REPRESENTING MEASURES IN COMPACT GROUPOIDS

BY

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1. Introduction

The objective of this paper is to describe an abstract theory of representing measures. To do this we consider a compact topological groupoid, i.e., X is compact Hausdorff and $\cdot : X \times X \to X$ is continuous. $(X, \cdot)$ is commutative if $x \cdot y = y \cdot x$ for every $x, y \in X$, and $(X, \cdot)$ is medial if $(w \cdot x) \cdot (y \cdot z) = (w \cdot y) \cdot (x \cdot z)$ for every $w, x, y, z \in X$. Observe that if X is a compact convex subset of a locally convex topological vector space with $\cdot$ as the midpoint function, then $(X, \cdot)$ is commutative and medial. Throughout this paper we shall refer to such a set as simply a compact convex set. With this example in mind define a real valued function f on a compact groupoid $(X, \cdot)$ to be convex if for every $x, y \in X$, $f(x \cdot y) \leq \frac{1}{2}f(x) + \frac{1}{2}f(y)$. Let $C(X)$ denote the continuous real valued functions on X and let C denote the continuous convex functions on X. An element $x \in X$ is called an idempotent if $x \cdot x = x$ and we call the set of all idempotents of X the core of X and denote it by core X. A class of functions K on a set S is said to separate points if for every $x, y \in S$ with $x \neq y$, there exists $f \in K$ such that $f(x) \neq f(y)$. We shall say that K is totally separating if $x, y \in S$ and $f(x) \geq f(y)$ for every $f \in K$ implies that $x = y$. $(X, \cdot)$ is said to be strongly separated by its convex functions if C separates the points in X and C is totally separating on core X. If $(X, \cdot)$ is a compact medial groupoid that is strongly separated by its convex functions, then $(X, \cdot)$ is called a compact mean space.

When X is a compact Hausdorff space then we shall let $\Omega(X)$ denote the regular Borel probability measures on X. Since $\Omega(X)$ is exactly those regular signed Borel measures μ in the closed unit ball of $C(X)^*$ such that $\int 1\, d\mu = 1$, $\Omega(X)$ is weak* compact. If $(X, \cdot)$ is a compact groupoid and $\mu, \nu \in \Omega(X)$, then

$$l(f) = \int f(x \cdot y)\, d\mu(x) \times \nu(y)$$

defines a norm one linear functional l on $C(X)$ such that $l(1) = 1$. Hence $l(f) = \int f\, d\phi$ for some $\phi \in \Omega(X)$. We shall denote the measure ϕ by $\mu * \nu$. $\mu * \nu$ is called the convex convolution of μ with ν. Now define a map $S: \Omega(X) \to \Omega(X)$ by $S(\mu) = \mu * \mu$ for every $\mu \in \Omega(X)$. Since $\cdot$ is continuous, it is easily verified that S is weak* continuous. If $\mu \in \Omega(X)$ and $x \in X$, then we say that μ represents x if for every $f \in C(X)$,

$$\lim_{n \to \infty} \int f\, dS^n(\mu) = f(x).$$

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In this paper we shall investigate representing measures in the case when $(X, \cdot)$ is a compact mean space. It will be shown that every measure in $\Omega(X)$ represents some point in core X and that core X is iseomorphic to a compact convex set. Some applications and examples will be given. Finally, we will develop a theory of compact groupoid valued integrals based on our notion of representing measures.

2. Measures representing points

In this section we shall assume that $(X, \cdot)$ is a compact mean space and we let C denote the continuous convex functions on S. To obtain a theory of representing measures we must first investigate some of the properties of the class of functions C.

PROPOSITION 2.1. (i) *If $f, g \in C$, and $\alpha \geq 0$, then $\alpha f + g \in C$ and* $\max(f, g) \in$ *C.*

(ii) *$C - C$ is dense in $C(X)$.*

(iii) *If $f \in C$ and $f \geq 0$, then $f^2 \in C$.*

(iv) *If $x, y \in X$ and $x \neq y$, then there exists $f \in C$ such that $f(x \cdot y) < \frac{1}{2}f(x) + \frac{1}{2}f(y)$.*

Proof. (i) is a routine verification and by (i), $C - C$ is a vector space. If $f_1, f_2, g_1, g_2 \in C$ then

$$\max(f_1 - g_1, f_2 - g_2) = \max(f_1 + g_2, f_2 + g_1) - (g_1 + g_2).$$

Hence $C - C$ is a lattice. Since $C - C$ contains constants and separate points, $C - C$ is dense in $C(X)$ by the Stone-Weierstrass theorem. (iii) is easily verified. If $x, y \in X$ such that $x \neq y$, then there exists $f \in C$ such that $f(x) \neq f(y)$. We may assume $f \geq 0$ since if $c \leq \inf f(x)$, then $f - c \geq 0$. By (iii), $f^2 \in C$. But also, $f^2(x \cdot y) < \frac{1}{2}f^2(x) + \frac{1}{2}f^2(y)$ as is easily shown.

If $\mu, \nu \in \Omega(X)$, we say $\mu \leq \nu$ if for every $f \in C$, $\int f\, d\mu \leq \int f\, d\nu$. It is clear that $\leq$ is a partial ordering of $\Omega(X)$.

