will always denote the residual space of σ. Similarly we associate P_i and R_i with any σ_i in $U_n(V)$. And res σ denotes dim $(\sigma - 1)V$.

For any σ in $U_n(V)$ we have $\sigma P = P$ and $\sigma R = R$. And

1.1. $P = \ker(\sigma - 1_V), R = (\sigma - 1_V)V$,

where 1_V denotes the identity map of V. For any $\Sigma \in U_n(V)$, $\Sigma\sigma\Sigma^{-1}$ has fixed space ΣP and residual space ΣR.

DEFINITION. A subspace W of V is called a hyperbolic plane if it is regular, two-dimensional, and isotropic. It follows from *La Géométrie des Groupes Classiques*, p. 21, that W is a hyperbolic plane if and only if W has a basis of isotropic vectors x, y such that $(x, y) = 1$.

DEFINITION. For two subspaces U and W of V with $(U, W) = 0$ and $U \cap W = 0$ we denote the direct sum $U \oplus W$ by $U \perp W$. For any group G whatever we denote the commutator subgroup of G by DG, and for a, b, in G we let $[a, b] = aba^{-1}b^{-1}$.

By a transvection τ is meant an element of $SL_n(V)$ that fixes a hyperplane pointwise. If $\tau \neq 1_V$, $(\tau - 1_V)V$ is a line L, called the proper line of τ. 1_V is considered a transvection having any line of V as proper line. Which transvections τ fall in $U_n(V)$? If $\tau \neq 1_V$, $\tau \in U_n(V)$, and P is the fixed hyperplane of τ, then $L = (\tau - 1_V)V = P^*$. Since det $\tau = 1$ it is easily seen $L \subseteq P = L^*$, and so L is an isotropic line. Now for an isotropic vector a of V and an element λ of F, define the map

$$\tau_{a,\lambda} \quad \text{by} \quad \tau_{a,\lambda}(x) = x + \lambda(x, a)\cdot a \quad \text{for all} \quad x \in V.$$

A computation shows each $\tau_{a,\lambda}$ is a transvection having Fa as its proper line, and $\tau_{a,\lambda} \in U_n(V)$ if and only if $\lambda + \lambda^* = 0$. Conversely each transvection τ in $U_n(V)$ with isotropic proper line Fa has the form $\tau = \tau_{a,\lambda}$ for λ a suitable element of F of trace zero. If $\Sigma \in U_n(V)$, one easily sees

$$\Sigma\tau_{a,\lambda}\Sigma^{-1} = \tau_{\Sigma a,\lambda} \quad \text{and} \quad (\tau_{a,\lambda})^{-1} = \tau_{a,-\lambda}.$$

The following propositions are now easily proved.

1.2. *Let $\tau_{a,\lambda}$ and $\tau_{b,\beta}$ be in $U_n(V)$. Then $\tau_{a,\lambda} = \tau_{b,\beta} \Leftrightarrow a = \alpha b$ with $\beta = \alpha\alpha^*\lambda$ for some $\alpha \in F$.*

1.3. *The product of two unitary transvections is a transvection if and only if the two unitary transvections have the same proper line.*

1.4. *Let $\tau_{a,\lambda}$ be a transvection in $U_n(V)$ and let $\sigma \in U_n(V)$. Then σ and $\tau_{a,\lambda}$ commute if and only if $\sigma a = \alpha a$ for some element α in F with $\alpha\alpha^* = 1$.*

1.5. *Two unitary transvections commute if and only if their proper lines are orthogonal.*

For any m-dimensional vector space W whatever we denote the scalar transformations of $GL_m(W)$ by $RL_m(W)$, or simply $RL(W)$.

1.6. *Let W b a two-dimensional vector space and suppose*

$$\sigma \in GL_2(W) - RL_2(W).$$

Then the centralizer $C_W(\sigma)$ of σ in $GL_2(W)$ is abelian.

Proof. Apply 2.6 of [6]. Q.E.D.

1.7. *Let $n \geq 2$ and suppose $\sigma \in U_n(V)$ fixes all isotropic lines of V. Then $\sigma \in RL(V)$.*

Proof. If dim $V = 2$, σ fixes at least 3 lines of V which implies $\sigma \in RL(V)$. If dim $V > 2$, σ fixes all hyperbolic planes since a hyperbolic plane can be spanned by isotropic vectors. Any anisotropic line is the intersection of two hyperbolic planes [3, p. 43] so σ fixes all lines. Q.E.D.

