ALGEBRAIC VECTOR BUNDLES OVER THE l-HOLE TORUS

BY

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Introduction

Moore [5] has shown that every finite dimensional continuous vector bundle over a 2-sphere is isomorphic to an algebraic bundle. In this paper we use similar methods to study algebraic bundles over the l-hole torus T_l (i.e., an orientable surface of genus l). The procedure followed will be to construct a continuous function from T_l to S^2 and pull back the bundles over S^2 via this map. This yields then a representation of the vector bundles over T_l in terms of idempotent matrices with entries in certain integral extensions of quotient rings of affine rings. We then proceed to calculate certain dimensions of these rings. While we have not been able to obtain a complete classification of the algebraic bundles over these rings we are able to show that there are infinitely many non-isomorphic projective modules of rank 1 (2) when the field used is the complex (real) numbers.

1. Functions from T_l to S^2

We will begin by giving a description of the l-hole torus T_l as the set of zeros of a polynomial in three variables. T_1 can be constructed by rotating the circle $x_1 = 0$, $x_3^2 + (x_2 - 4)^2 = 1$ about the line $x_1 = 0$, $x_2 = 2$, with the result that a point (x_1, x_2, x_3) from R^3 is on T_1 if and only if it satisfies the equation

$$((x_1^2 + (x_2 - 2)^2)^{1/2} - 2)^2 + x_3^2 = 1.$$

If we expand this equation and square both sides we add no new real roots, so T_1 is exactly the set of points (x_1, x_2, x_3) from R^3 which satisfy the equation

$$x_1^4 + (x_2 - 2)^4 + (x_3^2 + 3)^2 + 2x_1^2(x_2 - 2)^2 + 2x_1^2(x_3^2 + 3) + 2(x_2 - 2)^2(x_3^2 + 3) - 16x_1^2 - 16(x_2 - 2)^2 = 0.$$

We will denote this polynomial by $T_1(x_1, x_2, x_3)$.

If we identify R^3 with $C \times R$ in the usual way then the map $h_l\colon C \times R \to C \times R$ given by $h_l(z, x_3) = (z^l, x_3)$ is a continuous onto function. It is easy to check that $h_l^{-1}(T_1)$ will be an l-hole torus. In terms of polynomials this means that we replace x_1 by $X_{1,l} = \operatorname{Re}(x_1 + ix_2)^l$ and x_2 by $X_{2,l} = \operatorname{Im}(x_1 + ix_2)^l$ in the polynomial $T_1(x_1, x_2, x_3)$ and this yields a polynomial

$$T_l(x_1, x_2, x_3) = T_1(X_{1,l}, X_{2,l}, x_3) \in R[x_1, x_2, x_3]$$

Received August 25, 1972.

such that $T_l = \{(x_1, x_2, x_3) \in R^3 \mid T_l(x_1, x_2, x_3) = 0\}$. We note here that a straightforward computation shows that T_l is irreducible in the ring

$$C[x_1, x_2, x_3].$$

Next we will need a continuous function from T_1 onto S^2. This will be given by the function g_l: $T_l \to S^2$ defined by

$$g_l(x_1, x_2, x_3) = (x_1^2 + x_2^2 + x_3^2)^{-1/2}(x_1, x_2, x_3).$$

If we use cylindrical coordinates and let

$$A_l = \left\{(r, \theta, x_3) \in T_l \mid \theta \notin \left(\frac{\pi}{l}, \frac{2\pi}{l}\right)\right\}, \quad B_l = \left\{(r, \theta, x_3) \in S^2 \mid \theta \notin \left(\frac{\pi}{l}, \frac{2\pi}{l}\right)\right\}$$