LEMMA 2.2. *If $\mu, \nu \in \Omega(X)$, then $\mu * \nu \leq \frac{1}{2}\mu + \frac{1}{2}\nu$. In particular, $S(\mu) \leq \mu$ for every $\mu \in \Omega(X)$.*

Proof. If $f \in C$, then

$$\begin{aligned}\int f\, d\mu * \nu &= \int f(x \cdot y)\, d\mu(x) \times \nu(y) \\ &\leq \int (\tfrac{1}{2}f(x) + \tfrac{1}{2}f(y))\, d\mu(x) \times \nu(y) \\ &= \tfrac{1}{2}\int f\, d\mu + \tfrac{1}{2}\int f\, d\nu\end{aligned}$$

From this lemma we see that if $\mu \in \Omega(X)$ and $f \in C$, then $\int f\, dS^n\mu$ is a monotone decreasing sequence.

The next lemma relies on the notion of the support of a measure. If μ is a regular Borel measure on a compact Hausdorff space Y, then the support of μ is the smallest of the compact sets whose complement has measure zero. The support of μ is denoted supp μ and the following facts about supp μ are easily verified.

(i) If O is open in Y and $O \cap \operatorname{supp} \mu \neq \emptyset$, then $\mu(O) > 0$.

(ii) If $f, g \in C(Y)$, $f \geq g$, and for some $x \in \operatorname{supp} \mu$, $f(x) > g(x)$, then $\int f\, d\mu > \int g\, d\mu$.

Now let $e: X \to \Omega(X)$ be defined by letting $e(x)$ be the point mass measure at x. We note that for a point mass measure $e(x)$, $S(e(x)) = e(x \cdot x)$ and more generally $e(x) * e(y) = e(x \cdot y)$.

PROPOSITION 2.3. *If $\mu \in \Omega(X)$ and $S(\mu) = \mu$, then there exists $x \in$ core X such that $\mu = e(x)$.*

Proof. Suppose $\mu \in \Omega(X)$. Then the Baire sets in $X \times X$ are $\mu \times \mu$ measurable. Thus there exists a measure ν in $\Omega(X \times X)$ such that for every $f \in C(X \times X)$, $\int f\, d\nu = \int f\, d\mu \times \mu$. Now supp $\nu \supset$ (supp μ) $\times$ (supp μ). To see this suppose $x, y \in \operatorname{supp} \mu$. If $(x, y) \in O$ for an open set O, then there exists U, V open F_σ sets in X such that $(x, y) \in U \times V$ and $U \times V \subset O$. But then

$$\nu(O) \geq \nu(U \times V) = \mu(U) \times \mu(V) > 0.$$

(Actually, supp ν = (supp μ) $\times$ (supp μ), but we do not require that much.)

Now suppose $S(\mu) = \mu$. If $a, b \in \operatorname{supp} \mu$ and $a \neq b$, then there exists $f \in C$ such that $f(a \cdot b) < \frac{1}{2}f(a) + \frac{1}{2}f(b)$. But then

$$\begin{aligned}\int f\, dS(\mu) &= \int f(x \cdot y)\, d\mu(x) \times \mu(y) \\ &< \int (\tfrac{1}{2}f(x) + \tfrac{1}{2}f(y))\, d\mu(x) \times \mu(y) \\ &= \int f\, d\mu.\end{aligned}$$

This contradiction proves that supp μ must consist of a single point, so that $\mu = e(x)$ for some $x \in X$. But then $e(x) = S(e(x)) = e(x \cdot x)$. Hence $x \in$ core X.

PROPOSITION 2.4. *If $\mu \in \Omega(X)$, then there exists $x \in$ core X such that μ represents x.*

Proof. If $f \in C$, then $\langle \int f\, dS^n\mu \rangle$ is a monotone decreasing sequence of real numbers bounded below by $\inf f(X)$. Thus $\lim_{n\to\infty} \int f\, dS^n\mu$ exists. From this we deduce that $\langle \int f\, dS^n\mu \rangle$ has a limit for $f \in C - C$. Since $C - C$ is norm dense in $C(X)$, $S^n\mu$ must weak* converge to a linear functional in the closed unit ball of the dual of $C(X)$. Since $\{S^n\mu\} \subset \Omega(X)$ and $\Omega(X)$ is weak* compact, the limit must be a measure in $\Omega(X)$.

If $\nu = \lim_{n\to\infty} S^n\mu$, then by the continuity of S, $S\nu = S \lim_{n\to\infty} S^n\mu = \lim_{n\to\infty} S^{n+1}\mu = \nu$. Thus $\nu = e(x)$ for some $x \in \operatorname{core} X$.

By the above proposition we may define $\psi \colon \Omega(X) \to \operatorname{core} X$ by $\psi(\mu) = x$ if μ represents x. At this point we note that if X is a compact convex set and if $\cdot$ is the usual midpoint function, then our definition of representing measure agrees with the usual definition, i.e., if f is a continuous affine function on X then $f \in C \cap -C$, so that for $\mu \in \Omega(X)$,

$$f(\psi(\mu)) = \lim_{n\to\infty} \int f\, dS^n(\mu) = \int f\, d\mu.$$

PROPOSITION 2.5. *ψ is continuous.*

Proof. We first observe that core X is compact. Hence sets of the form $\{x \in \operatorname{core} X : f(x) < \alpha\}$ with $f \in C$ and α real form a subbase for the topology of X since C is totally separating on core X. If $\mu \in \Omega(X)$ and for $f \in C$, $f(\psi(\mu)) < \alpha$, then since $\lim_{n\to\infty} \int f\, d(S^n\mu) = f(\psi(\mu))$, there exists an integer n such that $\int f\, dS^n\mu < \alpha$. Since S^n is weak* continuous on $\Omega(X)$, $U = \{\nu \in \Omega(X) : \int f\, d(S^n\nu) < \alpha\}$ is a weak* open set in $\Omega(X)$. But $\mu \in U$ and if $\nu \in U$, then $f(\psi(\nu)) < \alpha$. Hence ψ is continuous.

