INTEGRAL REPRESENTATIONS OF FRACTIONAL POWERS OF INFINITESIMAL GENERATORS

BY

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Introduction

The main purpose of this paper is to give a class of integral representations for the fractional powers $(-A)^\alpha$, where $0 < \alpha$ and A is the infinitesimal generator of a bounded strongly continuous semigroup T_t of bounded linear operators on a Banach space X. The definition of $(-A)^\alpha$ used in [3] is

$$\lim_{\varepsilon \to 0} C \int_\varepsilon^\infty t^{-\alpha-1}(I - T_t)^r f \, dt \tag{0.1}$$

where $0 < \alpha < r$, r is a positive integer and C is an appropriate constant. For the case $0 < \alpha < 1$, $r = 1$, the above definition of $(-A)^\alpha$ can be motivated by noting that if $a < 0$, then by a simple change of variable

$$\int_0^\infty t^{-\alpha-1}(1 - e^{ta}) \, dt = (-a)^\alpha \int_0^\infty t^{-\alpha-1}(1 - e^{-t}) \, dt;$$

so $(-a)^\alpha$ is a constant times the integral on the left. Komatzu [2] has shown that the operator defined by (0.1) can also be represented in the form

$$\lim_{\varepsilon \to 0} C \int_\varepsilon^\infty t^{-\alpha-1}(-tA(I - tA)^{-1})^r f \, dt. \tag{0.2}$$

A similar motivation could be given for this integral representation.

In this paper we introduce "kernels"

$$S(\sigma_{(t)}) = \int_0^\infty T_u \, d\sigma_{(t)}(u)$$

where $d\sigma_{(t)}(u) = d\sigma(u/t)$ and show (see Theorem 1.4) that limits of the form

$$\lim_{\varepsilon \to 0} \int_\varepsilon^\infty t^{-\alpha-1} S(\sigma_{(t)}) f \, dt$$

all define the same operator as σ ranges over a wide class of measures. In Section 2 we show that the "kernels" in (0.1) and (0.2) correspond to special choices of σ within the class. In Section 3 we show that the class cannot be enlarged by establishing a converse to Theorem 1.4 (see Theorem 3.1). In Section 4 we define Lipschitz spaces corresponding to the "kernels" $S(\sigma_{(t)})$ and prove a Lions-Peetre type theorem, Theorem 4.1, relating these Lipschitz spaces to

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certain real interpolation spaces; special cases of this theorem were proved in [2], [3].

1. The integral representation

Throughout, T_t denotes a bounded strongly continuous semigroup of bounded linear operators on a Banach space X with norm $\| \quad \|$. The infinitesimal generator is denoted A and its domain space is denoted $D(A)$. We let M denote the class of all complex Borel measures on $[0, \infty)$; we will often refer to these as measures. For each semigroup T_t and measure σ we define the operator $S(\sigma)$ by

$$S(\sigma)f = \int_0^\infty T_u f \, d\sigma(u), \quad f \in X,$$

where the integral converges in X. For σ, μ in M, $\sigma * \mu$ denotes the usual convolution, $|\sigma|$ the total variation measure, $\sigma^{(k)}$ denotes $\sigma * \cdots * \sigma$ (k times), $\mathscr{L}\sigma$ denotes the usual Laplace transform of σ, and $\mathscr{L}M$ denotes the class of all Laplace transforms of measures in M. For $t > 0$, $\sigma_{(t)}(E) = \sigma(t^{-1}E)$ for each Borel set E in $[0, \infty)$. We let δ_t denote the measure which is the unit point mass at t, $t \geq 0$. By the extended measures, denoted M_e, we mean the class of set functions from the bounded Borel sets of $[0, \infty)$ to the complex numbers which are countably additive on the Borel sets of each bounded interval. The letter u usually denotes a point in $[0, \infty)$ and du the Lebesgue measure on $[0, \infty)$. For a locally integrable function h, $\sigma * h$ is the usual convolution of a measure and function and $\mathscr{L}h$ denotes the Laplace transform of h. We also write $\sigma * u^\alpha$ for $\sigma * h$, if $h(u) = u^\alpha$.