then g_l: $(T_l, A) \to (S^2, B)$ is a relative homeomorphism, that is, g_l is a homeomorphism from $T_l \backslash A$ to $S^2 \backslash B$. In order to prove this it is sufficient to show that g_l is a one-to-one map of $T_l \backslash A$ to $S^2 \backslash B$. So we assume there are two points (r, θ, x_3) and (r', θ, x_3') from $T_l \backslash A$ such that $g_l(r, \theta, x_3) = g_l(r', \theta, x_3')$. This would say that these two points lie on the same ray from the origin, so we can conclude that $rx_3' = r'x_3$. But $h_l(r, \theta, x_3) = (r^l, l\theta, x_3) \in T_1$, with $\pi < n\theta < 2\pi$, so it suffices to show that for $\pi < \phi < 2\pi$, ϕ fixed and $V \geq 0$, V is a strictly decreasing function of ρ, for (ρ, ϕ, V) a point on T_1. In cylindrical coordinates the equation of T_1 is $((\rho^2 \cos^2 \phi + (\rho \sin \phi - 2)^2)^{1/2} - 2)^2 + V^2 = 1$, and the derivative of V with respect to ρ yields,

$$2VV' = -\frac{((\rho^2 \cos^2 \phi + (\rho \sin \phi - 2)^2)^{1/2} - 2)(2\rho - 4 \sin \phi)}{((\rho^2 \cos^2 \phi + (\rho \sin \phi - 2)^2)^{1/2}}.$$

However, when $\pi < \phi < 2\pi$, $(\rho^2 \cos^2 \phi + (\rho \sin \phi - 2)^2)^{1/2} \geq 2$ and $\sin \phi \leq 0$, so $V' \leq 0$ and the function is strictly decreasing.

We now claim that the function g_l induces an isomorphism

$$g_l^*: H^2(S^2; Z) \to H^2(T_l; Z).$$

Since g_l is a relative homeomorphism it follows that g_{l*}: $H_2(T_l, A_l; Z) \to H_2(S^2, B_l; Z)$ is an isomorphism [8, p. 202]. By applying the excision axiom we get that $H_n(T_l, A_l; Z) = 0$ for $n \geq 1, n \neq 2$ and $H_2(T_l, A_l; Z) = Z$. Also, since A_l is the l-hole torus with a disk removed, its homology groups are the same as the groups of the space formed by joining $2l$ circles at a point. Thus the homology sequence for the pair (T_l, A_l), with integer coefficients, is

$$H_2(A_l) \xrightarrow{i_{2*}} H_2(T_l) \xrightarrow{j_{2*}} H_2(T_l, A_l) \xrightarrow{\partial_{2*}} H_1(A_l) \xrightarrow{i_{1*}} H_1(T_l) \xrightarrow{j_{1*}} H_1(T_l, A_l)$$

or

$$0 \xrightarrow{i_{2*}} Z \xrightarrow{j_{2*}} Z \xrightarrow{\partial_{2*}} Z \oplus Z \xrightarrow{i_{1*}} Z \oplus Z \xrightarrow{j_{1*}} 0.$$

Since i_{2*} is the zero map, j_{2*} is a monomorphism. Since j_{1*} is the zero map, i_{1*} is onto and thus is an isomorphism. But then $\operatorname{Im} \partial_{2*} = \ker i_{1*} = 0$, so j_{2*} is onto and thus is an isomorphism. A similar argument shows that the map

$$J_{2*}\colon H_2(S^2) \to H_2(S^2, B_l)$$

is an isomorphism. Finally, if we look at the commutative diagram

$$\begin{tikzcd} H_2(T_l) \arrow[r, "j_{2*}"] \arrow[d, "g_{l*}"'] & H_2(T_l, A_l) \arrow[d, "g_{l*}"] \\ H_2(S^2) \arrow[r, "J_{2*}"] & H_2(S^2, B_l) \end{tikzcd}$$

we see that $g_{l*}\colon H_2(T_k) \to H_2(S^2)$ is an isomorphism, and this allows us to conclude that $g_l^*\colon H^2(S^2; Z) \to H^2(T_l, Z)$ is also an isomorphism.