PROPOSITION 2.6. *If $f \in C(X \times X)$, $\mu \in \Omega(X)$,*

$$g_n(x) = \int f(x, y)\, d(S^n\mu)(y),$$

and $g(x) = f(x, \psi(\mu))$, then $g_n \in C(X)$ for every n and g_n converges uniformly to g.

Proof. It is easily seen that each $g_n \in C(X)$. We first suppose that $f(x, y)$ is convex in its second coordinate, i.e., for every $x, y, z \in X$, $f(x, y \cdot z) \le \frac{1}{2} f(x, y) + \frac{1}{2} f(x, z)$.

Then for every $x \in X$, $g_n(x)$ is a monotone decreasing sequence converging to $g(x)$. Since $g \in C(X)$, the convergence is uniform by Dini's theorem. Now let K be the class of all $f(x, y)$ such that $f(x, y)$ is convex in its second coordinate. If $f, h \in K$ and $\alpha \ge 0$ then it is easily shown that $\alpha f + h \in K$ and $\max(f, h) \in K$. Furthermore, if $h_1 \in C(X)$, $h_2 \in C$, and $f(x, y) = h_1(x) + h_2(y)$, then $f \in K$. Hence K separates points in $X \times X$. Thus by the Stone-Weierstrass theorem $K - K$ is dense in $C(X \times X)$. Since the result is true for each $f \in K - K$, a routine convergence argument proves the result for $f \in C(X \times X)$.

PROPOSITION 2.7. *If $\mu, \nu \in \Omega(X)$, then for each positive integer n, $S^n(\mu * \nu) = (S^n\mu) * (S^n\nu)$.*

Proof. We need only show that $S(\mu * \nu) = S(\mu) * S(\nu)$ since the result then follows by induction. If $f \in C(X)$,

$$\int f\, dS(\mu * \nu) = \int f((w \cdot x) \cdot (y \cdot z))\, d\mu(w) \times \nu(x) \times \mu(y) \times \nu(z)$$

$$= \int f((w \cdot y) \cdot (x \cdot z))\, d\mu(w) \times \mu(y) \times \nu(x) \times \nu(z)$$

$$= \int f\, d(S\mu) * (S\nu)).$$

This proves the equality.

PROPOSITION 2.8. *If $\mu, \nu \in \Omega(X)$, then $\psi(\mu * \nu) = \psi(\mu) \cdot \psi(\nu)$.*

Proof. If $f \in C(X)$, then

$$\int f\, dS^n(\mu * \nu) = \int f\, d(S^n\mu) * (S^n\nu)$$

$$= \int d(S^n\mu)(x) \int f(x \cdot y)\, d(S^n\nu)(y).$$

Now $\int f(x \cdot y)\, dS^n\nu(y)$ converges uniformly to $f(x \cdot \psi(\nu))$. Hence the limit of the above sequence equals

$$\lim_{n \to \infty} \int f(x \cdot \psi(\nu))\, d(S^n\mu)(x) = f(\psi(\mu) \cdot \psi(\nu)).$$

Thus $\psi(\mu * \nu) = \psi(\mu) \cdot \psi(\nu)$.

At this point we define the map $\eta\colon X \to \operatorname{core} X$ by $\eta(x) = \psi(e(x))$ for every $x \in X$. η is called the core map and η is continuous since ψ and e are continuous.

PROPOSITION 2.9. *If $x, y \in X$, then $\eta(x \cdot y) = \eta(x) \cdot \eta(y)$.*

Proof.

$$\eta(x \cdot y) = \psi(e(x \cdot y)) = \psi(e(x) * e(y)) = \psi(e(x)) \cdot \psi(e(y)) = \eta(x) \cdot \eta(y).$$

PROPOSITION 2.10. *Let $f \in C$ and $x \in \operatorname{core} X$; define $g(y) = f(x \cdot y)$ and $h(y) = f(x \cdot \eta(y))$. Then $g, h \in C$.*

Proof.

$$\begin{aligned} g(y \cdot z) &= f(x \cdot (y \cdot z)) = f((x \cdot x) \cdot (y \cdot z)) \\ &= f((x \cdot y) \cdot (x \cdot z)) \leq \tfrac{1}{2}f(x \cdot y) + \tfrac{1}{2}f(x \cdot z) \\ &= \tfrac{1}{2}g(y) + \tfrac{1}{2}g(z), \\ h(y \cdot z) &= f(x \cdot \eta(y \cdot z)) = f(x \cdot (\eta(y) \cdot \eta(z)) \\ &\leq \tfrac{1}{2}f(x \cdot \eta(y)) + \tfrac{1}{2}f(x \cdot \eta(z)) \\ &= \tfrac{1}{2}h(y) + \tfrac{1}{2}h(z). \end{aligned}$$

LEMMA 2.11. *If $f \in C$, then*

(i) $f\eta \in C$

and

(ii) *if* $\mu_1 \geq \mu_2$ *and* $\nu_1 \geq \nu_2$,

then

$$\int f\eta \, d\mu_1 * \nu_1 \geq \int f\eta \, d\mu_2 * \nu_2.$$

Proof. (i) is obvious. Now for $f \in C$

$$\begin{aligned}\int f\eta \, d\mu_1 * \nu_1 &= \int f(\eta(x) \cdot \eta(y)) \, d\mu_1(x) \times \nu_1(y) \\ &= \int d\nu_1(y) \int f(\eta(x) \cdot \eta(y)) \, d\mu_1(x) \\ &\geq \int d\nu_1(y) \int f(\eta(x) \cdot \eta(y)) \, d\mu_2(x) \\ &= \int f(\eta(x) \cdot \eta(y)) \mu_2(x) \times \nu_1(y) \\ &= \int f\eta \, d\mu_2 * \nu_1.\end{aligned}$$