1.1. Definition. For each measure σ and $\alpha > 0$ define the operator $B_\alpha(\sigma)$ on X by

$$B_\alpha(\sigma)f = \lim_{\varepsilon \to 0} \int_\varepsilon^\infty t^{-\alpha-1} S(\sigma_{(t)}) f \, dt$$

where the domain of $B_\alpha(\sigma)$ is all f in X for which the above limit exists in X.

1.2. Definition. A measure σ is an r-measure, where r is a positive integer, if

$$\int_0^\infty |u^{-1}(\sigma * u^\alpha)| \, du < \infty$$

for $0 < \alpha < r$.

1.3. Lemma. *If σ is an r-measure and $0 < \alpha < r$, then the integral*

$$\int_0^\infty \mathscr{L}\sigma(t) t^{-\alpha-1} \, dt$$

converges absolutely. (Denote the values of this integral by $C_\alpha(\sigma)$.)

Proof. It clearly suffices to show that $\mathscr{L}\sigma(s) = O(s^\alpha)$, as $s \to 0$, for each $0 < \alpha < r$. Clearly

$$\mathscr{L}(\sigma * u^\alpha)(s) = C\mathscr{L}\sigma(s)s^{-\alpha-1}.$$

This, and the fact that $H = u^{-1}\sigma * u^\alpha$ is in L_1, gives

$$|\mathscr{L}\sigma(s)| \leq Cs^{\alpha+1}\left[s^{-1}\int e^{-us}su|H(u)|\,du\right].$$

And, the right-hand side of this last inequality is $O(s^\alpha)$.

We now state the main theorem.

1.4. THEOREM. *If* $0 < \alpha < r$ *and* r *is a positive integer, then the operators* $C_\alpha(\sigma)^{-1}B_\alpha(\sigma)$ *are equal as* σ *ranges over the* r*-measures for which* $C_\alpha(\sigma) \neq 0$.

Before we give the proof of the theorem we will obtain the following five needed lemmas.

The first lemma is well known.

1.5. LEMMA. (i) *The map* $\mu \to S(\mu)$ *is a homomorphism from the convolution algebra of complex measures into the bounded linear operators on* X.

(ii) $S(\mu_{(\varepsilon)})f \to (\int d\mu)f$ *as* $\varepsilon \to 0$ *for each* f *in* X *and complex measure* μ.

1.6. LEMMA. *If* $\mu_1, \mu_2 \in M$, $\mu \in M_e$ *and* $\mu_1 * \mu, \mu_2 * \mu \in M$, *then*

$$S(\mu_1 * \mu)S(\mu_2) = S(\mu_1)S(\mu * \mu_2).$$

Proof. Apply Lemma 1.5 and the fact that the extended measures form an associate algebra.

The following lemma is easily proved.

1.7. LEMMA. *Suppose that*

(i) F *is a differentiable complex valued function on* $(0, \infty)$,

(ii) $\int_0^\infty |g(u)|u^{-1}e^{-tu}\,du < \infty$, *each* $t > 0$, *and*

(iii) $\mathscr{L}g(s) = F'(s)$, $s > 0$.

Then for some C,

$$\mathscr{L}(-u^{-1}g(u) + C\,\delta_0) = F(s), \quad s > 0.$$

1.8. LEMMA. *Given a measure* σ, *positive real numbers* α, ε *and a Borel subset* E *of* $[0, \infty)$, *let* $v_\varepsilon(\sigma, E) = \int_\varepsilon^\infty \sigma_{(t)}(E)t^{-\alpha-1}\,dt$. *Then* v_ε *is a measure such that*

(i) $\int_\varepsilon^\infty \mathscr{L}\sigma_{(t)}(s)t^{-\alpha-1}\,dt = \int e^{-us}\,dv_\varepsilon(u)$

(ii) $\int_\varepsilon^\infty t^{-\alpha-1}S(\sigma_{(t)})f\,dt = \int T_u f\,dv_\varepsilon(u)$.

Proof. From the definition of v_ε, we have

$$\int_\varepsilon^\infty \left(\int g(u)\,d\sigma_{(t)}(u)\right)t^{-\alpha-1}\,dt = \int g(u)\,dv_\varepsilon(u) \tag{1}$$

where g is the characteristic function of E. By using bounded pointwise convergence we obtain (1) for bounded continuous complex and vector valued functions g, which includes (i) and (ii) of the lemma.