1.8. *Let $n \geq 4$ and $\Sigma \in U_n(V)$ be such that $\Sigma(Fa) \neq Fa$ for some isotropic line Fa of V. Let $\tau_{a,\lambda}$ be a non-trivial transvection in $U_n(V)$. Then Σ and $\tau_{a,\lambda}\Sigma^{-1}\tau_{a,-\lambda}$ don't commute, if $\chi(F) \neq 2$.*

Proof. Suppose they did commute; then

$$\tau_{\Sigma a,\lambda}\, \tau_{a,-\lambda} = \tau_{a,\lambda}\, \tau_{\Sigma^{-1}a,-\lambda}\,.$$

Choose a vector X orthogonal to $\Sigma^{-1}a$ but not to a. Upon substitution of X in the above equation we obtain

$$-\lambda(X, a)a + \lambda(X, \Sigma a)\Sigma a - \lambda^2(a, \Sigma a)(X, a)\Sigma a = \lambda(X, a)a$$

which implies $a \in F\cdot\Sigma a$, a contradiction. Q.E.D.

DEFINITION. We say a subgroup S of $U_n(V)$ "has enough transvections" if for each isotropic line L of V there is a non-trivial transvection in S with proper line L.

We now define the projective unitary group, written $PU_n(V)$ or simply $PU(V)$, to be the quotient group of $U(V)$ by its center, i.e., $PU(V) = U(V)/RL(V) \cap U(V)$. Let $^-$ denote the natural map of $U(V)$ onto $PU(V)$. For any subset A of $U(V)$, $\bar{A}$ or A^- denotes the image of A under the map. We call $\bar{\sigma} \in PU(V)$ a projective transvection if one coset representative of $\bar{\sigma}$ in $U(V)$ is a transvection. If $n \geqq 2$ there cannot be two distinct coset representatives of $\bar{\sigma}$ which are transvections, and so we may define the proper line of a projective transvection $\bar{\sigma}$ to be the proper line of the unique coset representative of $\bar{\sigma}$ which is a transvection.

DEFINITION. Let G be a subgroup of $PU(V)$. We say G has enough (projective) transvections if for each isotropic line L of V there is a non-trivial projective transvection in G with proper line L. We define

$$\Delta = \{\sigma \in U(V) \mid \bar{\sigma} \in G\};$$

so Δ is a subgroup of $U(V)$ that has enough transvections and $RL(V) \cap U(V) \subseteq \Delta$. For the rest of this paper whenever G or Δ is mentioned we always understand them to be defined as above. If $A \subseteq \Delta$ (resp. $A \subseteq G$), then $C(A)$ is $C_\Delta(A)$ (resp. $C_G(A)$).

DEFINITION. We say an element of $U(V)$ is a quasi-symmetry if its residual space is an anisotropic line or if it is 1_V. An element of $PU(V)$ is a projective quasi-symmetry if it is the image under the $^-$ map of a quasi-symmetry. Finally we say an element of $U(V)$ is a shearing if it leaves a hyperplane pointwise fixed; projective shearings in $PU(V)$ are defined in the obvious way.

DEFINITION. Let W be a subspace of V. We define $E(W) = \{\sigma \in \Delta \mid R \subseteq W\}$ where R is the residual space of σ.

DEFINITION. Let $\sigma \in U(V) \cap SL_n(V)$. We call σ a plane rotation if its residual space R is a plane. σ is called a totally degenerate plane rotation or a hyperbolic rotation if R is respectively a totally degenerate plane or a hyperbolic plane.

1.9. *Let $\nu(V) \geq 3$, let P be a subspace of V of dimension at least $n - 2$. Write $P =$ rad $P \perp W$. Then W is isotropic.*

Proof. The hypotheses imply $n \geq 6$; and we have rad $P = P^* \cap P$. Since dim rad $P \leq 2$ and $\nu(V) \geq 3$, rad P cannot be a maximal totally degenerate subspace of V. So rad $P \subseteq T$ for some three-dimensional totally degenerate subspace T of V. It follows that

$$T \subseteq T^* \subseteq (\text{rad } P)^* = P + P^*.$$

Since $W \subseteq P + P^*$, a dimension argument implies $W \cap T \neq 0$. So W is isotropic. Q.E.D.

1.10. *Let $\nu(V) \geq 3$ and suppose P is a subspace of V of dimension at least $n - 2$. Then every isotropic line of P not in* rad P *lies in two distinct hyperbolic planes of P.*

Proof. As in (1), p. 42 of [3]. Q.E.D.

1.11. *Let σ and Σ be in $U(V)$ and suppose $\bar{\sigma}$ and $\bar{\Sigma}$ commute. Let $\sigma \mid W = \alpha$, $\alpha \in RL(W)$. If 2 dim $W > n$, then σ and Σ commute.*

Proof. Since $\bar{\sigma}$ and $\bar{\Sigma}$ commute, $\sigma = \beta \cdot \Sigma\sigma\Sigma^{-1}$ with $\beta \in RL(V)$. Choose non-zero x in $W \cap \Sigma(W)$. From the equation $\sigma(x) = \beta \cdot \Sigma\sigma\Sigma^{-1}(x)$ it follows $\alpha = \beta\alpha$ or $\beta = 1_V$. Q.E.D.