It is similarly shown that

$$\int f\eta \, d\mu_2 * \nu_1 \geq \int f\eta \, d\mu_2 * \nu_2.$$

PROPOSITION 2.12. *If $\mu \in \Omega(X)$, $x \in$ core X, and $\mu \geq e(x)$, then $\psi(\mu) = x$.*

Proof. If $f \in C$, then

$$\lim_{n \to \infty} \int f \, dS^n \mu = f(\psi(\mu)) = f(\eta(\psi(\mu))) = \lim_{n \to \infty} \int f\eta \, dS^n \mu.$$

Applying Lemma 2.11 we see that

$$\int f\eta \, dS^n \mu \geq f(\eta(x)) = f(x) \quad \text{for each } n.$$

Hence $f(\psi(\mu)) \geq f(x)$. Since C is strongly separating, $\psi(\mu) = x$.

PROPOSITION 2.13. *If $\mu, \nu \in \Omega(X)$, and $\mu \leq \nu$, then $\psi(\mu) = \psi(\nu)$.*

Proof. If $\psi(\mu) = x$, then $\mu \geq e(x)$. Therefore $\nu \geq e(x)$. Hence $\psi(\nu) = x$.

PROPOSITION 2.14. *If $\mu, \nu \in \Omega(X)$, then*

$$\psi(\tfrac{1}{2}\mu + \tfrac{1}{2}\nu) = \psi(\mu) \cdot \psi(\nu).$$

Proof. $\frac{1}{2}\mu + \frac{1}{2}\nu \geq \mu * \nu$. Since $\psi(\mu * \nu) = \psi(\mu) \cdot \psi(\nu)$,

$$\psi(\tfrac{1}{2}\mu + \tfrac{1}{2}\nu) = \psi(\mu) \cdot \psi(\nu).$$

3. Characterization of core X

If $(X, \cdot)$ and $(Y, \cdot)$ are topological groupoids and $\gamma: X \to Y$ is a continuous homomorphism of X onto Y, we say that Y is the homomorphic image of X. If γ is a homeomorphism, then γ is called an iseomorphism and X and Y are said to be iseomorphic. Recall that when we say that $(X, \cdot)$ is a compact convex set we mean that X is a compact convex subset of a locally convex topological vector space and $x \cdot y = \frac{1}{2}x + \frac{1}{2}y$. The main result of this section is the following.

THEOREM 3.1. *If $(X, \cdot)$ is a compact mean space then* (core $X, \cdot$) *is iseomorphic to a compact convex set.*

First observe that $\Omega(X)$ is a compact convex set and the map $\psi: \Omega(X) \to$ core X is a continuous homomorphism. To prove the above theorem we shall prove that the homomorphic image of a compact convex set is iseomorphic to a compact convex set and this will be Theorem 3.5. Throughout this section we assume that $(Y, \cdot)$ is a compact convex set and ψ: $Y \to X$ is a continuous homomorphism of Y onto X. We first prove three lemmas.

LEMMA 3.2. *If $\alpha \in [0, 1]$ and $x_1, x_2, y_1, y_2 \in Y$ such that $\psi(x_1) = \psi(x_2)$ and $\psi(y_1) = \psi(y_2)$, then*

$$\psi(\alpha x_1 + (1 - \alpha)y_1) = \psi(\alpha x_2 + (1 - \alpha)y_2).$$

Proof. The result is clear in the case that $\alpha = \frac{1}{2}$ since ψ is a homomorphism. Similarly the result holds for $\alpha = \frac{1}{4}$ and $\alpha = \frac{3}{4}$. Continuing in this way the equality is easily established for all dyadic rationals α, i.e., $\alpha = m/2^n$ where m and n are nonnegative integers. The proof is by induction on n. Since the dyadic rationals are dense in $[0, 1]$ and ψ is continuous, the equality holds for all $\alpha \in [0, 1]$.

LEMMA 3.3. *If $\alpha \in (0, 1]$, $x_1, x_2, y \in Y$ and*

$$\psi(\alpha x_1 + (1 - \alpha)y) = \psi(\alpha x_2 + (1 - \alpha)y),$$

then $\psi(x_1) = \psi(x_2)$.

Proof. Let E be the set of numbers $a \in [0, 1]$ such that

$$\psi(ax_1 + (1 - a)y) = \psi(ax_2 + (1 - a)y).$$

Since ψ is continuous, E is closed. Since $\alpha \in E$, sup $E > 0$. To complete the proof we shall show that sup $E = 1$. To do this we let $\beta = \sup E$ and we show that $2\beta/(\beta + 1) \in E$. Now

$$\psi\left(\frac{\beta x_1}{\beta + 1} + \frac{\beta x_2}{\beta + 1} + \frac{(1 - \beta)y}{\beta + 1}\right) = \psi\left(\frac{\beta x_2 + (1 - \beta)y}{\beta + 1}\right) + \frac{\beta x_1}{\beta + 1}$$

Since $\beta \in E$, $\psi(\beta x_2 + (1 - \beta)y) = \psi(\beta x_1 + (1 - \beta)y)$. Applying Lemma 3.2, the above is equal to

$$\psi\left(\frac{\beta x_1 + (1 - \beta)y}{\beta + 1} + \frac{\beta x_1}{\beta + 1}\right) = \psi\left(\frac{2\beta x_1}{\beta + 1} + \frac{(1 - \beta)y}{\beta + 1}\right)$$

By the same argument

$$\psi\left(\frac{\beta x_1}{\beta + 1} + \frac{\beta x_2}{\beta + 1} + \frac{(1 - \beta)y}{\beta + 1}\right) = \psi\left(\frac{2\beta x_2}{\beta + 1} + \frac{(1 - \beta)y}{\beta + 1}\right)$$

Thus $2\beta/(\beta + 1) \in E$.