1.9. LEMMA. *Corresponding to $\alpha > 0$ and a measure σ, define the extended measure λ and the measure ρ as follows:*

$$d\lambda(u) = \Gamma(\alpha)^{-1}u^{\alpha-1}\,du,$$

$$d\rho(u) = \Gamma(\alpha - 1)^{-1}u^{-1}\sigma * u^{\alpha}\,du.$$

If σ is an r-measure, $r > \alpha$ and $C_\alpha(\sigma) = 1$, then

(i) $\int d\rho = 1$

(ii) $\lambda * \nu_\varepsilon = \rho_{(\varepsilon)}$.

Proof. We see from the definition of λ and Lemma 1.8 (i) that

$$\mathscr{L}(\lambda * \nu_\varepsilon)(s) = s^{-\alpha}\int_\varepsilon^\infty \mathscr{L}\sigma_{(t)}(s)t^{-\alpha-1}\,dt = \int_{s\varepsilon} \mathscr{L}\sigma(t)t^{-\alpha-1}\,dt;$$

and, if we denote the latter as $F(s\varepsilon)$, then $F'(s) = -\mathscr{L}\sigma(s)s^{-\alpha-1}$. Thus,

$$F'(s) = -\mathscr{L}(\sigma * \Gamma(\alpha + 1)^{-1}u^{\alpha}).$$

We have assumed that σ is an r-measure, which means that $\int |\sigma * u^\alpha| u^{-1}\,du$ is finite. Thus, we conclude from Lemma 1.7 that for some constant C,

$$F(s) = \mathscr{L}(u^{-1}\sigma * \Gamma(\alpha - 1)^{-1}u^{\alpha} + C\,\delta_0), \quad s > 0.$$

However, $F(s) \to 0$ and $\mathscr{L}(u^{-1}\sigma * u^\alpha) \to 0$ as $s \to \infty$. Thus, $C = 0$. This shows that $F(s) = \mathscr{L}\rho(s)$. Since $F(\varepsilon s) = \mathscr{L}\rho_{(\varepsilon)}(s)$, we have

$$\lambda * \nu_\varepsilon = \rho_{(\varepsilon)}.$$

Also, $\int d\rho = F(0) = \int \mathscr{L}\sigma(t)t^{-\alpha-1}\,dt = C_\alpha(\sigma)$, and we have assumed that $C_\alpha(\sigma) = 1$. The lemma is proved.

Proof of Theorem 1.4. Let σ and σ' be two r-measures for which $C_\alpha \neq 0$. We may assume that $C_\alpha(\sigma) = C_\alpha(\sigma') = 1$. We must show that $B_\alpha(\sigma) = B_\alpha(\sigma')$. Suppose that f is in the domain of $B_\alpha(\sigma')$. Lemma 1.9 shows that the extended associativity, Lemma 1.6, holds in the following case:

$$S(\nu_\varepsilon(\sigma))S(\lambda * \nu_\eta(\sigma'))f = S(\lambda * \nu_\varepsilon(\sigma))S(\nu_\eta(\sigma'))f.$$

If we take the limit as $\eta \to 0$ and apply Lemma 1.9 and Lemma 1.5, we conclude that

$$S(\nu_\varepsilon(\sigma))f = S(\lambda * \nu_\varepsilon(\sigma))B_\alpha(\sigma')f.$$

If we now let $\varepsilon \to 0$, we see that f is in the domain of $B_\alpha(\sigma)$ and from 1.9 (ii) that $B_\alpha(\sigma)f = B_\alpha(\sigma')f$. This completes the proof of the theorem.

2. Special cases

Komatzu [2] shows that for appropriately normalized constants C, the limits (0.1) and (0.2) define the same operators. The following two lemmas show that Komatzu's result is part of Theorem 1.4.

2.1. LEMMA. *For $r = 1, 2, \ldots,$*

$$(I - T_t)^r = \int T_{ut}\, d\sigma(u),$$

where $\sigma = (\delta_0 - \delta_1)^{(r)}$; furthermore, σ is an r-measure.