2. Double centralizer results

2.1. *Let S be a subgroup of $U_n(V)$ that has enough transvections. Let σ be a non-involution in $S \cap SL_n(V)$ with residual and fixed spaces R, P with $R \not\subseteq P$. Suppose $n \geqq 2$ and* dim $R = 2$. *Then $E(R) \subseteq CDC(\sigma)$.*

Proof. Let $RL_2(R)$ denote the scalar transformations of $GL_2(R)$. Since $R \not\subseteq P$, $\sigma \mid R \neq 1_R$. Now $\sigma \mid R \notin RL_2(R)$ since if so, $1 = \det \sigma = \det \sigma | R$, which implies $\sigma \mid R = -1_R$. But σ is not an involution. Thus

$$\sigma \mid R \in GL_2(R) - RL_2(R),$$

and 1.6 tells us $C_R(\sigma \mid R)$ is abelian where $C_R(\sigma \mid R)$ denotes the centralizer of

$\sigma \mid R$ in $GL_2(R)$. Let $\sigma_1 \in DC(\sigma)$; so $\sigma_1(R) = R$ and

$$\sigma_1 \mid R \in DC(\sigma) \mid R \subseteq DC_R(\sigma \mid R) = 1_R .$$

So $\sigma_1 \mid R = 1_R$ and $R_1{}^* = P_1 \supseteq R$; any σ_2 in $E(R)$ will have its residual space $R_2 \subseteq R \subseteq R_1{}^*$. Thus σ_2 and σ_1 commute by 1.5 of [6] and so $E(R) \subseteq CDC(\sigma)$. Q.E.D.

2.2. *Let* $\nu(V) \geq 3$ *and let* $\sigma \in \Delta$ *have* $\dim R \leq 2$. *Then* $CDC(\bar{\sigma}) \subseteq E(R)^-$.

Proof. The hypotheses imply $n \geq 6$. Now let $L = Fa$ be any isotropic line of P not in rad P. By 1.10 we can choose two distinct hyperbolic planes $Fa + Fb$ and $Fa + Fc$ in P with b and c isotropic. Let τ_a, τ_b, τ_c, be non-trivial transvections in Δ with proper lines Fa, Fb, Fc. Let $f = [\tau_a, \tau_b]$ and $g = [\tau_a, \tau_c]$; f and g have residual spaces $Fa + Fb$ and $Fa + Fc$ respectively. Since σ fixes a, b, and c, f and g are in $DC(\sigma)$ and so $\bar{f}$ and $\bar{g}$ are in $DC(\sigma)^- \subseteq DC(\bar{\sigma})$.

Now let $\bar{\Sigma} \in CDC(\bar{\sigma})$. $\bar{\Sigma}$ and $\bar{f}$ commute and by 1.11, Σ and f commute since $n \geqq 6$. So Σ acts on the residual space of f and similarly Σ acts on the residual space of g. Hence Σ acts on their intersection, Fa. So Σ fixes all isotropic lines of P not in rad P. If rad $P \neq 0$, let K be a line of rad P and write $P = \operatorname{rad} P \perp W$. W is isotropic and regular by 1.9. Let L_0 be an isotropic line of W. All lines of $K \oplus L_0$ are isotropic and the only line of $K \oplus L_0$ in rad P is K. So Σ fixes all lines of $K \oplus L_0$, except possibly K. Therefore $\Sigma K = K$. So Σ fixes all isotropic lines of P.

Since W is isotropic and regular 1.7 implies

$$\Sigma \mid W = \lambda \quad \text{with} \quad \lambda \in RL(V) \cap U(V).$$

We claim $\Sigma \mid \operatorname{rad} P = \lambda$. This follows from considering the effect of Σ on $K \oplus L_0$. Hence $\Sigma \mid P = \lambda \cdot 1_P$ and so $\Sigma \in \lambda \cdot E(R)$. Thus

$$\bar{\Sigma} \in (\lambda \cdot E(R))^- = E(R)^-. \qquad \text{Q.E.D.}$$

2.3. *Let* $\nu(V) \geqq 3$ *and* $\sigma \in \Delta$. *Then* $CDC(\bar{\sigma})$ *is abelian if* σ *is a transvection or a totally degenerate plane rotation or quasi-symmetry. And* $CDC(\bar{\sigma})$ *is non-abelian if* σ *is a hyperbolic rotation with* $\sigma^2 \neq 1_V$.

Proof. If σ is a transvection or a totally degenerate plane rotation or a quasi-symmetry, 2.2 implies $CDC(\bar{\sigma}) \subseteq E(R)^-$. Since $\dim R = 1$ or R is totally degenerate, 1.4 of [6] implies $E(R)^-$ is abelian.

