BORDISM J-HOMOMORPHISMS

BY
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1. Introduction

In these notes we introduce various generalizations of the bordism J-homomorphisms of [3] and [6]. We compute ImJ in a few of the easier cases, and discuss the geometrical implications of our results. For all notation, see [7].

In §2, we set up a diagram of the form

$$\begin{array}{ccc} \pi_*(G_i/G_j) & & \\ \Big\downarrow & \searrow^{J} & \\ & & (MG_j)_* \\ & \nearrow_{J} & \\ (MG_k)_*(G_i/G_{j+}) & & \end{array}$$

for each triple of stable subgroups $G_k \leq G_j < G_i$ of Top. The case $G_i = F$, homotopy equivalences of the sphere, may often be incorporated. Both $\pi_*(G_i/G_j)$ and $(MG_k)_*(G_i/G_{j+})$ admit geometrical interpretations, and we also discuss the J-homomorphisms from this angle. A host of examples is given in §3.

We set up in §4 a homological formula. This eases our calculations with the MU_* J-homomorphisms, the cases we shall mainly deal with. Rationally, the sums are easy, so we do them in §5. Their simplicity notwithstanding, they do in many cases give good insight into the structure of ImJ.

Thus fortified, in §§6 and 7 we compute completely the images of $J:\pi_*(SO/U) \to MU_*$ and $J:MU_*(SO/U) \to MU_*$. In neither case is the image a direct summand. In §8, we explain our results in terms of unitary structures on manifolds.

The development of the ideas herein was much influenced by many enjoyable discussions with Bob Switzer and Reg Wood. Idar Hansen was also very helpful.

2. Constructions

Let us first consider a triple of stable subgroups $G_k \leq G_j < G_i$ of the orthogonal group O. We shall later, with due care, extend many of our constructions to include the cases PL, Top and F.

Choose integers ε and ε_i such that $G_i(\varepsilon_i n)$ and $G_j(n)$ act on $\mathbf{R}^{\varepsilon n}$ (e.g. if $G_i = U$ and $G_j = Sp$ then $\varepsilon = 4$ and $\epsilon_i = 2$). Then as explained in [6],

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the null homotopic composition

$$G_i(\varepsilon_i n)/G_j(n) \rightarrow BG_j(n) \rightarrow BG_i(\varepsilon_i n)$$

gives rise to a map of Thom complexes

$$S^{\varepsilon n} \wedge G_i(\varepsilon_i n)/G_j(n)_+ \rightarrow MG_j(n).$$

Letting $n \rightarrow \infty$ and adjointing, this may be written as

$$j: G_i/G_{j+} \rightarrow \Omega^\infty MG_j(\infty).$$

Then $j_*: \pi_*(G_i/G_j) \rightarrow (MG_j)_*$ is our relative J-homomorphism: if $G_i = O$ and $G_j = 1$ it reduces to the classical case.

Now consider the commutative diagram

$$\begin{array}{ccc}
G_i/G_{j+} & \xrightarrow{\ j\ } & \Omega^\infty MG_j(\infty) \\
\downarrow{\scriptstyle f_1} & & \uparrow \\
\Omega^\infty(MG_k(\infty) \wedge G_i/G_{j+}) & & p \\
\downarrow{\scriptstyle f_2} & & \big| \\
\Omega^\infty(MG_j(\infty) \wedge \Omega^\infty MG_j(\infty)) & \longrightarrow & \big|
\end{array} \quad .$$

Here f_1 is the composition

$$G_i/G_{j+} \hookrightarrow \Omega^\infty(S^\infty \wedge G_i/G_{j+}) \hookrightarrow \Omega^\infty(MG_k(\infty) \wedge G_i/G_{j+})$$

(if $G_k = 1$ this second inclusion is the identity) whilst f_2 is induced by the smash product of the inclusion $MG_k(\infty) < MG_j(\infty)$ with the map j. p arises from the sum operation on G_j-bundles; we shall restrict ourselves to considering groups for which this operations exists.

Applying the functor π_*, we arrive at

$$\begin{array}{ccc}
\pi_*(G_i/G_j) & & \\
\downarrow & \searrow{\scriptstyle J} & \\
 & & (MG_j)_*. \\
 & \nearrow{\scriptstyle J} & \\
(MG_k)_*(G_i/G_{j+}) & &
\end{array}$$

As an alternative approach for those suspicious of infinite loop spaces, this diagram may be induced from a corresponding diagram of spectra.

Furthermore, note that

$$(MG_k)_*(G_i/G_{j+}) = (MG_k)_*(G_i/G_j) \oplus (MG_k)_*$$

and $\pi_*(G_i/G_j)$ maps into the summand $(MG_k)_*(G_i/G_j)$. Also, J on the summand $(MG_k)_*$ is the standard homomorphism induced by $G_k < G_j$.

Thus we have the diagram

$$
\begin{array}{ccc}
\pi_* G_i/G_j & & \\
\downarrow & \searrow J & \\
(MG_k)_*(G_i/G_j) & \xrightarrow{J} & (MG_j)_* \\
\wedge & \nearrow J & \\
(MG_k)_*(G_i/G_{j+}) & &
\end{array} \tag{2.1}
$$

As indicated, we shall simply write J for all our homomorphisms, and trust the context makes clear the meaning in each case. All the author's attempts to incorporate the G_i, G_j and G_k labels have failed to avoid hopelessly complex notation.

Now let us interpret our groups and homomorphisms geometrically. Suppose $f: S^n \to G_i/G_j$ represents a class in $\pi_n(G_i/G_j)$. Then f is equivalent to a G_j-structure on the stable normal G_i-bundle of S^n, which is trivial. A homotopy of $f \simeq g$ in turn yields a tube with G_j structure on its stable normal G_i-bundle which restricts to the two given structures at its ends. So $\pi_*(G_i/G_j)$ may be thought of as the h-bordism group of G_i-spheres with G_j-structure imposed.

Then $J: \pi_*(G_i/G_j) \to (MG_j)_*$ simply considers such spheres as G_j-manifolds: the h-bordism relation is compatible with the usual bordism relation in $(MG_j)_*$.

Next, define a $(G_k, G_j: G_i)$ manifold as one which admits independent G_k and G_j structures which agree as G_i structures. Such manifolds can be organized into a $(G_k, G_j: G_i)$-bordism group, which we shall briefly label $\Omega_*^{(G_k, G_j: G_i)}$. Given a $(G_k, G_j: G_i)$-manifold M^n, let ν_k and ν_j respectively represent the G_k and G_j stable normal bundles. Then considered as a G_i-bundle, the (virtual) G_j-bundle $\nu_k - \nu_j$ is trivial. So we have a map $f: M^n \to G_i/G_j$.

Similarly, given any G_k manifold N^n with stable normal bundle η_k and a map $g: N^n \to G_i/G_j$, g represents a G_j-bundle γ_j over N^n which is trivial as a G_i-bundle. Thus η_k and $\eta_k - \gamma_j$ define a $(G_k, G_j: G_i)$ structure on N^n.

It is routine to check that the two functions so defined respect each bordism relation and give rise to mutually inverse homomorphisms between $\Omega_*^{(G_k, G_j: G_i)}$ and $(MG_k)_*(G_i/G_{j+})$. This yields our geometrical description of the latter group. Similar observations have been made by Stong [9].