2.2. LEMMA. *For $r = 1, 2, \ldots,$*

$$(-tA(I - tA)^{-1})^r = \int T_{ut}\, d\sigma(u)$$

where $\sigma = (\delta_0 - e^{-u}\, du)^{(r)}$; furthermore, σ is an r-measure.

The first assertion in Lemma 2.1 is clear; the second assertion follows from Lemmas 2.3 and 2.4 below. The first assertion of Lemma 2.2 follows from the identity

$$-tA(I - tA)^{-1} = I - (I - tA)^{-1},$$

the well-known representation [1, p. 32],

$$(\lambda - A)^{-1} = \int T_t e^{-\lambda t}\, dt, \quad \operatorname{Re} \lambda > 0,$$

for the resolvent and Lemma 1.5, which is needed for $r = 2, 3, \ldots.$ The second assertion follows from Lemmas 2.3 and 2.4 below.

To complete the proofs of Lemmas 2.1 and 2.2 we will establish the following lemmas.

2.3. LEMMA. *Suppose r is a positive integer and σ is a complex Borel measure on $[0, \infty)$ such that:*

(i) *there are positive real numbers a, C_0 such that $|\sigma| \leq C_0 u^{-r-1}\, du$ on $[a, \infty)$; and,*

(ii) *$\int_0^\infty u^k\, d\sigma(u) = 0$ for $k = 0, 1, \ldots, r - 1$.*

Then, σ is an r-measure.

2.4. LEMMA. *Let $\sigma_1, \ldots, \sigma_r$ be complex Borel measures on $[0, \infty)$ such that*

$$\int d\sigma_j = 0 \quad and \quad \int u^k\, d|\sigma_j| < \infty, \quad 1 \leq j \leq r.$$

If $\sigma = \sigma_1 * \cdots * \sigma_r$ and $0 \leq k < r$, then

$$\int u^k\, d|\sigma| < \infty \quad and \quad \int u^k\, d\sigma = 0.$$

Proof of 2.3. We must show that $|x^{-1}\sigma * x^{\alpha}|$ has a finite integral on $[0, \infty)$. The integral over $[0, a)$ is clearly finite, so we will obtain a bound for $|x^{-1}\sigma * x^{\alpha}|$ when $x > a$. We will first establish

$$(1) \qquad \sigma * x^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k \left[x^{\alpha-k} \int_0^x u^k \, d\sigma(u) \right] \quad \text{for } x > a.$$

But, this follows if we show that

$$(2) \qquad \lim \int_0^x R_n(u/x) \, d\sigma(u) = 0 \text{ (as } n \to \infty), \quad x > a,$$

where $R_n(v)$ is the remainder in the Taylor formula for $(1 - v)^{\alpha}$. From the integral form of the remainder we deduce the bound

$$|R_n(v)| \leq (\text{const})\, n^{-\alpha} x[1 + (1 - x)^{\alpha-1}], \quad 0 < v < 1;$$

and this shows that (2) holds provided that

$$(3) \qquad \int_0^x (ux^{-1}[1 + (1 - u/x)^{\alpha-1}]) \, d|\sigma|(u) < \infty, \quad x > a.$$

However, (3) follows from the fact that the integrand is bounded on $[0, a)$ and $|\sigma| \leq c_0 u^{-r-1} \, du$ on $[a, \infty)$. We have now established (1). By considering the cases $k < r$, $k = r$, and $k > r$ and applying (ii) to the first case we obtain

$$\left| x^{\alpha-k} \int_0^x u^k \, d\sigma(u) \right| \leq x^{\alpha-r}[c_0 + a^k|\sigma|([0, a]) + c_0 \log (x/a)] \quad \text{for } x > a, 1.$$

This, the fact that

$$\left| \binom{\alpha}{k} \right| \leq \text{const } k^{-\alpha-1}$$

and (1), shows that $|x^{-1}\sigma * x^{\alpha}|$ is bounded by an integrable function on $[a + 1, \infty)$, which completes the proof.