If σ is a hyperbolic rotation with $\sigma^2 \neq 1_V$, 2.1 implies $E(R) \subseteq CDC(\sigma)$, so $E(R)^- \subseteq (CDC(\sigma))^- \subseteq CDC(\bar{\sigma})$ since $C(\sigma)^- = C(\bar{\sigma})$ by 1.11. Since R is a hyperbolic plane and G has enough projective transvections, there are two non-commuting projective transvections in $E(R)^-$. Q.E.D.

3. Applications to automorphism theory

3.1. *Let* Λ *be an isomorphism of* G *into* $PU(V)$ *which maps each projective transvection* $\bar{\tau}$ *in* G *to a projective transvection having the same proper line. Then* Λ *equals the identity map on* G.

Proof. Let L be an isotropic line of V and $\bar{\tau}$ a non-trivial projective transvection in G having proper line L. Let $\bar{\sigma}$ be a typical element of G; write $\bar{\sigma}_1 = \Lambda\bar{\sigma}$ and let $\bar{\tau}_1 = \Lambda\bar{\tau}$ where τ_1 is a transvection having proper line L. Set $f = \sigma\tau\sigma^{-1}$ and $g = \sigma_1\tau_1\sigma_1^{-1}$; $\Lambda\bar{f}$ is a projective transvection with line σL and $\bar{g}$ is a projective transvection with proper line $\sigma_1 L$. Since $\Lambda\bar{f} = \bar{g}$, $\sigma_1\sigma^{-1}$ fixes all isotropic lines of V and 1.7 implies $\sigma = \lambda\sigma_1$ with $\lambda \in RL(V) \cap U(V)$. Hence $\bar{\sigma} = (\lambda\sigma_1)^{-} = \Lambda\bar{\sigma}$ for all $\bar{\sigma}$ in G. Q.E.D.

Definition. Let g be a semi-linear isomorphism of V onto V. We say g is a unitary semi-similitude if there is $\lambda \in F$ such that $(gx, gy) = \lambda(x, y)^u$ for all x, y in V where u is the field automorphism of g.

Given a unitary semi-similitude g we may define a map Λ_g of $U_n(V)$ onto $U_n(V)$ by $\Lambda_g(\sigma) = g\sigma g^{-1}$ for $\sigma \in U_n$. Similarly we define the map $\bar{\Lambda}_g$ of $PU_n(V)$ onto $PU_n(V)$ by $\bar{\Lambda}_g(\bar{\sigma}) = (\Lambda_g(\sigma))^{-}$ for all $\bar{\sigma} \in PU_n(V)$. It is easy to see Λ_g and $\bar{\Lambda}_g$ are automorphisms of $U_n(V)$ and $PU_n(V)$ respectively.

3.2. *Let g be a semi-linear isomorphism of V onto V and $n \geqq 2$. Then g is a unitary semi-similitude $\Leftrightarrow$ for all x, y in V, $(x, y) = 0$ implies $(gx, gy) = 0$.*

Proof. $\Rightarrow$ is clear. The converse implication is proved on page 18 of [3]. Q.E.D.

Remark. We assume $\chi(F) \neq 2$ in the rest of Section 3.

3.3. *Let Λ be an automorphism of G and σ a shearing in Δ with residual line L. Suppose $\nu(V) \geq 3$. Then $\Lambda\bar{\sigma}$ is a projective transvection or projective quasi-symmetry.*

Proof. We can assume $\bar{\sigma} \neq \bar{1}_V$. Put $\bar{\Sigma} = \Lambda\bar{\sigma}$; by 1.7 there is an isotropic line Fa in V such that $\Sigma Fa \neq Fa$. Let $\tau_{a,\lambda}$ be a non-trivial transvection in Δ with line Fa. Put $T = \tau_{a,\lambda}$; by 1.8, Σ and $T\Sigma^{-1}T^{-1}$ don't commute and the equation

$$[\Sigma, T] = \alpha T\Sigma^{-1}T^{-1}\Sigma \quad \text{with} \quad \alpha \in RL(V)$$

is impossible by a dimension argument. Put $\Lambda\bar{\tau} = \bar{T}$, $h = [\Sigma, T]$, and $f = [\sigma, \tau]$. Since $\bar{\Sigma}$ and $\bar{T}\bar{\Sigma}^{-1}\bar{T}^{-1}$ cannot commute, σ and $\tau\sigma^{-1}\tau^{-1}$ cannot commute; hence $L \neq \tau L$ and $(L, \tau L) \neq 0$. So $L + \tau L$ is the residual space of f.

Now h is the product of the transvections $\Sigma T\Sigma^{-1}$ and T^{-1} which have the district proper lines ΣFa and Fa. Hence h is a plane rotation with residual space $R = \Sigma Fa + Fa$. Because a is isotropic, R is either a hyperbolic plane or a totally degenerate plane; we will show R is a hyperbolic plane.