If $f \in C(X)$ such that for every $x, y \in X$, $f(x \cdot y) = \frac{1}{2}f(x) + \frac{1}{2}f(y)$, then f is called a continuous affine function. The following lemma is well known, but we include it for completeness.

LEMMA 3.4. *If $(X, \cdot)$ is a compact groupoid whose class of continuous affine functions separates points in X, then $(X, \cdot)$ is iseomorphic to a compact convex set.*

Proof. Let A be the class of continuous affine functions on X. If we let D be the product of the intervals $[\inf f(X), \sup f(X)]$ as f ranges over all $f \in A$, then D is a compact set in the product topology. Furthermore D is a convex set. We define $\delta\colon X \to D$ by $P\delta(x) = f(x)$ where P is the projection of D into $[\inf f(X), \sup f(X)]$. It is routinely verified that δ is continuous. δ is one-to-one since A separates points in X. Since X is compact δ is a homeomorphism and $\delta(X)$ is compact. It is easily verified that $\delta(x \cdot y) = \frac{1}{2}\delta(x) + \frac{1}{2}\delta(y)$. Thus $\delta(X)$ is a convex set in D and δ is an iseomorphism of X onto $\delta(X)$.

THEOREM 3.5. *If $(X, \cdot)$ is a compact groupoid and X is the homomorphic image of a compact convex set Y, then X is iseomorphic to a compact convex set.*

Proof. Let A denote all continuous affine functions on Y and let L be the set of $f \in A$ such that $f(x) = f(y)$ for every $x, y \in Y$ such that $\psi(x) = \psi(y)$. We shall prove the theorem by showing that if $\psi(x) \neq \psi(y)$ for some $x, y \in Y$, then there exists $f \in L$ such that $f(x) \neq f(y)$. This will complete the proof since for any $f \in L$, we may define $g\colon X \to R$ such that $g(\psi(x)) = f(x)$ for every $x \in Y$. Such a function g is continuous since if E is a closed subset of the reals, $g^{-1}(E) = \psi(f^{-1}(E))$ which is closed. It is routinely verified that g is affine on X. Hence if we prove the above assertion, then the continuous affine functions on X will separate points and we may apply Lemma 3.4.

First we must make some observations about A and Y. If $f \in A$, let $\|f\| = \sup |f|(Y)$. With this norm, A is a closed subspace of $C(Y)$ and is therefore a Banach space. Further define $\delta\colon Y \to A^*$ by $\delta(x)(f) = f(x)$ for every $f \in A$ and $x \in Y$. Then δ is an affine homeomorphism of Y onto a compact convex subset of the closed unit ball W of A^* in the weak* topology. For the sake of simplicity we may assume that Y is a compact convex subset of W such that every continuous affine function on Y is a linear functional in A and if $f \in A$, then $\|f\| = \sup \{|y(f)|\colon y \in Y\}$. We now show that if $E = \{\alpha x - (1 - \alpha)y\colon \alpha \in [0, 1]$, and $x, y \in Y\}$, then $E = W$. It is clear that $E \subset W$, and E is a compact convex set. If $w \notin E$, then there exists $x \in A$ such that

$$w(x) > \sup \{y(x)\colon y \in E\} = \sup \{|y(x)|\colon y \in Y\} = \|x\|.$$

But then $\|w\| > 1$ so that $w \notin W$.

Now let $H = \{\alpha x - \alpha y\colon x, y \in Y, \psi(x) = \psi(y)$, and α real$\}$. Now suppose that $x, y \in Y$ such that for some $\alpha \in [0, 1]$,

$$\alpha x - (1 - \alpha)y \in H \cap W.$$

We shall show that $\alpha = \frac{1}{2}$ and $\psi(x) = \psi(y)$. If c is a nonzero constant in A, then

$$(1 - 2\alpha)\cdot c = (\alpha x - (1 - \alpha)y)(c) = 0$$

since $\alpha x - (1 - \alpha)y \in W$. Hence $\alpha = \frac{1}{2}$. Now assume that for $x', y' \in Y$ with $\psi(x') = \psi(y')$ and for $\beta > 0$, $\frac{1}{2}x - \frac{1}{2}y = \beta x' - \beta y'$. But then

$$\left[\frac{\frac{1}{2}}{\frac{1}{2} + \beta}\right] x + \left[\frac{\beta}{\frac{1}{2} + \beta}\right] y' = \left[\frac{\frac{1}{2}}{\frac{1}{2} + \beta}\right] y + \left[\frac{\beta}{\frac{1}{2} + \beta}\right] x'$$

Thus $\psi(x) = \psi(y)$ by Lemma 3.3. We have thus shown that

$$W \cap H = \{\tfrac{1}{2}x - \tfrac{1}{2}y\colon x, y \in Y \quad \text{and} \quad \psi(x) = \psi(y)\}$$

so that $W \cap H$ is a weak* closed subset of W. We have also shown that if $\psi(x) \neq \psi(y)$, then $x - y \notin H$. By the Krein-Smulian theorem (see Dunford and Schwartz [1, p. 429]) H is a weak* closed subspace of A^*. Hence if $\psi(x) \neq \psi(y)$ for $x, y \in Y$, then there exists $f \in A$ such that $f(x) \neq f(y)$ where f annihilates H. Since f annihilates H, $f(a) = f(b)$ if $\psi(a) = \psi(b)$. This completes the proof.