2. Vector bundles and projective modules

Moore [5] has shown that there is a one-to-one correspondence between the set of integers and the set of all equivalence classes of complex one-plane bundles over S^2. If θ^2 represents the trivial two-plane bundle then the idempotent matrix

$$N_n = \frac{1}{h_n}\begin{bmatrix} (1 - x_3)^n & (x_1 - ix_2)^n \\ (x_1 + ix_2)^n & (1 - x_3)^n \end{bmatrix}, \quad n \geq 0,$$

where $h_n = (1 + x_3)^n + (1 - x_3)^n$, defines an endomorphism of θ^2 in terms of the coordinates of a point $(x_1, x_2, x_3) \in S^2$, and the image of θ^2 under this map is the bundle γ_n corresponding to the positive integer n. The bundle corresponding to the negative integer $-n$ is the image of the endomorphism of θ^2 defined by the matrix N_{-n} which is obtained from N_n by replacing x_2 by $-x_2$, so $N_{-n} = N_n$. It is also shown that the first chern class $c_1(\gamma_n) = n$.

In the real case we notice that the trivial real four plane bundle θ^4 over S^2 comes from the complex bundle θ^2 by restricting scalar multiplication to R. For $n \geq 0$ consider the matrix

$$M_n = \frac{1}{h_n}\begin{bmatrix} (1 - x_3)^n I & B_n \\ B_n' & (1 + x_3)^n I \end{bmatrix} \quad \text{where } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and if $a_1 + ia_2 = (x_1 + ix_2)^n$ then

$$B_n = \begin{bmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{bmatrix},$$

and B_n' is the transpose of B_n. Then M_n defines an endomorphism of θ^4 and if η_n is the image of M_n, there is a one-to-one correspondence from the set Z^+ of nonnegative integers and the set of equivalence classes of real two plane vector bundles over S^2. Since these bundles are restrictions of complex bundles they are oriented and the euler class is given by $e(\eta_n) = c_1(\gamma_n) = n$.

Since the maps $g_l\colon T_l \to S^2$ are continuous functions they pullback the vector bundles over S^2 described above to vector bundles over T_l.

PROPOSITION 2.1. (i) *There exists a bijection from* Z *into* $\mathrm{Vect}_C^1(T_l)$ *given by* $n \to g_l^*(\gamma_n)$ *for* $l = 1, 2, \ldots$.

(ii) *There exists an injection from* Z^+ *into* $\mathrm{Vect}_R^2(T_l)$ *given by* $n \to g_l^*(\eta_n)$ *for* $l = 1, 2, \ldots$.

Proof. Recall that $c_1(\gamma_n) = n$, so, since $c_1 g_l^*(\gamma_n) = g_l^*(c_1(\gamma_n)) = n$, and c_1 is a one-to-one correspondence from $\mathrm{Vect}_C^1(T_l)$ to $H^2(T_l, Z)$ [2, p. 234], it follows that $g_l^*(\gamma_n)$ is equivalent to $g_l^*(\gamma_k)$ if and only if $n = k$. Since g_l^* is an isomorphism this proves (i). Since all of the bundles η_n are orientable, we can use the same property of their Euler classes to show that the map in (ii) is an injection.

We note here that the bundle $g_l^*(\eta_n)$, $n \geq 0$, is the image of the trivial bundle $C^2 \times T_l$ under the endomorphism of $C^2 \times T_l$ defined at the point

$$(x_1, x_2, x_3) \in T_l$$

by the idempotent matrix

$$g_l^* N_n = \frac{1}{h_{ln}} \begin{bmatrix} (1 - g_{l3})^n & (g_{l1} + i g_{l2})^n \\ (g_{l1} - i g_{l2})^n & (1 - g_{l3})^n \end{bmatrix}$$

where the g_{li} are the components of the function g_l and $h_{ln} = (1 + g_{l3})^n + (1 - g_{l3})^n$. As above, $g_l^* N_{-n} = (g_l^* N_n)^-$. If we again let