Proof of Lemma 2.4. From known properties of the Laplace transform we have

$$\begin{aligned} \mathscr{L}(u^k\sigma)(s) &= \int e^{-su} u^k \, d\sigma(u) \\ (1) \qquad &= (-1)^k \frac{d^k}{ds^k} \int e^{-su} \, d\sigma(u) \\ &= (-1)^k \frac{d^k}{ds^k} (\mathscr{L}\sigma_1(s) \cdots \mathscr{L}\sigma_r(s)). \end{aligned}$$

The kth derivative of $\mathscr{L}\sigma$ is a sum of terms of the form

$$(2) \qquad (\mathscr{L}\sigma_1)^{(k_1)} \cdots (\mathscr{L}\sigma_r)^{(k_r)}$$

where each $k_j \geq 0$ and $k_1 + \cdots + k_r = k$. Since each factor in (2) is bounded, which follows from the assumption that each $\int u^k \, d|\sigma_j|$ is finite, and $k_j = 0$ for some j and $\mathscr{L}\sigma_j(0) = 0$, the conclusion of the lemma follows.

3. A converse theorem

Our purpose in this section is to prove Theorem 3.1, which is a converse to Theorem 1.4. Again, we let r denote a positive integer and let $0 < \alpha < r$. For convenience, let

$$C_{r,\alpha} = C_\alpha((\delta_0 - \delta_1)^{(r)}), \qquad B = C_{r,\alpha}B_\alpha((\delta_0 - \delta_1)^{(r)}).$$

By the right shift semigroup on L_1 we mean the semigroup T_t defined by: $T_tf(u) = f(u - t)$ for $u \geq t$, $Tf(u) = 0$ for $0 \leq u < t$, for each f in $L_1[0, \infty)$. Recall that M denotes the complex Borel measures on $[0, \infty)$.

3.1. THEOREM. *If σ is a measure such that $B_\alpha(\sigma) = B$ for the right shift semigroup on L_1, then σ is an r-measure.*

To prove the theorem we need the following lemma.

3.2. LEMMA. *Suppose r is a positive integer, $0 < \alpha < r$ and T_t is the right shift semigroup on L_1. For f in L_1,*

$$\lim_{\varepsilon \to 0} C_{r,\alpha} \int_\varepsilon^\infty t^{-\alpha}(I - T_t)^r f t^{-1} \, dt$$

exists in L_1 if and only if $s^\alpha \mathscr{L}f(s)$ is in $\mathscr{L}L_1$. If the limit exists, its Laplace transform is $s^\alpha \mathscr{L}f(s)$.

Proof. It is convenient to consider $\mathscr{L}L_1$ instead of L_1. If g is in $\mathscr{L}L_1$, the norm of g is defined as the L_1 norm of f where $\mathscr{L}f = g$. We consider $\mathscr{L}M$ in the same manner. We first observe that

$$(1) \qquad C_{r,\alpha} \int_\varepsilon^\infty t^{-\alpha}(1 - e^{-st})^r t^{-1} \, dt g(s) = F(\varepsilon s)s^\alpha g(s)$$

where

$$F(s) = C_{r,\alpha} \int_s^\infty t^{-\alpha}(1 - e^{-t})^r t^{-1} \, dt.$$

It follows from Lemma 1.7 that $F \in \mathscr{L}M$ and $F(0) = 1$ by choice of $C_{r,\alpha}$. If we assume that $g(s)$ and $s^\alpha g(s)$ are in $\mathscr{L}L_1$, then the limit on the left-hand side of (1) exists in L_1 because of the right-hand side of (1) and the limit is $s^\alpha g(s)$. Now suppose that g is in $\mathscr{L}L_1$ and the limit of the left-hand side of (1) exists. Then, $F(\varepsilon s)s^\alpha g(s)$ converges to some h in $\mathscr{L}L_1$; but, $F(\varepsilon s)s^\alpha g(s)$ converges pointwise to $s^\alpha g(s)$. Thus, $s^\alpha g(s)$ is in $L\mathscr{L}_1$ and is the limit of the left-hand side of (1). This proves the lemma.