Note that the summand $(MG_k)_*(G_i/G_j)$ corresponds to the subgroup of $\Omega_*^{(G_k, G_j: G_i)}$ consisting of those classes representable by a manifold G_k-bordant to zero. In either case, the J-homomorphism takes a $(G_k, G_j: G_i)$-manifold and considers its class in $(MG_j)_*$: again the bordism relations are compatible.

The proof that these descriptions agree with our earlier homotopy constructions is a simple exercise in understanding the Pontrjagin-Thom construction.

Thus $\mathrm{Im}\{J:\pi_*(G_i/G_j) \to (MG_j)_*\}$ consists of those G_j-bordism classes representable by G_i-spheres. $\mathrm{Im}J$ is a subgroup.

Similarly, $\mathrm{Im}\{J:(MG_k)_*(G_i/G_{j+}) \to (MG_j)_*\}$ consists of those G_j-bordism classes representable by a G_j-manifold M which can be given a new G_k-structure so as to agree with the original up to G_i. $\mathrm{Im}J$ is a subring. Note that if $G_k = G_j$ then J is epic.

$\mathrm{Im}\{J:(MG_k)_*(G_i/G_j) \to (MG_j)_*\}$ is the subring of the above in which each M is G_k-bordant to zero when invested with its new G_k-structure. Note that if $G_k = G_j$, then $\mathrm{Im}J$ is an ideal in $(MG_j)_*$.

Finally, let us mention the exotic cases. Since fibre homotopically trivial bundles have trivial Thom complexes, if $G_i = F$ we still have

$$j:F/G_{j+} \to \Omega^\infty MG_j(\infty).$$

Further, both Top and PL bundles can be invested with a sum operation, so either is admissible as G_j. Hence diagram (2.1) also exists for these choices: the extra problems are those of interpretation, and differ from case to case. Since many of our examples in §3 are of this nature, we shall discuss each one there. Not only do PL, Top and F give rise to some of the most interesting examples of J, but also they are amongst the most difficult to compute.

3. Some examples

(3.1) $G_i = F, G_j = G_k = PL$ *or* Top. $\pi_*(F/PL)$ represents the bordism group of almost framed PL-manifolds, and is well known to be periodic of order 4 (see [11]). $J:\pi_*(F/PL) \to MPL_*$ simply inserts such classes into the PL bordism ring, and Im J has recently been implicitly computed by Brumfiel, Madsen and Milgram [1]. It consists of a $\boldsymbol{Z}_2$ in dimensions $4n - 2$, $n \neq 2^j$, and in dimensions $4n$, $n > 1$.

$\pi_*(F/\mathrm{Top})$ represents the bordism group of almost framed topological manifolds, and has the same structure as $\pi_*(F/PL)$. Interpreting J in low dimensions needs care, since $\pi_*(\Omega^\infty \mathrm{Top}(\infty))$ is converted to the topological bordism ring by transversality. However, the results for $\mathrm{Im}J$ coincide with the PL case [1].

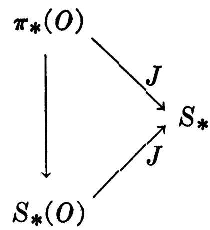
$MPL_*(F/PL_+)$ is, according to Sullivan, the correct setting in which to regard surgery problems [11]. A representative $f:M^n \to F/PL$ (or surgery problem) gives rise to a normal map of PL manifolds $L^n \to M^n$, and $J:MPL_*(F/PL_+) \to MPL_*$ sends $[f]_{PL}$ to $[L^n]_{PL}$. Note f may also be thought of as giving rise to a manifold with two distinct PL structures which agree as spherical fibrations. In fact

$$\mathrm{Im}\{J:MPL_*(F/PL) \to MPL_*\}$$

consists of the ideal of those PL bordism classes representable by a PL manifold L^n admitting a degree 1 normal map to a PL boundary M^n.

Similar remarks, made with care, hold in the topological case.

(3.2) $G_i = O, G_j = G_k = 1$. In the diagram



the upper J is the classical J homomorphism, and the lower is the standard stable J-homomorphism. Stable J has been discussed, for example, in [8], and is shown to be epic on the 2-primary component. It is conjectured, with supporting evidence, that this is true for all primes p. If this is correct it means that each framed bordism class contains a manifold reframable to zero. $S_*(O_+)$ is the biframed bordism ring.

(3.3) $G_i = O, G_j = G_k = SO$. This example is at the other end of the smooth bordism scale from (3.2). Corresponding to the stable J of (3.2) is $J: MSO_*(O/SO) \to MSO_*$. This J has as image those oriented bordism classes representable by a manifold which can be re-oriented so as to be an oriented boundary. Clearly J is zero. Such a basic difference between (3.2) and (3.3) prompts the question as to what happens in between these two extreme cases, and attempting to give an answer will be the main function of the following sections.

(3.4) $G_i = F, G_j = G_k = O$. $\pi_*(F/O)$ represents the bordism ring of almost framed smooth manifolds. $MO_*(F/O_+)$ is the bordism ring of manifolds with distinct smooth structures which agree up to fibre homotopy equivalence. More recognisably, $f: M^n \to F/O$ gives rise to a smooth normal map of degree 1, say $f': N^n \to M^n$, i.e. a smooth surgery problem.

Thus $\operatorname{Im}\{J: MO_*(F/O) \to MO_*\}$ is the ideal of those smooth bordism classes representable by a manifold N^n which admits a smooth, normal map f' into a smooth boundary M^n.

(3.5) LEMMA. $\operatorname{Im} J = 0$.

Proof. Let us compute a Stiefel-Whitney number of such an N^n. For $\omega_\alpha \in HZ_2^{|\alpha|}(BO)$,

$$\omega_\alpha(N^n) = \langle [N^n], \omega_\alpha(\nu_N) \rangle$$

where ν_N is the stable normal bundle of N and $[N^n]$ is the fundamental $\mathbf{Z}_2$ class. Then we have $f'_*[N^n] = [M^n]$ and $f'^*\nu_N \simeq \nu_M$ so

$$\omega_\alpha(N^n) = \langle [N^n], \omega_\alpha(f'^*\nu_M) \rangle = \langle [M^n], \omega_\alpha(\nu_M) \rangle = \omega_\alpha(M^n) = 0.$$

So N^n is itself a boundary, as sought. □

Thus both J's are zero in

$$\begin{array}{ccc} \pi_*(F/O) & & \\ \downarrow & \overset{J}{\searrow} & \\ & & MO_*. \\ & \overset{J}{\nearrow} & \\ MO_*(F/O) & & \end{array}$$

(3.6) $G_j = U$.

(i) $G_k = U, G_i = F$. We have the diagram

$$\begin{array}{ccc} \pi_*(F/U) & & \\ \downarrow & \overset{J}{\searrow} & \\ & & MU_*. \\ & \overset{J}{\nearrow} & \\ MU_*(F/U) & & \end{array}$$

Neither of the J-homomorphisms is zero, since they factor through others known to be non-trivial (see (ii) below). However, computation seems most awkward. $\pi_*(F/U)$ can be described as the bordism group of almost framed U-manifolds, whilst $MU_*(F/U_+)$ represents surgery problems of U-manifolds.

(ii) $G_i = O$ (*or* SO), $G_j = U$, $G_k = 1$ *or* U. These choices furnish many of our computable cases. We have

$$\begin{array}{ccc} \pi_*(SO/U) & & \\ \downarrow & \overset{J}{\searrow} & \\ S_*(SO/U) & \xrightarrow[J]{} & MU_*. \\ \downarrow & \overset{J}{\nearrow} & \\ MU_*(SO/U) & & \end{array}$$

$\pi_*(SO/U)$ is the h-bordism group of oriented smooth spheres with U structure, and ImJ measures which U-bordism classes are so representable. It transpires that $[CP^1]_U \in \text{Im}J$, so J is not zero.