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B_{ln} = \begin{bmatrix} a_{l1} & -a_{l2} \\ a_{l2} & a_{l1} \end{bmatrix},$$

where $(a_{l1} + i a_{l2})^n = (g_{l1} + i g_{l2})^n$, then the bundle $g_l^*(\eta_n)$ is the image of the endomorphism of $R^4 \times T_l$ defined at the point $(x_1, x_2, x_3) \in T_l$ by the idempotent matrix

$$g_l^*(M_n) = \frac{1}{h_{ln}} \begin{bmatrix} (1 - g_{l3})^n I & B_{ln} \\ B'_{ln} & (1 + g_{l3})^n I \end{bmatrix}.$$

Denote by (T_l) the ideal in either $R[x_1, x_2, x_3]$ or $C[x_1, x_2, x_3]$ generated by the polynomial $T_l(x_1, x_2, x_3)$. Since $T_l(x_1, x_2, x_3)$ is irreducible over $C[x_1, x_2, x_3]$ it is irreducible over $R[x_1, x_2, x_3]$, and the ideal (T_l) is a prime ideal in either ring. Let

$$A_l = R[x_1, x_2, x_3]/(T_l) \quad \text{and} \quad CA_l = C[x_1, x_2, x_3]/(T_l), \quad l = 1, 2, \ldots.$$

Since (T_l) is a prime ideal all of these rings are integral domains. The standard inclusion map of R into C extends to an injection of A_l into CA_l for each l, and if we identify A_l with its image in CA_l, then CA_l is a free A_l module with basis $\{1, i\}$. Next, we adjoin $\alpha = (x_1^2 + x_2^2 + x_3^2)^{1/2}$ to these rings to get the rings $B_l = A_l[\alpha]$ and $CB_l = CA_l[\alpha]$ which are integral extensions of A_l and CB_l, respectively.

Let $S_l = \{f \in B_l \mid f(x) \neq 0 \text{ for all } x \in T_l\}$; then S_l is a multiplicative system in B_l as well as in CB_l, and we can form the rings of quotients $(B_l)_{S_l}$ and $(CB_l)_{S_l}$. Since B_l and CB_l are integral domains, so are $(B_l)_{S_l}$ and $(CB_l)_{S_l}$. If we denote the ring of continuous F-valued functions on T_l by $\mathscr{C}_F(T_l)$, where $F = C$ or R, then, in the usual way, we can identify $(B_l)_{S_l}$ and $(CB_l)_{S_l}$ with subrings of $\mathscr{C}_R(T_l)$ and $\mathscr{C}_C(T_l)$.

Swan [9] has shown that the section functor Γ defines a one-to-one correspondence between the set of equivalence classes of n-plane F vector bundles over a normal space X and isomorphism classes of projective modules of rank n over the ring of continuous F-valued functions on X. Since both h_{ln} and α are not zero at any point on T_l, they are in S_l, and thus the matrices $g_l^* N_n$ have all their entries in the rings $(CB_l)_{S_l}$ and the matrices $g_l^* M_n$ have all their entries in the rings $(B_l)_{S_l}$; hence they define the projective modules

$$Q_{l,n} = (CB_l)^2_{S_l} g_l^* N_n \quad \text{and} \quad P_{l,n} = (B_l)^4_{S_l} g_l^* M_n.$$

Since $\Gamma(g_l^*(\gamma_n))$ and $Q_n \otimes \mathscr{C}_C(T_l)$ are both the image of $\mathscr{C}_C^2(T_l)$ under the map defined by $g_l^* N_n$, they are isomorphic as $\mathscr{C}_C(T_l)$ modules. Similarly, $P_n \otimes \mathscr{C}_R(T_l)$ is isomorphic to $\Gamma(g_l^*(\eta_n))$. If we let $\mathscr{P}(\Lambda, n)$ denote the set of isomorphism classes of rank n projective Λ modules, then the above can be summarized as follows:

THEOREM 2.2. (i) *The map* $n \to Q_{l,n}$ *gives an injection of* Z *into* $\mathscr{P}((CB_l)_{S_l}, 1)$ *for each* $l = 1, 2, \ldots$.