We will now prove the theorem. From the hypothesis and the lemma we conclude that

$$\lim_{\varepsilon \to 0} \int_{\varepsilon}^{\infty} t^{-\alpha} \mathscr{L}\sigma(ts)t^{-1}\, dtg(s) = s^{\alpha}g(s) \tag{2}$$

in $\mathscr{L}L_1$ if $g(s)$ and $s^{\alpha}g(s)$ are in $\mathscr{L}L_1$. It is easy to show that $(1+s)^{-\alpha}$ and $(s/(1+s))^{\alpha}$ are in $\mathscr{L}M$. This and (2) imply that

$$\lim_{\varepsilon \to 0} (1+s)^{-\alpha} \int_{\varepsilon} t^{-\alpha}\mathscr{L}\sigma(ts)t^{-1}\, dt\, g(s) = (s/(1+s))^{\alpha} g(s), \tag{3}$$

g in $\mathscr{L}L_1$.

The left-hand side of (3) is $(s/(1+s))^{\alpha}F(\varepsilon s)$ where

$$F(s) = \int_{s}^{\infty} t^{-\alpha}\mathscr{L}\sigma(t)t^{-1}\, dt.$$

Since the norm of $\mathscr{L}M$ is an operator norm, we conclude from (3) and the uniform boundedness principle, that

$$\|(s/(1+s))^{\alpha}f(\varepsilon s)\|_{\mathscr{L}M} \leq \text{const.}, \quad \varepsilon > 0. \tag{4}$$

Since the norm in (4) is not changed if s is replaced by ns, $n = 1, 2, \ldots$, if we let $\varepsilon = n^{-1}$, we have

$$\|(ns/(1+ns))^{\alpha}f(s)\|_{\mathscr{L}M} \leq \text{const.}, \quad n = 1, 2, \ldots.$$

This shows that F is in $\mathscr{L}M$. Since F is in $\mathscr{L}M$ and

$$F'(s) = -\Gamma(\alpha+1)^{-1}\mathscr{L}(\sigma * u^{\alpha}),$$

it is clear that $u^{-1}\sigma * u^2$ is in L_1. This completes the proof.

4. A Lions-Peetre type theorem

For each measure σ, $0 < \alpha < 1$, and $1 \leq q \leq \infty$ we let lip (σ, α, q) denote the space of f in X for which the norm

$$\|f\| + \left(\int [t^{-\alpha}\|S(\sigma_{(t)})f\|]^{q} t^{-1}\, dt\right)^{1/q}$$

is finite. We will refer to these spaces as the lip spaces. The spaces $(X_0, X_1)_{\alpha q J}$ and $(X_0, X_1)_{\alpha q K}$ are defined in [1, p. 166]; we will refer to them as the J spaces and K spaces, respectively.

The object of this section is to prove the following theorem.

4.1. THEOREM. *Suppose that σ is a complex measure on $[0, \infty)$ which satisfies:*

(i) *there exist a, $C > 0$ such that $|\sigma| \leq Cu^{-3}\, du$ on $[a, \infty)$;*

(ii) $\int d\sigma = 0$.

Then, lip $(\sigma, \alpha, q) = (X, D(A))_{\alpha q J}$ *for* $0 < \alpha < 1$, $1 \leq q \leq \infty$; *and, the norms of these spaces are equivalent.*

Proof. For convenience, let S_t denote $S(\sigma_{(t)})$. We will first show that the J space is continuously embedded in the lip space. Since the J space is equal to the K space [1, p. 173], it suffices to show that the K space is embedded in the lip space. By an argument similar to that of [1, p. 194] it suffices to show

$$\|S_t f\| \leq \left(M \int d|\sigma|\right) \|f\|, \quad f \in X; \tag{1}$$

$$\|S_t f\| \leq t\left(M \int u \, d|\sigma|(u)\right) \|Af\|, \quad f \in D(A); \tag{2}$$

where $M = \sup \|T_t\|$. It is clear that (1) holds. To prove (2) we first note that since

$$h^{-1}(S_{t+h}f - S_t f) = \int_0^\infty T_{ut}\left((uh)^{-1} \int_0^{uh} T_s Af \, ds\right) u \, d\sigma(u)$$