4. Applications

In this section we shall investigate some of the consequences of Theorem 3.1. If S is a set and C is a class of real valued functions on S, then C is preconvex if for every $(w, x, y, z) \in S^4$, there is a unique $p \in S$ such that $f(p) \leq \frac{1}{4}(f(w) + f(x) + f(y) + f(z))$, for all $f \in C$. Observe that a preconvex class of functions is automatically totally separating.

THEOREM 4.1. *If X is a compact Hausdorff space and C is a preconvex class of real valued continuous functions on X, then there exists a compact convex set Y and a homeomorphism $\phi: Y \to X$ such that $f\phi$ is convex for every $f \in C$.*

Proof. If $x, y \in X$, define $x \cdot y$ to be the unique point z such that $\frac{1}{2}f(x) + \frac{1}{2}f(y) \geq f(z)$ for every $f \in C$. This point is obtained by applying the preconvexity of C to $(x, x, y, y) \in X^4$. We first prove that $\cdot$ is continuous. If $x, y \in X, f \in C$, and $\alpha > 0$, then let

$$U(f, \alpha) = \{p \in X : f(p) < \tfrac{1}{2}(f(x) + f(y)) + \alpha\}.$$

If x and y are fixed, then since C is preconvex

$$\bigcap \{\mathrm{cl}\,(U(f, \alpha) : f \in C, \alpha > 0\} = \{x \cdot y\}.$$

Thus sets of the form $U(f, \alpha)$ form a subbase for the neighborhood base at $x \cdot y$. Now for any fixed $U(f, \alpha)$, let

$$W = \{a : f(a) < f(x) + \alpha\} \times \{b : f(b) < f(y) + \alpha\}.$$

Then W is open in $X \times X$, $(x, y) \in W$ and $\cdot$ maps W into $U(f, \alpha)$. Thus $\cdot$ is continuous. Now if $(w, x, y, z) \in X^4$, then

$$\tfrac{1}{4}(f(w) + f(x) + f(y) + f(z)) \geq f((w \cdot x) \cdot (y \cdot z))$$

and

$$\tfrac{1}{4}(f(w) + f(x) + f(y) + f(z)) \geq f((w \cdot y) \cdot (w \cdot z))$$

for every $f \in C$. Hence $(w \cdot x) \cdot (y \cdot z) = (w \cdot y) \cdot (x \cdot z)$. The proof that $\cdot$ is commutative is similar. If $x \in X$, then $f(x \cdot x) \leq f(x)$ for every $f \in C$. Hence $x \cdot x = x$. Thus core $X = X$. Thus $(X, \cdot)$ is a compact mean space such that core $X = X$. Hence $(X, \cdot)$ is iseomorphic to a compact convex set by Theorem 3.1. It is clear that such an iseomorphism carries the functions in C into convex functions.

LEMMA 4.2. *If $(X, \cdot)$ is a compact groupoid whose continuous convex functions are totally separating, then every element of X is an idempotent.*

Proof. If $x \in X$ and f is convex, then $f(x) \geq f(x \cdot x)$. Since the continuous convex functions are totally separating $x = x \cdot x$.

Now if $(X, \cdot)$ is a compact groupoid, then a pseudometric d on X is convex if for every $(x, y, z) \in X^3$,

$$d(x, y \cdot z) \leq \tfrac{1}{2}d(x, y) + \tfrac{1}{2}d(x, z).$$

PROPOSITION 4.3. *If $(X, \cdot)$ is a compact commutative medial groupoid and the topology of X is given by a family $\mathfrak{F}$ of convex pseudometrics, then $(X, \cdot)$ is iseomorphic to a compact convex set.*

Proof. Suppose $(x, y) \in X \times X$ and $x \neq y$. Then there exists $d \in \mathfrak{F}$ such that $d(x, y) > 0$. If we define $f(y) = d(x, y)$ then $f \in C$ and $f(x) < f(y)$.

Hence C is totally separating. By Theorem 3.1 $(X, \cdot)$ is iseomorphic to a compact convex set.

5. Extensions of the groupoid (X, ·)

There are four fairly straightforward ways of extending a compact medial groupoid $(X, \cdot)$ to another such groupoid $(Y, \cdot)$. By an extension we shall mean a map $K\colon X \to Y$ such that K is an iseomorphism onto its image in Y. The first of these which we call a type one extension occurs if Y is any compact Hausdorff space, $K\colon X \to Y$ is a homeomorphism onto its image, and there exists $\gamma\colon Y \to Y$ such that $\gamma^2 = \gamma$ and $\gamma(Y) = K(X)$. In that case we define $\cdot$ on $K(X)$ in the obvious way so that K is an iseomorphism onto $K(X)$. $\cdot$ is then extended to Y by $x \cdot y = \gamma(x) \cdot \gamma(y)$. It is not difficult to show that by this definition $(Y, \cdot)$ is a medial groupoid.