Note $\Sigma T\Sigma^{-1} = \tau_{\Sigma a,\lambda}$ and $T^{-1} = \tau_{a,-\lambda}$; from these formulas it follows $\Sigma T\Sigma^{-1}$ and T^{-1} both fix the plane R. Thus either $\Sigma T\Sigma^{-1}$ and T^{-1} both induce 1_R or they induce non-trivial transvections on R with distinct proper lines (depending on whether $(a, \Sigma a) = 0$ or $(a, \Sigma a) \neq 0$). In any case $h \mid R = [\Sigma, T] \mid R \neq -1_R$ since $\chi(F) \neq 2$. By 1.7 of [6] $h^2 \neq 1_V$; and surely $h^2 \neq \gamma \cdot 1_V$ with $\gamma \neq 1$. So $\bar{h}$ is not an involution and since $\Lambda\bar{f} = \bar{h}$, f is not an involution. So

f satisfies the hypotheses of 2.1 and so $E(L + \tau L) \subseteq CDC(f)$. Thus

$$E(L + \tau L)^{-} \subseteq (CDC(f))^{-} \subseteq CD(C(f))^{-} = CDC(\bar{f})$$

since $C(f)^{-} = C(\bar{f})$ by 1.11. Thus $CDC(\bar{f})$ is non-abelian, since both $\bar{\sigma}$ and $\bar{\tau}\bar{\sigma}^{-1}\bar{\tau}^{-1}$ are in $E(L + \tau L)^{-}$ and they don't commute. Hence $CDC(\bar{h})$ is non-abelian. So if R were totally degenerate 2.3 would imply $CDC(\bar{h})$ is abelian, a contradiction. Thus R is a hyperbolic plane.

Finally we show $\Lambda\bar{\sigma}$ is a projective shearing. We saw above

$$\bar{\sigma} \in E(L + \tau L)^{-} \subseteq (CDC(f))^{-} \subseteq CD(C(f))^{-} = CDC(\bar{f}).$$

So we have that $\Lambda\bar{\sigma} \in CDC(\Lambda\bar{f}) = CDC(\bar{h})$. By 2.2,

$$\bar{\Sigma} = \Lambda\bar{\sigma} \in CDC(\bar{h}) \subseteq E(R)^{-};$$

thus we may assume the residual space of Σ is contained in R. If R is the residual space of Σ and $\Sigma \mid R$ is a scalar, then since $CDC(\bar{h}) \subseteq E(R)^{-}$ we see $\bar{\Sigma}$ centralizes $CDC(\bar{h})$ which contradicts the fact $\bar{\sigma} \notin CCDC(\bar{f})$; if R is the residual space of Σ and $\Sigma \mid R$ isn't a scalar, the proof of 2.1 shows $E(R) \subseteq CDC(\Sigma)$, contradicting the fact $CDC(\bar{\sigma})$ is abelian. So Σ has residual space a line. Q.E.D.

Thus 3.3 says any automorphism Λ of G maps projective shearings to projective shearings under the assumptions we made. For the rest of Section 3 let us assume the hypotheses of Theorem 3.3 are in force; we are going to show that in fact Λ maps projective transvections to projective transvections. Note if σ_1 and σ_2 are shearings $\neq 1_V$ with residual spaces L_1 and L_2, then $\sigma_1\sigma_2 = \sigma_2\sigma_1$ iff $L_1 = L_2$ or $(L_1, L_2) = 0$. We also see that if $\bar{\sigma} \in G$ is a non-trivial shearing with residual space the line L, then $CC(\bar{\sigma}) = E(L)^{-}$.

DEFINITION. For a subspace W of V, let $S(W)$ be all projective shearings in G whose residual lines are contained in W. For a subset X of G, let $C'(X)$ be all projective shearings in G which commute with each element of X.

LEMMA 1. *Let $\bar{\sigma}$ and $\bar{\sigma}_2$ be non-trivial commuting shearings in G with distinct residual lines L_1 and L_2. Then*

$$C'C'(\bar{\sigma}_1, \bar{\sigma}_2) \subseteq S(L_1 + L_2); \quad \textit{and} \quad C'C'(\bar{\sigma}_1, \bar{\sigma}_2) = S(L_1 + L_2)$$

if $\bar{\sigma}_1$ and $\bar{\sigma}_2$ are both transvections.

Proof. Clearly

$$C'(\bar{\sigma}_1, \bar{\sigma}_2) = S((L_1 + L_2)^{*}) \cup S(L_1) \cup S(L_2).$$

Thus $C'C'(\bar{\sigma}_1, \bar{\sigma}_2) \subseteq S(L_1 + L_2)$. However if σ_1 and σ_2 are both transvections then $C'(\bar{\sigma}_1, \bar{\sigma}_2) = S((L_1 + L_2)^{*})$ and this implies

$$S(L_1 + L_2) = C'C'(\bar{\sigma}_1, \bar{\sigma}_2). \qquad \text{Q.E.D.}$$

LEMMA 2. *Under the hypotheses of* 3.3, *if $\bar{\sigma}_1$ is a projective transvection in G then $\Lambda\bar{\sigma}_1$ is also a projective transvection.*

Proof. If char $F = p > 0$, then each transvection $\neq 1$ has order p while no quasi-symmetry can have order p. So we can assume char $F = 0$.