$S_*(SO/U_+)$ is the bordism group of (framed, U:SO)-manifolds, whilst $MU_*(SO/U_+)$ is the bordism ring of oriented smooth manifolds with two distinct U-structures. Thus Im$\{J:MU_*(SO/U) \to MU_*\}$ is the ideal of bordism classes representable by a U-manifold which can be given a second U-structure so as to be U bordant to zero. It is this J-map in particular which falls midway between (3.2) and (3.3). If J is epic (as seems likely in the case (3.2)) it means that in some sense the underlying manifolds are of minor importance in MU_* whilst the U-structures capture most of the information. If ImJ is small, as in the oriented case, then the converse is true. In fact, ImJ turns out to be an interesting proper ideal, as explained in §7.

(3.7) *$G_j = Sp$; G_i and G_k as in* (3.6). We have

$$\begin{array}{ccc} \pi_*(SO/Sp) & & \\ & \searrow^{J} & \\ \downarrow & & MSp_* \\ & \nearrow_{J} & \\ MSp_*(SO/Sp) & & \end{array}$$

as our main sources of interest. The upper J was investigated in [6], and turned out to have image an interesting $\oplus \boldsymbol{Z}_2$-summand. The lower ImJ is more mysterious, being the ideal of those Sp bordism classes represented by a manifold which can be given a new Sp structure so as to be an Sp-boundary.

Note that in all such cases it is not necessarily sufficient to choose some G-manifold representing an element of MG_* and check whether it admits a second and bounding G-structure. Even though it does not, some other G bordant manifold might admit such a change, whence the bordism class will still be in ImJ.

4. A homology formula

In this section we derive a formula relating to

$$J : MU_*(SO/U) \to MU_*,$$

(and implicitly to $J : \pi_*(SO/U) \to MU_*$) which will be vital for our subsequent sums.

For any $x \in MU_k(SO/U)$, choose N sufficiently large such that x is in the image of $MU_k(SO(2N)/U(N))$. We have the Thom complex diagram

$$\begin{array}{ccc} S^{2N} \wedge SO(2N)/U(N)_+ & \xrightarrow{M(i)} & MU(N) \\ \cup & & \cup \\ SO(2N)/U(N) & \xrightarrow[i]{} & BU(N). \end{array}$$

Then as explained in §2, $J(x) = \mu M(i)_*(g_{2N} \otimes x)$, where g_{2N} is the canonical generator of $MU_{2N}(S^{2N})$ and

$$\mu : MU_{k+2N}(MU(N)) \to MU_k$$

is induced by the product in $\mathbf{MU}$.

Now let H_* denote the integral homology functor, and write $\underline{h}$ for the associated hurewicz homomorphism. In our case,

$$\underline{h} : MU_*(SO(2N)/U(N)_+) \to H_*(\mathbf{MU}) \otimes H_*(SO(2N)/U(N)_+).$$

Remember also that

$$H_*(BU_+) = \boldsymbol{Z}[b_1, b_2, \cdots] \quad \text{and} \quad H_*(\mathbf{MU}) = \boldsymbol{Z}[b_1', b_2', \cdots],$$

with $\varphi: b_1' \rightarrowtail b_i$ the accompanying (multiplicative) Thom isomorphism. Then we have

(4.1) PROPOSITION. $\underline{h}J(x) = \mu_*(1 \otimes \varphi^{-1} i_*)\underline{h}(x)$ *in* $H_k(\mathbf{MU})$.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} MU_*(S^{2N} \wedge SO(2N)/U(N)_+) & \xrightarrow[M(i)_*]{} & MU_*(MU(N)) \\ \downarrow{\scriptstyle \underline{h}} & & \downarrow{\scriptstyle \underline{h}} \\ H_*(\mathbf{MU}) \otimes H_*(S^{2N} \wedge SO(2N)/U(N)_+ & \xrightarrow[1 \otimes M(i)_*]{} & H_*(\mathbf{MU}) \otimes H_*(MU(N)). \end{array}$$

Then $\underline{h}(g_{2N} \otimes x) = \sum b'_\alpha \otimes h_{2N} \otimes d_\alpha$, where b'_α ranges over all products of the b_i's in dimensions $\leq k$, h_{2N} is the usual generator for $H_{2N}(S^{2N})$, and the d_α's are certain elements in $H_{k-2|\alpha|}(SO(2N)/U(N)_+)$: so

$$\underline{h}M(i)_*(g_{2N} \otimes x) = \sum b'_\alpha \otimes M(i)_*(h_{2N} \otimes d_\alpha).$$

But in homology, $\varphi M(i)_*(h_{2N} \otimes d_\alpha) = i_* d_\alpha$, since there is essentially a unique Thom class.
Thus

$$\begin{aligned} \underline{h}M(i)_*(g_{2N} \otimes x) &= \sum b'_\alpha \otimes \varphi^{-1} i_* d_\alpha \\ &= (1 \otimes \varphi^{-1} i_*) \sum b'_\alpha \otimes d_\alpha \\ &= (1 \otimes \varphi^{-1} i_*)\underline{h}(x). \end{aligned}$$

Finally,

$$\begin{aligned} \underline{h}(Jx) &= \underline{h}\mu M(i)_*(g_{2N} \otimes x) \\ &= \mu_* \underline{h}M(i)_*(g_{2N} \otimes x) \\ &= \mu_*(1 \otimes \varphi^{-1} i_*)\underline{h}(x). \quad \square \end{aligned}$$

In fact, if $\mathbf{E}$ denotes $\mathbf{KU}$ or $\mathbf{MU}$ the above diagram still exists, and we might compute a formula for $\underline{e}J(x) \in E_k(\mathbf{MU})$. However, the non-uniqueness of the Thom class in these cases makes the results far less easy to work with.

We could also arrive at (4.1) by geometrical considerations. For given that x is represented by $f: M^k \to SO(2N)/U(N)$, computing $\underline{h}J(x)$ amounts to finding the chern numbers of M^k equipped with the stable normal bundle $\nu_M \oplus i_* f$. This method is equally simple to the above.

Since $\underline{h}: MU_* \to H_*(\mathbf{MU})$ is well understood (and monic), we have now reduced our task to evaluating $\underline{h}$ on $MU_*(SO/U_+)$ (and $\pi_*(SO/U_+)$), and calculating $i_*: H_*(SO/U) \to H_*(BU)$.

5. Rational calculations

Before getting embroiled in the precise details of our computations, it is both easy and illuminating to tensor various of our problems with the rationals $\mathbf{Q}$. Indeed, these are the best results we are able to obtain in certain cases.

Let us first consider

$$\begin{array}{ccc}
\pi_*(SO/U)\otimes Q & & \\
\downarrow & \searrow^{J\otimes 1} & \\
S_*(SO/U)\otimes Q & \xrightarrow{J\otimes 1} & MU_*\otimes Q. \\
\downarrow & \nearrow_{J\otimes 1} & \\
MU_*(SO/U)\otimes Q & &
\end{array}$$

In the light of (4.1), to compute the lower map we must know how $i_*\otimes 1$: $H_*(SO/U)\otimes\mathbf{Q}\to H_*(BU)\otimes Q$ works.