and (i) is assumed, we have

$$\frac{d}{dt} S_t f = \int_0^\infty T_{ut} Afu \, d\sigma(u) \tag{3}$$

where the derivative exists in X. The inequality (2) now follows from (3) and the identity

$$S_t f = \int_0^t \frac{d}{du} S_u f \, du.$$

We will now show that the lip space is embedded in the J space. As in [3], we first note that we can assume that A has a bounded inverse; because, if the semigroup T_t is replaced with the semigroup $e^{-t}T_t$, then the lip space and the J space are unchanged except that the norms are replaced by equivalent norms. This is clear for the J space since $e^{-t}T_t$ has infinitesimal generator $A - I$. We now show why the lip space is the same. Let

$$S_t' = \int e^{-tu} T_{tu} \, d\sigma(u).$$

It suffices to show that

$$\int [t^{-\alpha}\|(S_t - S_t')f\|]^q t^{-1} \, dt < \infty, \quad f \in X.$$

This will hold if we show

$$\int (ut)^{-\alpha}(1 - e^{-ut})u^\alpha \, d|\sigma|(u) = O(t^{1-\alpha}), \quad \text{as } t \to 0. \tag{4}$$

But, the integral in (4) is dominated by

$$C \int_0^{1/t} (ut)^{1-\alpha} u^\alpha \, d|\sigma|(u) + 2 \int_{1/t}^\infty (tu)^{-\alpha} u^\alpha \, d|\sigma|(u)$$

and we can apply (iii) and (ii) to show that both terms in this sum are $O(t^{1-\alpha})$ as $t \to 0$. We have now shown that we can assume that A has a bounded inverse, which we denote by A^{-1}.

By the definition of the J space, to show that f is in the J space and the J norm is dominated by a multiple of the lip norm, it suffices to show that there is a function $v(t)$ such that:

(5) $v(t)$ is strongly measurable as a function with values in $D(A)$;
(6) $\int [t^{-\theta}\|v(t)\|]^q t^{-1}\, dt < \infty$;
(7) $\int [t^{-\theta+1}\|v(t)\|_{D(A)}]^q t^{-1}\, dt < \infty$;
(8) $\int v(t)t^{-1}\, dt = f$.

Before we define $v(t)$ we first note that if τ is a measure such that (a) τ has finite support and (b) $\int d\tau = 0$, then

(9) $\tau * \sigma$ is a 2-measure.

The assumptions we have made for σ and τ together with Lemma 2.4 show that the conditions of Lemma 2.3 are satisfied by $\tau * \sigma$ for the case $r = 2$; thus, $\tau * \sigma$ is a 2-measure. Clearly, we can choose τ so that we also satisfy (c) $C_1(\tau * \sigma) = 1$ ($C_1(\sigma)$ is defined in 1.3). Now define $v(t)$ by

$$v(t) = t^{-1}S(\tau_{(t)})S(\sigma_{(t)})A^{-1}f.$$

First consider (8). Since $A^{-1}f \in D(A)$, to prove (8) it suffices to show that

$$\int t^{-1}S(\tau_{(t)})S(\sigma_{(t)})gt^{-1}\, dt = Ag, \quad g \in D(A). \tag{10}$$

Since $S(\tau_{(t)})S(\sigma_{(t)}) = S((\tau * \sigma)_{(t)})$, $\tau * \sigma$ is a 2-measure, $(\delta_0 - \delta_1)^{(2)}$ is a 2-measure (see 2.1), it follows from Lemma 2.1 and Theorem 1.4 that the integral in (10) is equal to

$$\lim_{\varepsilon \to 0} C_1((\delta_0 - \delta_1)^{(2)})^{-1} \int_\varepsilon^\infty t^{-1}(I - T_t)^2 g t^{-1}\, dt;$$

and, in [2, 2.3, p. 93] it is shown that this latter limit is Ag.

We now turn to the other conditions, (5), (6), (7). Both (5) and (7) are obvious. To prove (6) it suffices to show that $\|t^{-1}S(\tau_{(t)})A^{-1}\|$ is bounded; and, this follows from the fact that $\int u\, d|\tau|(u) < \infty$ and that, consequently, (2) holds with f replaced by $A^{-1}f$ and σ replaced by τ.

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