(ii) *The map* $n \to P_{ln}$ *gives an injection of* Z^+ *into* $\mathscr{P}((B_l)_{S_l}, 2)$ *for each* $l = 1, 2, \ldots$.

We record some properties of these modules in the following propositions.

PROPOSITION 2.3. $Q_{l,n}$ *is not stably trivial for any* n *or* l.

Proof. If Q_{ln} were stably trivial, since free $(CB_l)_{S_l}$ modules correspond to free $\mathscr{C}_C(T_l)$ modules under the map $\_\_ \otimes \mathscr{C}_C(T_l)$, this would say that $\Gamma(g_l^*(\gamma_n))$ were stably trivial. This in turn would imply that the bundles $g_l^*(\gamma_n)$ were stably trivial, that is, for some trivial bundle θ^m, $g_l^*(\gamma_n) \oplus \theta^m = \theta^{m+1}$. But $c_1(g_l^*(\gamma_n)) = n$, so $c_1(g_l(\gamma_n) \oplus \theta^m) = n$, a contradiction.

PROPOSITION 2.4. *For* n *and* l *odd,* $P_{l,n}$ *is not stably trivial.*

Proof. The proof is the same as above, using the fact that $w_2(g_l^*(\eta_n)) \equiv c_1(g_l^*(\gamma_n)) \bmod 2$.

PROPOSITION 2.5. *For each* l *and* $n > 0$, $P_{l,n}$ *is indecomposable.*

Proof. If $P_{l,n}$ were decomposable then the vector bundle $g_l^*(\eta_n)$ would also have to be able to be written as a Whitney sum of 1-plane bundles. We will show that this is impossible in the next section.

3. Real 1-plane bundles over T_l

The first Stiefel-Whitney characteristic class gives an isomorphism

$$w_1\colon \mathrm{Vect}_R^1\,(X) \to H^1(X; Z_2)$$

[2, p. 234].
Thus, since $H^1(T_l; Z_2) = Z_2^{2l}$, we must find 2^{2l} nonequivalent 1-plane bundles over each T_l.

We will first classify the bundles over T_1. For the present we shall consider T_1 as $S^1 \times S^1$. It is well known that $S^1 = RP^1$, the real projective line, so we may consider T_1 as $RP^1 \times RP^1$. It is also well known that the bundle $\xi = (E, p, RP^1)$, with $E = \{([x], \lambda x) \in RP^1 \times R^2 \mid \lambda \in R\}$, is a nontrivial 1-plane bundle over RP^1 with $w_1(\xi) = \alpha$, where α is the nonzero element of $H^1(RP^1; Z_2)$, [4, pp. 2 and 7]. Let p_1 and p_2 be the projection maps from T_1 onto the first and second coordinates, respectively. Let $\beta_1 = p_l^*(\xi)$ and $\beta_2 = p_2^*(\xi)$, and let e_1 and e_2 be the injections of RP^1 into $RP^1 \times RP^1$. Then the sequence

$$H^1(RP^1; Z_2) \underset{e_1{*}}{\overset{p_1{*}}{\rightleftarrows}} H^1(RP^1 \times RP^1; Z_2) \underset{p_2{*}}{\overset{e_2{*}}{\rightleftarrows}} H^1(RP^1; Z_2)$$

is split exact in either direction [4, Appendix A, p. 16]. So

$$\begin{aligned} w_1(\beta_1) &= w_1(p_1^*(\xi)) = p_1^*(w_1(\xi)) = p_1^*(\alpha) \\ &= (\alpha, 0) \in H^1(RP^1; Z_2) \oplus H^1(RP^1; Z_2) = Z_2 \oplus Z_2, \end{aligned}$$