From Cartan [2] we learn the following:

(5.1) THEOREM. *$i^*\otimes 1{:}H^*(BU)\otimes\mathbf{Q}\to H^*(SO/U)\otimes\mathbf{Q}$ is epic, and $H^*(SO/U_+)$ is polynomial over $\mathbf{Z}$, generated by classes $y_{2i+1}=\frac{1}{2}i^*c_{2i+1}$.* □

(5.2) COROLLARY. *$MU_*(SO/U_+)\cong MU_*\otimes H_*(SO/U_+)$, which is also true rationally.*

Proof. The associated Atiyah-Hirzebruch-Whitehead spectral sequence collapses. □

(5.3) COROLLARY. *$i_*{:}H_*(SO/U)\to H_*(BU)$ is monic. Rationally $\operatorname{Im}(i_*\otimes 1)$ is the polynomial subalgebra of $H_*(BU)$ generated by $\{b_1, b_3, \cdots, b_{2n+1}, \cdots\}$. We shall write this as $\tilde{Q}[b_1, b_3, \cdots, b_{2n+1}, \cdots]$.* □

Regard the commutative diagram

$$\begin{array}{ccc}
\pi_*(SO/U)\otimes\mathbf{Q}\to S_*(SO/U)\otimes\mathbf{Q} & \xrightarrow{\underline{h}\otimes 1} & H_*(SO/U)\otimes\mathbf{Q} \\
\downarrow & & \downarrow \\
MU_*(SO/U)\otimes\mathbf{Q} & \xrightarrow{\underline{h}\otimes 1} & H_*(\mathbf{MU})\otimes H_*(SO/U)\otimes\mathbf{Q} \\
 & & \downarrow 1\otimes i_*\otimes 1 \\
\downarrow J & & H_*(\mathbf{MU})\otimes H_*(BU)\otimes\mathbf{Q} \\
 & & \downarrow (\mu_*\otimes 1)(1\otimes\varphi^{-1}\otimes 1) \\
MU_*\otimes\mathbf{Q} & \xrightarrow{\underline{h}\otimes 1} & H_*(\mathbf{MU})\otimes\mathbf{Q}
\end{array}$$

where each of the hurewicz homomorphisms is well known to be a rational isomorphism. So to compute $\mathrm{Im}(J\otimes 1)$, it suffices to know

$$\mathrm{Im}(\mu_*\otimes 1)(1\otimes i_*\varphi^{-1}\otimes 1) \leq H_*(\mathbf{MU})\otimes\mathbf{Q}.$$

(5.4) PROPOSITION. *We can choose generators* $\{x_1, x_2, \cdots, x_n, \cdots\}$ *for* MU_* *such that*

$$\mathrm{Im}\{J\otimes 1: MU_*(SO/U)\otimes\mathbf{Q}\to MU_*\otimes\mathbf{Q}\}$$

is the ideal $(x_1, x_3, \cdots, x_{2n+1}, \cdots)$.

Proof. By (5.3),

$$\begin{aligned}\mathrm{Im}(\mu_*\otimes 1)(1\otimes i_*\varphi^{-1}\otimes 1) &= \mathbf{Q}[b_1', b_2', b_3', \cdots]\otimes\tilde{\mathbf{Q}}[b_1', b_3', \cdots]\\ &= (b_1', b_3', \cdots, b_{2n+1}', \cdots)\lhd H_*(\mathbf{MU})\otimes\mathbf{Q}.\end{aligned}$$

Then choosing, for example, Cohen's generators for MU_* (see Stong [10]) yields our result. □

(5.5) COROLLARY. $\mathrm{Im}\{J\otimes 1: S_*(SO/U)\otimes\mathbf{Q}\to MU_*\otimes Q\}$ *is the subalgebra* $\tilde{\mathbf{Q}}[x_1, x_3, x_5, \cdots]$. □

(5.6) COROLLARY. $\mathrm{Im}\{J\otimes 1: \pi_*(SO/U)\otimes\mathbf{Q}\to MU_*\otimes\tilde{\mathbf{Q}}\}$ *is a subgroup* $\mathbf{Q}$ *in each dimension* $\equiv 2, 6\ (8)$ *with generators* x_{4k+1}, x_{4k+3} *mod decomposables.*

Proof. We have the commutative diagram

$$\begin{array}{ccc}\pi_*(SO/U)\otimes\mathbf{Q} & \xrightarrow{\ i_*\otimes 1\ } & \pi_*(BU)\otimes\mathbf{Q}\\ \Big\downarrow{\scriptstyle \underline{h}\otimes 1} & & \Big\downarrow{\scriptstyle \underline{h}\otimes 1}\\ H_*(SO/U)\otimes\mathbf{Q} & \xrightarrow[\ i_*\otimes 1\ ]{} & H_*(BU)\otimes\mathbf{Q}.\end{array}$$

$\pi_*(BU)\otimes\mathbf{Q}$ is $\mathbf{Q}$ in even dimensions, and $i_*\otimes 1$ is an isomorphism in dimensions $\equiv 2,6\ (8)$. Also, $\underline{h}\otimes 1$ maps the generator of $\pi_{2n}(BU)\otimes\mathbf{Q}$ into $p_n\otimes 1\in H_{2n}(BU)\otimes\mathbf{Q}$, where p_n is the n-th primitive $b_{(0,\ldots,0,1)}$ and is dual to c_n. Now

$$p_n\otimes 1 = (-1)^n n b_n\otimes 1$$

mod decomposables, so applying $(\underline{h}\otimes 1)^{-1}$ into MU_*, we get our result. □

Let us now briefly mention some symplectic cases. Firstly, observe that the Sp analogue to (4.1) still holds good, and for $x\in MSp_k(SO/Sp)$,

$$\underline{h}J(x) = \mu_*(1\otimes\varphi^{-1}i_*)\underline{h}(x) \quad\text{in}\quad H_k(\mathbf{MSp}).$$

This time

$$i: SO/Sp\to BSp,\qquad \varphi: H_*(\mathbf{MSp})\rightarrowtail\!\!\!\twoheadrightarrow H_*(BSp_+)$$

and μ is induced by the product in $\mathbf{MSp}$.

Now $\pi_*(SO/Sp)$ is 2-primary (see [6]), and CO/Sp is simply connected. Thus $H_*(SO/Sp)\otimes\mathbf{Q} = 0$, so we have $S_*(SO/Sp)\otimes\mathbf{Q} = 0$ also. Further,

$$\underline{h}\otimes 1 : MSp_*(SO/Sp)\otimes \mathbf{Q} \rightarrowtail\!\!\!\twoheadrightarrow H_*(\mathbf{MSp})\otimes H_*(SO/Sp)\otimes \mathbf{Q}$$

whence $MSp_*(SO/Sp)\otimes \mathbf{Q} = 0$.

We deduce

(5.7) PROPOSITION. *Both J-homomorphisms*

$$\begin{array}{ccc} S_*(SO/Sp) & & \\ \downarrow & \searrow^{J} & \\ & & MSp_* \\ & \nearrow_{J} & \\ MSp_*(SO/Sp) & & \end{array}$$

have images contained in $\mathit{Tors}MSp_*$. □

In a way this is a surprise. It means that MSp_* does not sit pleasantly between S_* and MSO_* for these purposes in the same way that MU_* does. Maybe MSp_* is such a problem because Sp-manifolds admit so few alternative Sp structures.