PROPOSITION 5.1. *If $(X, \cdot)$ and $(Y, \cdot)$ are as above, then* core $Y = \gamma($core $X)$ *and if $(X, \cdot)$ is a compact mean space, then $(Y, \cdot)$ is a compact mean space.*

Proof. Core $Y = \gamma($core $X)$ is obvious. It is clear that the family of continuous convex functions on $K(X)$ is strongly separating. If g is a continuous convex function on $K(X)$, then the function h defined by $h(x) = g(\gamma(x))$ makes h a continuous convex function on Y. Also if $f \in C(Y)$ such that $f(Y) \subset [0, 1]$ and $f(K(X)) = \{0\}$, then f is convex. Using functions of these two types it is clear that the continuous convex functions strongly separate points in Y.

A type two extension is the extension from $(X, \cdot)$ to $(\Omega(X), *)$. Recall that $e\colon X \to \Omega(X)$ where $e(x)$ is the point mass measure at x. We have already observed that e is an iseomorphism of X into $\Omega(X)$. Furthermore, it is easily verified that $*$ is medial. Also we have already shown that core $\Omega(X) = e($core $X)$ in Proposition 2.3.

PROPOSITION 5.2. *If $(X, \cdot)$ is a compact mean space, then $(\Omega(X), *)$ is also a compact mean space.*

Proof. If we let C denote the continuous convex functions on X, then $C - C$ is dense in $C(X)$. Now if $f \in C$, we may define g on $\Omega(X)$ by $g(\mu) = \int f\, d\mu$ for each $\mu \in \Omega(X)$. g is convex by Lemma 2.2. The class of such functions separates points in $\Omega(X)$ since $C - C$ is dense in $C(X)$ and is totally separating on $e(X)$. Hence the continuous convex functions strongly separate points in $\Omega(X)$.

A type three extension of X extends X to the hyperspace of X which is denoted by 2^X. The hyperspace of X is the set of closed subsets of X. The hyperspace topology on 2^X is the weakest topology on 2^X such that $(\sup f)$ is continuous for every $f \in C(X)$. With this topology 2^X is a compact Hausdorff space. If $E, F \in 2^X$, then we define

$$E * F = \{x \cdot y\colon x \in E \quad \text{and} \quad y \in F\}.$$

A straightforward argument using nets and applying the compactness of X shows that $E * F \in 2^X$ and that $*$ is continuous. It is also clear that $*$ is medial. We define $K: X \to 2^X$ by $K(x) = \{x\}$, and we see that K is an iseomorphism of X into 2^X. Now if $E \in \text{core } 2^X$, then $E * E = E$. If we define a set E to be convex if $E * E = E$, then core 2^X is the set of all closed convex subsets of X.

PROPOSITION 5.3. *If X is a compact convex set, then $(2^X, *)$ is a compact mean space.*

Proof. We first observe that if $f \in C(X)$, then $\inf f = -\sup(-f)$ so that $\inf f$ is also continuous on 2^X. Now if $E, F \in 2^X$ and $x \in E$, but $x \notin F$, then there exists a continuous convex function f such that $f(x) < \inf f(F)$. But then $\inf f$ separates E and F. It is easily verified that $\inf f$ and $\sup f$ are convex on 2^X if f is convex on X. Hence the class of convex functions on 2^X is separating. If $E, F \in \text{core } 2^X$ then E and F are convex. Suppose $E \subset F$ and $E \neq F$. Then there exists a continuous affine function on X such that $\sup f(E) < \sup f(F)$. The case when $E \not\subset F$ is handled above so that the continuous convex functions on 2^X are strongly separating. This completes the proof since $(2^X, *)$ is medial.

If $\{X_\alpha: \alpha \in I\}$ is an indexed family of compact groupoids, then the product X of these is also a compact groupoid where $\cdot$ is defined on X coordinate-wise. If each X_α is medial, then X is medial.

Furthermore, if f is a continuous convex function on X_α, then the function g defined on X, by $g(x) = f(x_\alpha)$ where x_α is the αth coordinate of x makes g a continuous convex function on X. Since it is clear that core $X = \pi\{\text{core } X_\alpha: \alpha \in I\}$, it is easily seen that the class of continuous convex functions on X is strongly separating if this is true for each X_α.

Now suppose that $(X, \cdot)$ and $(Y, \cdot)$ are compact medial groupoids whose continuous convex functions are strongly separating. If $y_0 \in \text{core } Y$, then we may define $\gamma: X \to X \times Y$ by $\gamma(x) = (x, y_0)$. We shall call such an extension a type four extension. We note that if $\{y_0\} = \text{core } Y$, then $\gamma(\text{core } X) = \text{core } (X \times Y)$.

Using these four types of extensions initially applied to compact convex sets and using products as well it is possible to construct a rather large variety of examples of compact mean spaces other than compact convex sets. However, it is probable that there are many compact mean spaces which are not constructible in this way.

6. Compact groupoid valued integrals

In this section we assume that $(X, \cdot)$ is a compact commutative medial groupoid whose continuous convex functions are strongly separating. We further suppose that (Y, B, μ) is a probability measure space. A function f from Y into X is measurable if $f^{-1}(O) \in B$ for every open F_σ set O in X. There are a number of definitions equivalent to this. In particular, any one of the following three conditions is equivalent to the measurability of f:

(i) $f^{-1}(E) \in B$ for every Baire set E.

(ii) The real valued function $g(f(x))$ is measurable for every $g \in C(X)$.