and similarly, $w_1(\beta_2) = (0, \alpha) \in Z_2 \oplus Z_2$. Finally, the bundle $\beta_1 \otimes \beta_2$ (see [2] for a definition) has $w_1(\beta_1 \otimes \beta_2) = w_1(\beta_1) + w_1(\beta_2) = (\alpha, \alpha)$. Thus, with θ_R^1, these give a representative of each possible equivalence class in $\mathrm{Vect}_R^1\,(T_1)$.

The l-hole torus can be constructed by removing an open disk from the l-1-hole torus and attaching a 1-hole torus which also has had a disk removed. We note that if D is an open disk contained in T_l then the inclusion map $i\colon T_l\backslash D \to T_l$ induces an isomorphism $i^*\colon H^1(T_l; Z_2) \to H^1(T_l\backslash D; Z_2)$. It follows then that if the bundles $\theta^1 = \beta_1, \beta_2, \ldots, \beta_{2l}$ are representative of each of the possible equivalence classes of bundles over T_l, then the $B_j' = i^*(\beta_j)$ for $j = 1, \ldots, 2^{2l}$ give representatives of all the equivalence classes of 1-plane bundles over $T_l\backslash D$.

The classification of the 1-plane bundles over T_l will now proceed by induction. As above, T_l can be considered as $T_{l-1}\backslash D_a \cup T_1\backslash D_b$ for D_a and D_b open disks. If ζ_j is a 1-plane bundle over $T_{l-1}\backslash D_a$ and δ_k is a 1-plane bundle over $T_1\backslash D_b$ then by the clutching construction [1, p. 20], $\zeta_j \cup \delta_k$ is a 1-plane bundle over T_l. If we let $A = \partial(T_{l-1}\backslash D_a)$ which is identified in T_l with $\partial(T_1\backslash D_b)$, then the Mayer-Vietoris sequence of the triple $(T_{l-1}\backslash D_a, T_1\backslash D_b, A)$ yields the exact sequence

$$H^{0\#}(A; Z_2) \longrightarrow H^1(T_l; Z_2) \xrightarrow{\ \psi\ } H^1(T_{l-1}\backslash D_a; Z_2) + H^1(T_1\backslash D_b; Z_2).$$

Since A is $\partial(T_1\backslash D) = S^1$, $H^{0\#}(A; Z_2) = 0$, ψ is a monomorphism, and since

$$H^1(T_l; Z_2) = Z_2^{2l} = H^1(T_{l-1}\backslash D_a; Z_2) \oplus H^1(T_1\backslash D_b; Z_2),$$

it follows that ψ is an isomorphism. The map ψ is induced by the injections

$$i_1\colon T_{l-1}\backslash D_a \to T_l \quad \text{and} \quad i_2\colon T_1\backslash D_b \to T_l,$$

and it is easy to check that

$$\psi(w_1(\zeta_j \cup \delta_k)) = (i_1^*(w_1(\zeta_j)), i_2^*(w_1(\delta_k)) = (w_1(\zeta_j), w_1(\delta_k)).$$

In this way we get all possible elements of Z_2^{2l} as images of w_1 of bundles of the form $\zeta_j \cup \delta_k$. Thus the classification of 1-plane bundles over T_l is reduced to the classification of 1-plane bundles over T_{l-1} and over T_1, so by induction we are done.