6. $J:\pi_*SO/U \rightarrow MU_*$

In this section we determine the image of the above J-homomorphism. We know already from (5.6) that it consists of a copy of $\boldsymbol{Z}$ in each dimension $\equiv$ 2, 6 (8). The only problem is: does ImJ constitute a direct summand?

By virtue of (4.1), we wish to evaluate

$$\pi_*(SO/U) \underset{\underline{h}}{\longrightarrow} H_*(SO/U) \underset{i^*}{\longrightarrow} H_*(BU).$$

Now there is the commutative diagram

$$\begin{array}{ccc} \pi_k(SO/U) & \xrightarrow{i_*} & \pi_k(BU) \\ \downarrow{\scriptstyle \underline{h}} & & \downarrow{\scriptstyle \underline{h}} \\ H_k(SO/U) & \underset{i_*}{\longrightarrow} & H_k(BU), \end{array}$$

where $\pi_{2j}(BU) = \boldsymbol{Z}$ on z^j, say, and $\pi_{2j+1}(BU) = 0$.

(6.1) LEMMA. $\pi_{4k+2}(SO/U) = \boldsymbol{Z}$, *and denoting the generator by* a_{2j+1} *then* $i_*a_{4j+1} = 2z^{4j+1}$, $i_*a_{4j+3} = z^{4j+3}$ *describes completely* Im$i_* < \pi_*(BU)$.

Proof. Consider the homotopy exact sequence of $SO/U \rightarrow BU \rightarrow BSO$. □

(6.2) COROLLARY. $\underline{h}$ ImJ $= \boldsymbol{Z}$ *on* $2(4j)!p'_{4j+1}$ *in* $H_{8j+2}(\mathbf{MU})$ *and* $\boldsymbol{Z}$ *on* $(4j+2)!p'_{4j+3}$ *in* $H_{8j+6}(\mathbf{MU})$.

Proof.

$$\underline{h}Ja_{4j+1} = \varphi^{-1}i_*\underline{h}a_{4j+1} = \varphi^{-1}\underline{h}(2z^{4j+1}) = -2(4j)!p'_{4j+1}$$

whilst $\underline{h}Ja_{4j+3} = -(4j+2)!p'_{4j+3}$ similarly. □

We now have to test the corresponding bordism classes for divisibility in MU_*. The standard procedure is to check the KU hurewicz images in $KU_*(\mathbf{MU})$. By courtesy of the Hattori-Stong theorem (see, e.g. [4]) $J(a_{2j+1})$ is divisible by an integer n iff $\underline{ku}Ja_{2j+1}$ is also. In order to evaluate $\underline{ku}\ \mathrm{Im}J < KU_*(MU)$, we shall prove a modified version of (4.1).

Let $f: S^{4j+2} \to SO/U$ represent a_{2j+1}. Write $\bar{S}^{4j+2}$ for S^{4j+2} invested with its non-trivial U-structure, so that ν is induced by $i\cdot f: S^{4j+2} \to BU$. Then $[\bar{S}^{4j+2}]_U = Ja_{2j+1}$.

Also, there is a standard orientation class $t_U \in KU^0(\mathbf{MU})$ as described by Atiyah, Bott and Shapiro. This gives rise to a Thom isomorphism $\psi: KU_*(\mathbf{MU}) \rightarrowtail\!\!\!\twoheadrightarrow KU_*(BU_+)$.

(6.3) THEOREM. *$\underline{ku}\ Ja_{2j+1} = \psi^{-1}(i_*\underline{ku}a_{2j+1} + td(\bar{S}^{4j+2})z^{2j+1})$ in $KU_{4j+2}(\mathbf{MU})$ where td denotes the Todd genus.*

Proof. Consider the diagram of Thom complexes

$$\begin{array}{ccccc} S^{2n+4j+2} & \xrightarrow{\ c\ } & M(\nu) \simeq S^{2n} \wedge S_+^{4j+2} & \xrightarrow{\ M(\nu)\ } & MU(n) \\ & \nwarrow & \cup & & \cup \\ & & \bar{S}^{4j+2} & \xrightarrow[\nu = i\cdot f]{} & BU(n). \end{array}$$

Here c collapses S^{n+4j+2} onto the Thom complex of the normal bundle of the embedded sphere. Write g_i and $\bar{g}_i$ for the respective generators of $KU_i(S^i)$. and $KU^i(S^i)$. Then by definition

$$\underline{ku}(Ja_{2j+1}) = M(\nu)_*c_*g_{2n+4j+2} = \psi^{-1}(\nu_*\psi_1(c_*g_{2n+4j+2}))$$

where $\psi_1: KU_{*+2n}(S^{2n} \wedge S_+^{4j+2}) \rightarrowtail\!\!\!\twoheadrightarrow KU_*(S_+^{4j+2})$ is the Thom isomorphism induced by $M(\nu)^*t_U \in KU^{2n}(S^{2n} \wedge S_+^{4j+2})$.

Now $c_*g_{4j+2} = g_{2n} \otimes g_{4j+2}$, whilst

$$M(\nu)^*t_U = \lambda z^{4j+2}\bar{g}_{2n} \otimes \bar{g}_{4j+2} + \bar{g}_{2n} \otimes 1$$

for some integer λ.

Since $c^*M(\nu)^*t_U = \lambda z^{4j+2}\bar{g}_{2n+4j+2}$, $\lambda = td(\bar{S}^{4j+2})$.

Thus

$$\begin{aligned} \psi_1(c_*g_{4j+2}) &= g_{4j+2} \cap (\lambda z^{4k+2}\bar{g}_{4j+2} + 1) \\ &= \lambda z^{4j+2} + g_{4j+2}. \end{aligned}$$

In fact this is the KU orientation class of $\bar{S}^{4j+2}$, so we have

$$\underline{ku}Ja_{2j+1} = \psi^{-1}(\nu_*g_{4j+2} + td(\bar{S}^{4j+2})z^{4j+2})$$

and

$$\nu_*g_{4j+2} = i_*f_*g_{4j+2} = i_*\underline{ku}a_{2j+1}$$

as sought. □

(6.4) COROLLARY.

$$\underline{ku}Ja_{4j+1} = \psi^{-1}\{td(\bar{S}^{8j+2})z^{4j+1} + 2\underline{ku}(z^{4j+1})\} \quad in \quad KU_{8j+2}(\mathbf{MU}),$$

and

$$\underline{ku}Ja_{4j+3} = \psi^{-1}\{td(\bar{S}^{8j+6})z^{4j+3} + \underline{ku}(z^{4j+3})\} \quad in \quad KU_{8j+6}(\mathbf{MU}).$$

Proof. Combine (6.1) and (6.3). □

So now we are reduced to evaluating $td(\bar{S}^{4j+2})$ and describing

$$\underline{ku}:\pi_*(BU) \rightarrow KU_*(BU).$$

(6.5) LEMMA.

$$td(\bar{S}^{4j+2}) = 0 \quad for \quad j > 0$$
$$= 1 \quad for \quad j = 0.$$

Proof. The only dimension $4j + 2$ in which the Todd polynomial in $H^*(BU)\otimes\mathbf{Q}$ contains the term c^{2j+1} is when $j = 0$. In that case, the coefficient is $\frac{1}{2}$, and

$$td(\bar{S}^2) = \langle -c_1/2, \varphi\underline{h}a_2\rangle = \langle -c_1/2, -2b_1\rangle = 1. \quad \square$$