(iii) For every O open and E closed with $E \subset O$, there exists $F \in B$ such that $f^{-1}(E) \subset F \subset f^{-1}(O)$. Now for every measurable f let μf^{-1} denote the Baire measure on X defined by $(\mu f^{-1})(E) = \mu(f^{-1}(E))$. It should be noted that every Baire measure has a unique extension to a regular Borel measure, so that μf^{-1} can be identified with a regular Borel measure on X. It is further noted that if g is a real valued Baire measurable function on X, then $\int g\, d\mu f^{-1} = \int g(f(x))\, d\mu(x)$. Now if f is a Baire measurable function from Y to X, then we define $\int f\, d\mu = \psi(uf^{-1})$.

Proposition 6.1. *If f and g are measurable, then $f \cdot g$ is measurable and $\int f \cdot g\, d\mu = \int f\, d\mu \cdot \int g\, d\mu$.*

Proof. Suppose O is open and E is closed with $E \subset O$. Then there exists $U_1, \ldots, U_n$ and $V_1, \ldots, V_n$ all open F_σ sets such that

$$\{(x, y) \colon x \cdot y \in E\} \subset \bigcup_{i=1}^{n} U_i \times V_i$$

and

$$\bigcup_{i=1}^{n} U_i \times V_i \subset \{(x, y) \colon x \cdot y \in O\}.$$

Thus

$$(f \cdot g)^{-1}(E) \subset \bigcup_{i=1}^{n} f^{-1}(U_i) \cap g^{-1}(V_i) \subset (f \cdot g)^{-1}(O).$$

This shows that there exists $W \in B$ such that

$$(f \cdot g)^{-1}(E) \subset W \subset (f \cdot g)^{-1}(O).$$

Now suppose h is a continuous convex function on X. Then

$$\begin{aligned} \int h\, d\mu(f \cdot g)^{-1} &= \int h(f(x) \cdot g(x))\, d\mu \\ &\le \int \tfrac{1}{2}h(f(x)) + \tfrac{1}{2}h(g(x))\, d\mu \\ &= \tfrac{1}{2}\int h\, d\mu f^{-1} + \tfrac{1}{2}\int h\, d\mu g^{-1}. \end{aligned}$$

Thus

$$\tfrac{1}{2}\mu f^{-1} + \tfrac{1}{2}\mu g^{-1} \ge \mu(f \cdot g)^{-1}.$$

But also

$$\tfrac{1}{2}\mu f^{-1} + \tfrac{1}{2}\mu g^{-1} \ge (\mu f^{-1}) \cdot (\mu g^{-1}).$$

Thus

$$\psi(\mu(f \cdot g)^{-1}) = \psi(\mu f^{-1} \cdot \mu g^{-1}) = \psi(\mu f^{-1}) \cdot \psi(\mu g^{-1}).$$

This establishes the equality.

PROPOSITION 6.2. (Jensen's Inequality). *If f is a measurable function from Y to X and g is a continuous convex function on X, then*

$$\int g(f(x))\,d\mu(x) \geq g\left(\int f\,d\mu\right).$$

Proof.

$$\int g(f(x))\,d\mu(x) = \int g\,d\mu f^{-1} \geq g(\psi(\mu f^{-1})) = g\left(\int f\,d\mu\right).$$

PROPOSITION 6.3. *If $\langle f_n \rangle$ is a sequence of measurable functions converging pointwise to f, then f is measurable and*

$$\lim_{n\to\infty} \int f_n\,d\mu = \int f\,d\mu.$$

Proof. Suppose $g \in C(X)$. Then gf_n is B-measurable for each n and

$$\lim_{n\to\infty} g(f_n(x)) = g(f(x))$$

for every $x \in X$. Hence gf is B-measurable. Thus f is measurable. Also, if $g \in C(X)$, then

$$\lim_{n\to\infty} \int g\,d(\mu f_n^{-1}) = \lim_{n\to\infty} \int gf_n\,d\mu = \int gf\,d\mu = \int g\,d\mu f^{-1}$$

by the bounded convergence theorem.

But then μf_n^{-1} converges weak* to μf^{-1}. Since ψ is continuous,

$$\lim_{n\to\infty} \int f_n\,d\mu = \lim_{n\to\infty} \psi(\mu f_n^{-1}) = \psi(\mu f^{-1}) = \int f\,d\mu.$$

7. Open questions

There are a number of questions that I have been unable to resolve. These are some of them:

(i) If X is a compact convex set, can one characterize all extensions $\gamma: X \to Y$ where $(Y, \cdot)$ is a compact mean space and core $Y = \gamma(X)$? What if X is a single point?

(ii) If $(X, \cdot)$ is a compact mean space such that $X * X = X$, does it follow that $X =$ core X?

(iii) If $(X, \cdot)$ is a compact mean space and $x \notin$ core X, does there exist a continuous convex function f such that $f(x) > \sup f(\text{core } X)$? Is this true in the case when $X = \Omega(Y)$ for Y a compact convex set?

(iv) If $(X, \cdot)$ is a compact mean space, under what conditions on X is $(X, \cdot)$ iseomorphic to $\Omega(Y)$ or 2^Y for some compact convex set Y?

(v) If $(X, \cdot)$ is a compact mean space, is $(2^X, *)$ a compact mean space? Can Proposition 5.2 be generalized at all?

(vi) Is there a reasonable theory of representing measures when $(X, \cdot)$ is not a compact mean space? What if we drop the condition that C be totally separating on core X?

(vii) O. H. Keller [3] proved that if X is a metrizable infinite dimensional compact convex set, then X is homeomorphic to the Hilbert cube. Is it possible to apply the results of Section 4 to obtain a topological Characterization of the Hilbert cube?

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