We may suppose $\bar{\sigma}_1 \neq \bar{1}$; let $\bar{\sigma}_1$ have residual line L_1 and choose an isotropic line L_2 in V such that $(L_2, L_1) = 0$ and $L_2 \neq L_1$. Choose a non-trivial projective transvection $\bar{\sigma}_2$ in G with line L_2. Let the shearings $\Lambda\bar{\sigma}_1$ and $\Lambda\bar{\sigma}_2$ have lines L_1' and L_2'.

Then the totally degenerate plane $L_1 + L_2$ contains an infinite number of distinct pairwise orthogonal isotropic lines. Thus

$$S(L_1 + L_2) = C'C'(\bar{\sigma}_1, \bar{\sigma}_2)$$

contains an infinite number of distinct pairwise commuting projective transvections with pairwise distinct double centralizers. So $C'C'(\Lambda\bar{\sigma}_1, \Lambda\bar{\sigma}_2)$ contains an infinite number of distinct pairwise commuting projective shearings with pairwise distinct double centralizers. We know

$$C'C'(\Lambda\bar{\sigma}_1, \Lambda\bar{\sigma}_2) \subseteq S(L_1' + L_2')$$

so the plane $L_1' + L_2'$ contains an infinite number of pairwise distinct lines $K_1 \cdots K_m \cdots$ such that $(K_i, K_j) = 0$ if $i \neq j$. This implies the plane $L_1' + L_2'$ is totally degenerate which implies $\Lambda\bar{\sigma}_1$ is a projective transvection as desired, Q.E.D.

So we see (under the hypotheses of 3.3) the automorphism Λ of G carries projective transvections to projective transvections.

For L an isotropic line define $\bar{T}(L)$ to be the group of all projective transvections in G having proper line L. It follows from 1.3 that $\bar{T}(L)$ is a maximal group of projective transvections in G and that every maximal group of projective transvections in G has the form $\bar{T}(L)$ for some isotropic line L. Let Λ be an automorphism of G; by Lemma 2, $\Lambda\bar{T}(L)$ is a maximal group of projective transvections in G and hence there is a unique isotropic line L' such that $\Lambda\bar{T}(L) = \bar{T}(L')$. The map $L \to L'$ is easily seen to be a bijection of the isotropic lines of V. Since the commuting of two projective transvections is equivalent to the orthogonality of their proper lines, we see $(L_1, L_2) = 0$ if and only if $(L_1', L_2') = 0$ for any two isotropic lines L_1 and L_2. And the bijection inverse to $L \to L'$ is the one induced by the automorphism Λ^{-1} of G.

Now let L_1 and L_2 be two distinct orthogonal isotropic lines, and $\bar{\tau}_1$ and $\bar{\tau}_2$ be two nontrivial projective transvections in G with residual lines L_1 and L_2 respectively. We saw above that $C'C'(\bar{\tau}_1, \bar{\tau}_2)$ is equal to the set of all projective transvections in G whose residual lines fall in the totally degenerate plane $L_1 + L_2$. From this centralizer computation and the fact Λ preserves projective transvections, it follows that if $L_3 \subset L_1 + L_2$ then $L_3' \subset L_1' + L_2'$, and $L_1' + L_2'$ is a totally degenerate subspace of V.

3.4. *Suppose $\{L_i\}$, $1 \leqq i \leqq r$, is a finite set of pair-wise orthogonal isotropic lines. Then*

$$L_1 \subseteq L_2 + \cdots + L_r \Leftrightarrow L_1' \subseteq L_2' + \cdots + L_r'.$$

Proof. Consideration of Λ^{-1} shows it is enough to prove

$$L_1 \subseteq L_2 + \cdots + L_r \Rightarrow L_1' \subseteq L_2' + \cdots + L_r'.$$

The above remarks establish this for $r = 3$ and it trivially holds for $r = 1$ or 2. So let $r \geqq 4$ and induct on r. If $L_1 = L_r$ the theorem is trivial. Assume $L_1 \neq L_r$ so $L_1 \subseteq K + L_r$ where K is a line contained in $L_2 + \cdots + L_{r-1}$. K is isotropic so

$$K' \subseteq L_2' + \cdots + L_{r-1}'$$

by induction. And $L_1' \subseteq K' + L_r'$ since the theorem holds for $r = 3$. Hence

$$L_1' \subseteq K' + L_r' \subseteq L_2' + \cdots + L_r'. \qquad \text{Q.E.D.}$$