Finally, we need

(6.6) LEMMA. *$\underline{ku}(z^{2j+1})$ is indivisible in $KU_{4j+2}(BU)$.*

Proof. Write $\underline{ku}(z^{2j+1}) = \sum \lambda_\alpha z^{4j+2-2|\alpha|}\beta_\alpha$, where $\lambda_\alpha \in \boldsymbol{Z}$ and $\{\beta_\alpha\}$ denotes the usual basis for $KU_*(BU)$ dual to the Atiyah-Chern classes $\gamma_\alpha \in KU^{2|\alpha|}(BU)$. Then

$$\lambda_1 z^{4j} = \langle \underline{ku}z^{2j+1}, \gamma_1\rangle = z^{2j+1},$$

so $\lambda_1 = 1$ and $\underline{ku}(z^{2j+1})$ is indivisible. □

(6.7) COROLLARY. *Ja_{4j+3} is indivisible in MU_{8j+6}.*

Proof. From (6.4), (6.5) and (6.6), $\underline{ku}Ja_{4j+3} = \psi^{-1}\underline{ku}(z^{4j+3})$ is indivisible. □

(6.8) COROLLARY. *Ja_{4j+1} is divisible by 2 in MU_{8j+2}, except when $j = 0$. Ja_1 is indivisible.*

Proof. From (6.4) and (6.5),

$$\underline{ku}Ja_{4j+1} = 2\psi^{-1}\underline{ku}(z^{4j+1}) \quad \text{when} \quad j > 0,$$

whilst $\underline{ku}Ja_1 = \psi^{-1}(z + 2\underline{ku}(z))$. Then apply (6.6). □

Thus our final result is that ImJ is a direct summand $\boldsymbol{Z}$ in dimensions 2, $8j + 6$, and is a $\boldsymbol{Z}$ divisible by 2 in dimensions $8j + 2, j > 0$. We could with some trouble express the generators of ImJ in terms of MU_* : this seems both complicated and unrewarding. The philosophy should be that these U-spheres represent simple and natural bordism classes in their own right.

As a tail-piece, let us remark that from the relation between MSU_* and MU_* (see, e.g. [10]), we can deduce

$$\text{Im}\,\{J:\pi_*(SO/SU) \rightarrow MSU_*\}$$

is also a $\boldsymbol{Z}$ direct summand in dimensions $8j + 6$, and a $\boldsymbol{Z}$ divisible by 2 in dimensions $8j + 2$ ($\pi_2(SO/SU) = 0$).

7. $J:MU_*(SO/U) \to MU_*$

In this section we determine the image of the above J-homomorphism. We know already from (5.4) that we can choose generators

$$\{x_1, x_2, \cdots, x_k, \cdots\}$$

for MU_* such that ImJ is a subideal of $(x_1, x_3, \cdots, x_{2k+1}, \cdots)$.

Before refining (5.4) further, we need to establish a hold on $MU_*(SO/U)$. The following theorem is due to Floyd and Stong [private communication].

(7.1) THEOREM. *Let $f:BU \to SO/U$ denote the lift of the classifying map of $\zeta - \bar{\zeta}$. Then $f_* : MU_*(BU) \to MU_*(SO/U)$ is epic.*

Proof. Recall from (5.1) how $i^*:H^*(BU) \to H^*(SO/U)$ works. With the same notation

$$f^* y_{2i+1} = \tfrac{1}{2} f^* i^* c_{2i+1} = \tfrac{1}{2} c_{2i+1}(\zeta - \bar{\zeta}) \quad \text{in} \quad H^{4i+2}(BU).$$

But $c(\zeta - \bar{\zeta}) = c(\zeta)\cdot c(\bar{\zeta})^{-1}$, where c is the total chern class. So

$$c(\zeta - \bar{\zeta}) = (1 + c_1 + c_2 + \cdots)(1 - c_1 + c_2 - \cdots)^{-1}$$
$$= 1 + 2c_1 + 2c_2 + \cdots + 2c_k + \cdots$$

mod decomposables. Therefore $c_{2i+1}(\zeta - \bar{\zeta}) = 2c_{2i+1}$ and $f^* y_{2i+1} = c_{2i+1}$, both mod decomposables. Thus f^* is monic onto a certain direct summand of $H^*(BU)$. Thus

$$f_* : H_*(BU) \to H_*(SO/U)$$

is epic. But $MU_*(BU)$ and $MU_*(SO/U)$ are free over MU_* (see (5.2)), so

$$f_* : MU_*(BU) \to MU_*(SO/U)$$

is epic also. □

(7.2) COROLLARY. *Let $bf_i \in MU_{2i}(BU)$ be the usual generator (see [7]), and write $cp_i \in MU_{2i}(BU)$ for the class $[CP^i \to BU]_U - [CP^i]_U$. Then $MU_*(SO/U)$ is generated as an algebra over MU_* either by the elements $\{f_* bf_i\}$ or by the elements $\{f_* cp_i\}$.* □

It is purely a matter of convenience which generating set we choose, and we shall work with the $f_* cp_i$'s because they allow easier algebraic manipulation and admit better geometric interpretation. Let us relabel $f_* cp_i$ as dp_i.

So we have

(7.3) COROLLARY. ImJ = *$(Jdp_1, Jdp_2, \cdots, Jdp_i, \cdots)$ as an ideal in MU_*.* □

(7.4) LEMMA. *In $H_{2i}(\mathbf{MU})$,*

$$\underline{h}(Jdp_i) = \{(\underline{b}')^{-i}(\hat{b}')^{-1}\}_i - (\underline{b}')_i^{-i-1}$$

where

$$\underline{b} = 1 + b_1 + \cdots + b_k + \cdots$$

and

$$\hat{b} = 1 - b_1 + b_2 + \cdots + (-1)^k b_k + \cdots$$

in $H_*(BU)$.

Proof. Following (4.1),

$$\underline{h}(Jdp_i) = \mu_*(1 \otimes \varphi^{-1}\underline{h}(i_* dp_i)),$$

so we must first evaluate

$$\underline{h}(i_* dp_i) \in H_*(\mathbf{MU}) \otimes H_*(BU).$$

Well, $\underline{h}(cp_i) = \sum_{t=1}^{i} (\underline{b}')_{i-t}^{-i-1} \otimes b_t$ (see [7]), whence

$$\underline{h}(i_* dp_i) = \underline{h}(i_* f_* cp_i) = \sum_{t=1}^{i} (\underline{b}')_{i-t}^{-i-1} \otimes i_* f_* b_t - (\underline{b}')_i^{-i-1}.$$

But $i \cdot f : BU \to BU$ classifies $\zeta - \bar{\zeta}$, so $i_* f_*(\underline{b}) = (\underline{b})(\hat{b})^{-1}$. Therefore

$$\underline{h}(i_* dp_i) = \sum_{t=1}^{i} (\underline{b}')_{i-t}^{-i-1} \otimes \{\underline{b}(\hat{b})^{-1}\}_t - (\underline{b}')_i^{-i-1}$$

and

$$\begin{aligned}\mu_*(1 \otimes \varphi^{-1}\underline{h}(i_* dp_i)) &= \sum_{t=1}^{i} (\underline{b}')_{i-t}^{-i-1}\{\underline{b}'(\hat{b}')^{-1}\}_t - (\underline{b}')_i^{-i-1} \\ &= \sum_{t=0}^{i} (b')_{i-t}^{-i} (\hat{b}')_t^{-1} - (\underline{b}')_i^{-i-1} \\ &= \{(\underline{b}')^{-i}(\hat{b}')^{-1}\}_i - (\underline{b}')_{-i}^{-i-1}. \qquad \square\end{aligned}$$