THEOREM 3.1. *No nontrivial orientable real* 2*-plane bundle over* T_l *can be decomposed into the Whitney sum of real* 1*-plane bundles.*

Proof. Let η be an orientable 2-plane bundle over T_l and suppose $\eta = \delta_1 \oplus \delta_2$, where each δ_i is a 1-plane bundle over T_l. Since η is orientable, $w_1(\eta) = 0$ [2, p. 244], so $0 = w_1(\eta) = w_1(\zeta_1 \oplus \zeta_2) = w_1(\zeta_1) + w_1(\zeta_2)$, which implies that $w_1(\zeta_1) = w_1(\zeta_2)$ or $\zeta_1 \cong \zeta_2$. Thus it suffices to show that $\zeta \oplus \zeta = \theta^2$ for any 1-plane bundle ζ over T_l. By our construction $\zeta = \delta_1 \cup \cdots \cup \delta_l$ where each δ_i is a 1-plane bundle over $T_1\backslash D$. Since

$$\zeta \oplus \zeta = (\delta_1 \cup \cdots \cup \delta_l) \oplus \delta_1 \cup \cdots \cup \delta_l) = (\delta_1 \oplus \delta_1) \cup \cdots \cup (\delta_l \oplus \delta_l)$$
[1, p. 22],

it suffices to show that $\delta \oplus \delta = \theta^2$ for any 1-plane bundle over $T_1 - D$. There are two possible cases: (a) $\delta = p_i^*(\xi)$ for $i = 1$ or 2, but then $\delta \oplus \delta = p_i^*(\xi) \oplus p_i^*(\xi) = p_i^*(\xi \oplus \xi) = p_i^*(\theta^2) = \theta^2$, or (b) $\delta = p_1^*(\xi) \otimes p_2^*(\xi)$, but then

$$\begin{aligned}
\delta \oplus \delta &= (p_1^*(\xi) \otimes p_2^*(\xi)) \oplus (p_1^*(\xi) \otimes p_2^*(\xi)) \\
&= p_1^*(\xi) \otimes (p_2^*(\xi) \oplus p_2^*(\xi)) \\
&= p_1^*(\xi) \otimes \theta^2 \\
&= (p_1^*(\xi) \otimes \theta^1) + (p_1^*(\xi) \otimes \theta^1) \\
&= p_1^*(\xi) \oplus p_1^*(\xi) \\
&= \theta^2.
\end{aligned}$$

4. Dimensions

The projective modulus of the domain Λ, proj mod Λ, is the least integer k such that every projective Λ module is the direct sum of a free module and a module of rank $\leq k$ [3].

THEOREM 4.1. *proj mod* $(B_l)_{S_l} = 2$ *for* $l = 1, 2, \ldots$.

Proof. Serre [7] has shown that for a commutative integral domain Λ,

$$\text{proj mod } \Lambda \leq \dim \max \Lambda$$

where $\max \Lambda$ is the maximum spectrum of Λ. It is well known that $\dim \max \Lambda \leq$ K-dim Λ, when K-dim Λ is the Krull dimension of Λ, so proj mod $\Lambda \leq K$-dim Λ. Now, K-dim $R[x_1, x_2, x_3] = 3$ and since $(T_l(x_1, x_2, x_3))$ is a prime ideal for each l,

$$K\text{-dim } R[x_1, x_2, x_3]/(T_l(x_1, x_2, x_3)) = 2.$$

It is also well known that for any multiplicatively closed set S contained in a ring Λ, K-dim $\Lambda_S \leq K$-dim Λ. Also it follows from the "Lying-over theorem" and the "Going up theorem" [6, p. 30] that if S is an integral extension of Λ then K-dim $S = K$-dim Λ. Thus K-dim $B_l = K$-dim $A_l = 2$, and K-dim $(B_l)_{S_l} \leq$ 2. But we have shown in Proposition 2.5 that the projective modules $P_{l,n}$ are irreducible and have rank 2. Thus

$$2 \leq \text{proj mod } (B_l)_{S_l} \leq K\text{-dim } (B_l)_{S_l} \leq 2.$$

Acknowledgement. This paper represents part of the author's doctoral dissertation submitted to Syracuse University in July 1970. The research was partially supported by a National Science Foundation grant. The author wishes to express his sincere appreciation to Professor David Lissner for his help and encouragement during the writing of this paper.

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