Now 3.4 implies the bijection $L \to L'$ of isotropic lines induced by Λ maps any finite set of independent pairwise orthogonal isotropic lines to independent pairwise orthogonal isotropic lines. For W any m-dimensional totally degenerate subspace of V, choose a base $x_1, \cdots, x_m$ for W and define W' to be the m-dimensional (totally degenerate) subspace of V spanned by the independent lines $(Fx_1)', \cdots, (Fx_m)'$. Using 3.4 it is easily seen W' is independent of the particular base $x_1, \cdots, x_m$ chosen for W. So Λ induces a well-defined dimension-preserving map $W \to W'$ of the totally degenerate subspaces of V. This map is easily seen to be a dimension-preserving bijection of the totally degenerate subspaces of V. And the bijection inverse to $W \to W'$ is the one induced by the automorphism Λ^{-1} of G.

Now the bijection $W \to W'$ of totally degenerate subspaces that Λ induces satisfies the hypothesis of the theorem given on page 82 of [3]. Applying that theorem we conclude there is a unitary semi-similitude g of V onto V such that $gW = W'$ for all totally degenerate subspaces W of V of dimension $\nu(V) - 1$.

Let L be an isotropic line of V. Choose a maximal totally degenerate subspace W of V with $L \subseteq W$. Then $L = \bigcap_\alpha W_\alpha$ where $\{W_\alpha\}$ is the family of all subspaces of W of dimension $\nu(V) - 1$ which contain L. We have $gL = \bigcap_\alpha gW_\alpha = \bigcap_\alpha W_\alpha' = L'$. Hence $gL = L'$ for all isotropic lines L of V. One easily sees $\bar{\Lambda}_{g^{-1}} \circ \Lambda$ is an isomorphism of G into $PU_n(V)$ that satisfies the hypotheses of 3.1. Hence $\Lambda = \bar{\Lambda}_g$ and we have proved

3.5. THEOREM. *Let G be a subgroup of $PU_n(V)$ which has enough projective transvections and suppose $\nu(V) \geqq 3$, $\chi(F) \neq 2$. Let Λ be an automorphism of G. Then there is a unitary semi-similitude g of V onto V such that $\Lambda = \bar{\Lambda}_g \mid G$.*

3.5a. COROLLARY. *Let $\nu(V) \geqq 3$, $\chi(F) \neq 2$ and let S be a subgroup of $U_n(V)$ that has enough transvections. Let Λ be an automorphism of S. Then there is a homomorphism χ of S into the center of $U_n(V)$ and a unitary semi-similitude g of V such that $\Lambda\sigma = \chi(\sigma)\cdot g\sigma g^{-1}$ for all $\sigma \in S$.*

Proof. Λ induces an automorphism $\bar{\Lambda}$ of $\bar{S}$ given by $\bar{\Lambda}(\bar{\sigma}) = (\Lambda\sigma)^-$ for all $\sigma \in S$. Thus $\bar{\Lambda} = \bar{\Lambda}_g$ by 3.5 and so $(\Lambda\sigma)^- = (\Lambda_g(\sigma))^-$ for all $\sigma \in S$, or $\Lambda\sigma =$

$\chi(\sigma) \cdot \Lambda_g(\sigma)$, $\chi(\sigma)$ a scalar in $U_n(V)$. Since Λ is an automorphism, χ is a homomorphism. Q.E.D.

4. The automorphisms of the unitary congruence groups

Let $\mathfrak{o}$ be an integral domain of any characteristic and for $\alpha \in \mathfrak{o}$ let the map $\alpha \to \alpha^*$ be a non-trivial automorphism of $\mathfrak{o}$ of period two. Let F be the quotient field of $\mathfrak{o}$; the automorphism * of $\mathfrak{o}$ has a natural extension to an automorphism of F which we again denote by *.

Let V be an n-dimensional vector space over F and (x, y) be a non-degenerate hermitian form defined on V taking values in F. Let E denote the fixed field of the * map.

4.1. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be any two fractional ideals of $\mathfrak{o}$. Then there is a non-zero λ in F of trace zero such that $\lambda\mathfrak{a} \subseteq \mathfrak{b}$.*

Proof. It is enough to prove this under the assumption $\mathfrak{a} \subseteq \mathfrak{o}$, $\mathfrak{b} \subseteq \mathfrak{o}$. Let β be a non-zero element of $\mathfrak{b}$. Since $\mathfrak{o}^* = \mathfrak{o}$, $\beta\beta^* \in \mathfrak{b}$. Clearly we can choose a non-zero t in $\mathfrak{o}$ of trace zero. Then $\lambda = \beta\beta^* t$ does the job. Q.E.D.