(7.5) COROLLARY. *In* $H_*(\mathbf{MU})$,

$$\underline{h}Jdp_{2i} = -2\sum_{t=0}^{i-1} (\underline{b}')_{2i+1}^{-2i}(\underline{b}')_{2i-2t-1}^{-1},$$

$$\underline{h}Jdp_{2i+1} = -2\sum_{t=0}^{i} (\underline{b}')_{2i-2t}^{-2i-1}(\underline{b}')_{2t+1}^{-1}.$$

Proof. This follows from (7.4) plus the observation that

$$(\hat{b}')_{2k}^{-1} = (\underline{b}')_{2k}^{-1} \quad \text{and} \quad (\hat{b}')_{2k+1}^{-1} = -(\underline{b}')_{2k+1}^{-1}. \qquad \square$$

Our remaining task is to choose generators for MU_* in such a way that ImJ may be written as simply as possible as an ideal. We need a few preliminary results and some notation.

(7.6) THEOREM (Milnor [5]). *For each* n *there exists* $y_n \in MU_{2n}$ *with* $y_n = \mu(n)b_n'$ *mod decomposables in* $H_{2n}(\mathbf{MU})$, *where*

$$\begin{aligned}\mu(n) &= 1 \quad \textit{if} \quad n+1 \textit{ is not a prime power} \\ &= p \quad \textit{if} \quad n+1 \textit{ is a power of a prime } p.\end{aligned}$$

These y_n*'s may be taken as algebra generators of* MU_*, *but are not canonically defined.* $\square$

(7.7) COROLLARY. Jdp_{2i-1} *is a generator of* MU_* *whenever* $i = 2^{m-1}$, $m > 1$.

Proof. Inspecting (7.5) shows that $\underline{h}(dp_{2^m-1}) = -2b_{2^m-1}'$ mod decomposables. Then apply (7.6). $\square$

This result, along with (5.4), suggests that (7.8) is indeed the correct theorem.

Next, let ε be defined on positive odd numbers by

$$\varepsilon(2n-1) = 1 \quad \text{if} \quad n = 2^{m-1}$$
$$= 2 \quad \text{otherwise.}$$

Then we have

(7.8) THEOREM. *We can choose generators* $\{y_i \in MU_{2i} \mid i \in \mathbf{Z}^+\}$ *such that*

$$\mathrm{Im}J = (y_1, y_3, 2y_5, \cdots, \varepsilon(2n-1)y_{2n-1}, \cdots).$$

Proof. Choose y_{2m-1} to be Jdp_{2m-1} for $m > 0$, by courtesy of (7.7). Fix some other choice for the remaining y_i's according to the recipe of (7.6). Stong [10] is a good source for such generators.

Now define the following ideals in MU_*:

$$\mathrm{Im}_n J = (Jdp_1, Jdp_2, \cdots, Jdp_n)$$
$$I_n = (y_1, y_3, \cdots, \varepsilon(n)y_n), \qquad n \text{ odd}$$
$$= (y_1, y_3, \cdots, \varepsilon(n-1)y_{n-1}), \quad n \text{ even.}$$

Note that $I_{2n} = I_{2n-1}$, $\mathrm{Im}_\infty J = \mathrm{Im}J$ and $I_\infty = I$, our conjectured ideal.

Since $Jdp_1 = y_1$, we have that $\mathrm{Im}_1 J = I_1$ and we can embark on an inductive proof.

Suppose $\mathrm{Im}_n J = I_n$. There are two cases to consider.

Case A. Assume n is even, say $n = 2l$. Then

$$\mathrm{Im}_{2l+1} J = (\mathrm{Im}_{2l} J, Jdp_{2l+1}) \quad \text{and} \quad I_{2l+1} = (I_{2l}, \varepsilon(2l+1)y_{2l+1}).$$

If $2l + 2 = 2^m$, $Jdp_{2l+1} = y_{2l+1}$ and $\varepsilon(2l+1) = 1$. In this case we have at once that $\mathrm{Im}_{n+1} J = I_{n+1}$.

Otherwise, consider Jdp_{2l+1}. There must be an expression

$$Jdp_{2l+1} = \textstyle\sum_{|\alpha|=2l+1} \lambda_\alpha y_\alpha,$$

where $\alpha = (\alpha_1, \cdots, \alpha_r)$ is some sequence of non-negative integers, $y_\alpha = y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_p^{\alpha_p}$ and $\lambda_\alpha \in \mathbf{Z}$.

We now interrupt our proof for a lemma.

(7.9) LEMMA. *If α contains no non-zero terms of the form α_{2m-1}, then* $2 \mid \lambda_\alpha$.

Proof. We shall proceed by induction on $\sigma(\alpha) = \alpha_1 + \cdots + \alpha_p$ (which attains a maximum of $2l + 1$). If $\sigma(\alpha) = 1$, $\alpha = (0, \cdots, 0, 1)$ and $y_\alpha = y_{2l+1}$. Also, $\underline{h}(y_{2l+1}) = b'_{2l+1}$ mod decomposables in $H_{4l+2}(\mathbf{MU})$, whence

$$\underline{h}(\textstyle\sum \lambda_\alpha y_\alpha) = \lambda_{(0,\ldots,0,1)} b'_{2l+1}$$

mod decomposables. But $\underline{h}(Jdp_{2l+1}) = -2b'_{2l+1}$ mod decomposables, so $\lambda_{(0,\ldots,0,1)} = -2$. Let induction commence!

Suppose our result is true whenever $\sigma(\alpha) \leq k$, and find an

$$\alpha' = (\alpha_2', \cdots, \alpha_q') \quad \text{with} \quad \sigma(\alpha') = k + 1.$$

(If there is no such α', the following argument will still apply to the first $\sigma(\alpha') > k$). Consider the equation $\underline{h}(Jdp_{2l+1}) = \sum \lambda_\alpha \, \underline{h}(y_\alpha)$. Then in the right hand side,

$$\underline{h}(y_{\alpha'}) = \prod_{\alpha_i'} (\mu(i) b_i' + \text{decomposables})^{\alpha_i'}$$
$$= \mu b_{\alpha'}' + \text{higher decomposables},$$

where $\mu = \prod_{i \neq 2^m - 1} \mu(i)$ and is odd. But $b_{\alpha'}'$ will not occur in $\sum \lambda_\alpha \, \underline{h}(y_\alpha)$ for any other α with $\sigma(\alpha) \geq k$. However, it might arise in $\lambda_{\alpha''} \, \underline{h}(y_{\alpha''})$ where $\sigma(\alpha'') \leq k$. Then if α'' contains no non-zero terms of the form α_{2^m-1}'', $\lambda_{\alpha''}$ is divisible by 2 by induction.

On the other hand, if α'' does contain some non-zero α_{2^m-1}'', then $y_{\alpha''}$ involves $y_{2^m-1}^z$ where $z = \alpha_{2^m-1}''$. Hence $2 \mid \underline{h} y_{\alpha''}$. So whenever $b_{\alpha'}'$ occurs in $\sum \lambda_\alpha \, \underline{h}(y_\alpha)$ outside of $\underline{h}(y_{\alpha'})$, it has an even coefficient.