DEFINITION. An $\mathfrak{o}$-module M is bounded if there exists an $\mathfrak{o}$-linear isomorphism of M into some free $\mathfrak{o}$-module of finite dimension.

Let M be a bounded $\mathfrak{o}$-module contained in V such that $FM = V$ where $FM = \{\alpha x \mid \alpha \in F, x \in M\}$.

For any vector $a \in V$, define the coefficient of a, $\mathfrak{c}_a$, as $\{\alpha \in F \mid \alpha a \in M\}$. $\mathfrak{c}_a$ is a fractional ideal of $\mathfrak{o}$. And if ρ is a non-zero linear functional on V, then $\rho(M)$ is a fractional ideal of $\mathfrak{o}$.

We define the three integral unitary groups $T_n(M)$, $U_n^+(M)$, and $U_n(M)$ as follows:

$$U_n(M) = \{\sigma \in U_n(V) \mid \sigma M = M\}, \qquad U_n^+(M) = U_n(M) \cap SL_n(V)$$

and $T_n(M)$ is the group generated by all transvections in $U_n(M)$.

Now let $\mathfrak{a}$ be a non-zero ideal in $\mathfrak{o}$. Put

$$\mathfrak{a} \cdot M = \{\textstyle\sum_i \alpha_i x_i \mid \alpha_i \in \mathfrak{a}, x_i \in M\}$$

and define the unitary congruence groups to be

$$U_n(M; \mathfrak{a}) = \{\sigma \in U_n(M) \mid (\sigma - 1_V)M \subseteq \mathfrak{a} \cdot M\}$$

$$U_n^+(M; \mathfrak{a}) = U_n(M; \mathfrak{a}) \cap SL_n(V)$$

and $T_n(M; \mathfrak{a})$ is the group generated by all transvections in $U_n(M; \mathfrak{a})$. We see $T_n(M; \mathfrak{a}) \subseteq U_n^+(M; \mathfrak{a}) \subseteq U_n(M; \mathfrak{a})$ are normal subgroups of $U_n(M)$, and $T_n(M; \mathfrak{o}) = T_n(M)$, $U_n^+(M; \mathfrak{o}) = U_n^+(M)$, $U_n(M; \mathfrak{o}) = U_n(M)$.

The projective unitary congruence groups

$$PTL_n(M; \mathfrak{a}), \quad PU_n^+(M; \mathfrak{a}), \quad PU_n(M; \mathfrak{a})$$

are defined to be the images of $T_n(M; \mathfrak{a})$, $U_n^+(M; \mathfrak{a})$, $U_n(M; \mathfrak{a})$ respectively under the $^-$ map of $U_n(V)$ onto the quotient of $U_n(V)$ mod its center.

If $\tau_{a,\lambda} \in U_n(V)$, it is easy to see

$$\lambda(M, a) \subseteq \mathfrak{c}_a \cdot \mathfrak{a} \Rightarrow \lambda(M, a)a \subseteq \mathfrak{a} \cdot M \Rightarrow \tau_{a,\lambda} \in TL_n(M; \mathfrak{a}).$$

4.2. *For $n \geqq 2$, $TL_n(M; \mathfrak{a})$ has enough transvections.*

Proof. Let $L = Fa$ be an isotropic line of V; (x, a) is a non-zero linear functional on V and so (M, a) is a fractional ideal of $\mathfrak{o}$. Using 4.1, choose a non-zero λ in F of trace zero such that $\lambda(M, a) \subseteq \mathfrak{c}_a \cdot \mathfrak{a}$; since λ has trace zero, $\tau_{a,\lambda} \in U_n(V)$. The above remarks show $\tau_{a,\lambda} \in TL_n(M; \mathfrak{a})$. Q.E.D.

4.3. *Let $\nu(V) \geqq 3$, $\chi(F) \neq 2$; let G be one of the groups*

$$PU_n(M; \mathfrak{a}), \quad PU_n^+(M; \mathfrak{a}) \quad or \quad PTL_n(M; \mathfrak{a}),$$

and suppose Λ is an automorphism of G. Then there is a unitary semi-similitude g of V onto V such that $\Lambda = \bar{\Lambda}_g | G$.

Proof. 4.2 implies G has enough projective transvections. Now apply 3.5. Q.E.D.

4.4. Theorem. *Let $\nu(V) \geqq 3$, $\chi(F) \neq 2$, let S be one of the unitary congruence groups $U_n(M; \mathfrak{a})$, $U_n^+(M; \mathfrak{a})$ or $TL_n(M; \mathfrak{a})$, and suppose Λ is an automorphism of S. Then there is a homomorphism χ of S into $RL(V) \cap U_n(V)$ and a unitary semi-similitude g of V onto V such that $\Lambda\sigma = \chi(\sigma) \cdot \Lambda_g(\sigma)$ for all $\sigma \in S$.*

Proof. Apply 3.5a and 4.2. Q.E.D.

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