Finally, $2 \mid \underline{h}(Jdp_{2l+1})$, so the coefficients of $b_{\alpha'}'$ are even in $\underline{h}(Jdp_{2l+1})$, and an even integer $+ \lambda_{\alpha'} \mu$ in $\sum \lambda_\alpha \, \underline{h}(y_\alpha)$. But μ is odd, so $2 \mid \lambda_{\alpha'}$. □

Utilizing this, we can write

$$Jdp_{2l+1} = -2y_{2l+1} + 2\sum_{\beta \neq (0,\ldots,0,1)} \tilde{\lambda}_\beta \, y_\beta + \sum \lambda_\gamma \, y_\gamma$$

where the first sum is over those β with no non-zero β_{2^m-1}, and the second is over those γ with at least one non-zero γ_{2^m-1}. Notice each β must contain some non-zero β_{2k+1}, $k < l$. So $Jdp_{2l+1} + 2y_{2l+1} \in I_{2l}$ and we have shown

$$(\mathrm{Im}_{2l} J, Jdp_{2l+1}) = (I_{2l}, \varepsilon(2l+1) y_{2l+1}).$$

Therefore $\mathrm{Im}_{n+1} J = I_{n+1}$.

Case B. Assume n is odd, say $n = 2l - 1$. Then

$$\mathrm{Im}_{n+1} J = (\mathrm{Im}_{2l-1} J, Jdp_{2l}),$$

whilst $I_{n+1} = I_n$ which in turn is $\mathrm{Im}_n J$ by assumption. So to complete our induction we must show $Jdp_{2l} \in \mathrm{Im}_{2l-1} J$.

Well, as before $Jdp_{2l} = \sum_{|\alpha|=2l} \lambda_\alpha \, y_\alpha$ for certain integers λ_α. By repeating the methods of (7.9), we can write $Jdp_{2l} = 2\sum \tilde{\lambda}_\beta \, y_\beta + \sum \lambda_\gamma \, y_\gamma$, where β and γ are as before, Even though $|\beta| = 2l$, there can be no β for which each β_{2k+1} is zero. For if there were such a β, $\underline{h}(y_\beta)$ would contain b_β', and by inspection of (7.5) no such term can appear in $\underline{h}(Jdp_{2l})$. This means each y_β involves some y_{2k+1} with $k < l$, whence $Jdp_{2l} \in \mathrm{Im}_{2l-1} J$. □

(7.10) COROLLARY. $2MU_{4*+2} \leq \mathrm{Im}J$.

Proof. $2MU_2 \leq \mathrm{Im}J$, since $2[CP^1]_U \in \mathrm{Im}J$. Suppose $2MU_{4k-2} \leq \mathrm{Im}J \; \forall \, k \leq n$. Then

$$2MU_{4k+2} = \oplus_{i=0}^{n} 2y_{2i+1}(MU_{4n-4i}) \oplus \oplus_{j=1}^{n} y_{2j}(2MU_{4n-4j+2}).$$

Hence $2MU_{4n+2}$ also is $\leq \mathrm{Im}J$. □

8. Geometrical interpretations

In conclusion, let us see what our results say about changing U-structures on manifolds. We also offer a few comments on the symplectic case. I am particularly grateful for conversations with Jim Alexander, Pierre Conner, Stan Kochman and Bob Strong on these matters.

Recall the interpretation of $\mathrm{Im}J \leq MU_*$ given in (3.6(ii)). Jdp_i is, representable as $[DP^i]_U$ for a certain U-manifold DP^i. In fact $DP^i = C\bar{P}^i - CP^i$, where $C\bar{P}^i$ denotes CP^i with the normal structure $\nu = -i\zeta - \bar{\zeta}$, suitably stabilised so as to preserve orientation: with the new structure $CP^i - CP^i$, DP^i is clearly a U-boundary. Note that (7.7) tells us that $[DP^{2^m-1}]_U$ is a generator for MU_*. Thus we can write

$$\mathrm{Im}J = ([DP^1]_U, \cdots, [DP^{2n+1}]_U, \cdots).$$

A more geometrical proof of (7.8) can now be given with the help of

(8.1) LEMMA. *For each triple $G_k \leq G_j \leq G_i$, the composite*

$$(MG_k)_*(G_i/G_j) \xrightarrow{J} (MG_j)_* \to (MG_i)_*$$

is zero.

Proof. Any manifold in $\mathrm{Im}J$ admits a new G_k-structure with respect to which it bounds. However, the G_i-structure remains unchanged, so the manifold is also a G_i-boundary. □

(8.2) COROLLARY. $\mathrm{Im}J \leq MU_*$ *is a subideal of* $\mathrm{Ker}\,\{MU_* \to MSO_*\}$. □

However, according to Stong [10] and making use of (7.5), this kernel I may be expressed as

$$(y_1, y_3, 2y_5, \cdots, \varepsilon(2n-1)y_{2n-1}, \cdots)$$

where $y_{2^m-1} = [DP^{2^m-1}]_U$ and the other y_i's are suitably chosen. So we have only to show that each $2y_{2n-1}$ in fact exists in $\mathrm{Im}J$.

(8.3) LEMMA. *A U-manifold M^{4n-2} admits a second U-structure, say $\tilde{M}^{4n-2}$, $\tilde{\nu}$ such that*

$$[M^{4n-2}]_U = -[\tilde{M}^{4n-2}]_U \quad in \quad MU_{4n-2}.$$

Proof. Choose $\tilde{\nu}$ to be the complex conjugate of ν, stabilised so as to retain the original orientation. Since $c_i(\tilde{\nu}) = (-1)^i c_i(\nu)$, the new chern numbers must be the negatives of the old. □

(8.4) COROLLARY. $\mathrm{Im}J = I$.

Proof. $[M^{4n-2} - \tilde{M}^{4n-2}]_U = 2[M^{4n-2}]_U$ in MU_{4n-2}, and $M^{4n-2} - \tilde{M}^{4n-2}$ can clearly be given a new and bounding U-structure. So $2y_{2n-1} \in \mathrm{Im}J$ for all n and $I \leq \mathrm{Im}J$. But $\mathrm{Im}J \leq I$ from (8.2). □

In many ways these geometrical ideas are unsatisfying. Rather than taking differences of known manifolds, we hoped more natural examples might arise as generating ImJ. It still seems worth searching for some construction which does the job more neatly, maybe using the basis $\{f_* bf_i\}$ of (7.2) for $MU_*(SO/U)$.

Also, although the geometrical viewpoint is particularly concise in the unitary case, it appears that the homological methods of §7 (using at least KO_*) will be needed to shed light on the more exotic cases such as

$$J: MSp_*(G/Sp) \to MSp_*,$$

where $G = SO$ or U.

Finally, recall from [0], J. Alexander's family of indecomposables

$$\{\mu_k \in MSp_{8k-3}, k > 0\}.$$

It is extremely interesting to observe

(8.5) PROPOSITION. *For each* $k > 0$,

$$\mu_k \in \mathrm{Im}\,\{J: MSp_{8k-3}(SU/Sp) \to MSp_{8k-3}\}$$

Proof. The construction of each μ_k proceeds precisely by taking a manifold which is an Sp-boundary, and changing its Sp-structure so as to leave it unaltered as an SU-bundle. □

This prompts

(8.6) CONJECTURE. $J: MSp_*(SO/Sp) \to MSp_*$ *or (more attractively)*

$$J: MSp_*(U/Sp) \to MSp_*$$

is an epimorphism of 2-*components*. □

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