SPECTRAL DECOMPOSITION AND DUALITY

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Introduction

The purpose of this paper is to improve our previous result [3] concerning the duality of decomposable operators. In that paper we have proved that the dual of a 2-decomposable operator is also 2-decomposable. We shall prove here that the dual of a 2-decomposable operator is actually decomposable. This result has some interesting consequences. The first one is that on a reflexive Banach space, any 2-decomposable operator is decomposable, thus improving a result contained in [1] and answering positively a question raised in [4]. A second one is that the dual of any decomposable operator is a decomposable operator. A similar result for a more restrictive notion of decomposability was obtained in [5]. Some other consequences are related to the quasinilpotent equivalence of 2-decomposable operators.

The paper consists of four sections. In Section 1 we give some definitions and auxiliary results. In Section 2 we prove a general decomposition theorem for continuous linear functionals which will be used essentially in the proof of our main theorem and which seems to be interesting by itself. Finally, Section 3 contains the main result of the paper, and Section 4, its consequences.

1. Preliminaries

We begin by recalling some definitions from the theory of spectral decompositions. Let X be a complex Banach space and $L(X)$ be the space of all continuous linear operators on X.

DEFINITION 1. [2], [4] (a) An operator $T \in L(X)$ is said to be *m-decomposable* (m is a natural number, $m \geq 2$) if for every finite covering $\{G_1, \ldots, G_k\}$ of the spectrum $\sigma(T)$ of T consisting of $k \leq m$ open sets, there exist k maximal spectral subspaces $Y_1, \ldots, Y_k$ of T such that:

(i) $X = \sum_{j=1}^{k} Y_j$,

(ii) $\sigma(T \mid Y_j) \subset G_j$ $(1 \leq j \leq k)$.

(b) T is said to be *decomposable* if it is m-decomposable for every number m.

A *maximal spectral subspace* Y of T is a (closed linear) subspace invariant for T, and containing any other invariant subspace with a smaller spectrum (i.e., $TZ \subset Z$ and $\sigma(T \mid Z) \subset \sigma(T \mid Y)$ imply $Z \subset Y$).

It is easy to see that some results proved in [2] for decomposable operators remain valid for 2-decomposable operators. Thus, denoting the resolvent of T, by $R(\cdot\,; T)$, for any $x \in X$, the analytic function $z \to R(z; T)x$ defined on the resolvent set, $\rho(T)$, has a single-valued maximal extension. We denote by

Received April 9, 1975.

$z \to x_T(z)$ this extension, by $\rho_T(x)$ (the resolvent set of x with respect to T) its domain of definition, and by $\sigma_T(x)$ (the spectrum of x with respect to T) the complement of $\rho_T(x)$ in C, $\sigma_T(x) = C \backslash \rho_T(x)$.

For an arbitrary set $F \subset C$ we denote $X_T(F) = \{x : x \in X, \sigma_T(x) \subset F\}$. If T is 2-decomposable and F is a closed set, then $X_T(F)$ is closed and $\sigma(T \mid X_T(F)) \subset F$; thus it is a maximal spectral subspace of T. Conversely if Y is a maximal spectral subspace for T and we denote $F = \sigma(T \mid Y)$, then $Y = X_T(F)$.

DEFINITION 2. Two operators $T, S \in L(X)$ are *quasinilpotent equivalent* if

$$\lim_{n \to \infty} \left\| \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} T^k S^{n-k} \right\|^{1/n} = \lim_{n \to \infty} \left\| \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} S^k T^{n-k} \right\|^{1/n} = 0.$$

It is known [2] that if T, S are decomposable operators, then they are quasinilpotent equivalent if and only if $X_T(F) = X_S(F)$ for any closed set F.

Some formal similarities of the conditions (i)–(ii) with the corresponding properties of the partition of the unity suggest the question if indeed a 2-decomposable operator is decomposable. In [1] it was proved, by using simple arguments of topological dimension theory, that any 3-decomposable operator is decomposable. We shall improve this result by using arguments of duality. Let $T \in L(X)$ be a 2-decomposable operator and let us denote by X' the dual of X and by T' the dual of T, so that $(T'u)(x) = u(Tx)$, $u \in X'$, $x \in X$. Then T' is also 2-decomposable and we have $X'_{T'}(F) = X_T(\mathscr{C}F)^\perp$ for any closed set $F \subset C$ [3]. The following proposition gives us an equivalent condition for the decomposability of T' which is easier to handle.

PROPOSITION 1. *Suppose T is 2-decomposable. Then T' is decomposable if and only if for any finite family $\{F_1, \ldots, F_k\}$ of closed sets with void intersection, we have $X' = X_T(F_1)^\perp + \cdots + X_T(F_k)^\perp$.*

Proof. If T' is decomposable and $F_1, \ldots, F_k$ are closed sets such that $\bigcap_{j=1}^k F_j = \emptyset$, then the open sets $G_1 = C \backslash F_1, \ldots, G_k = C \backslash F_k$ cover C. If $\{U_1, \ldots, U_k\}$ is another open covering of C such that $\overline{U}_j \subset G_j$ $(1 \leq j \leq k)$ then, by the decomposability of T', we have

$$X' = \sum_{j=1}^{k} X'_{T'}(\overline{U}_j) = \sum_{j=1}^{k} X_T(\mathscr{C}\overline{U}_j)^\perp.$$

On the other hand, $X_T(\mathscr{C}\overline{U}_j) \supset X_T(F_j)$, whence

$$\sum_{j=1}^{k} X_T(\mathscr{C}\overline{U}_j)^\perp \subset \sum_{j=1}^{k} X_T(F_j)^\perp \subset X',$$

and therefore $X' = \sum_{j=1}^k X_T(F_j)^\perp$. Conversely, let us suppose the condition stated in the proposition is satisfied. Let $\{G_1, \ldots, G_k\}$ be an arbitrary finite open covering of C and $\{U_1, \ldots, U_k\}$ be another open covering of C such that $\overline{U}_j \subset G_j$ $(1 \leq j \leq k)$. The closed sets $F_j = \mathscr{C}U_j$ $(1 \leq j \leq k)$ have void intersection and, consequently, we have

$$X' = X_T(F_1)^\perp + \cdots + X_T(F_k)^\perp;$$

on the other hand $X_T(F_j)^\perp \subset X_T(\mathscr{C}\overline{U}_j)^\perp = X'_{T'}(\overline{U}_j)$ so that

$$X' = X'_{T'}(\overline{U}_1) + \cdots + X'_{T'}(\overline{U}_k)$$

and the decomposability of T' is proved (see [2, Chapter 2, Notes and Remarks]). Proposition 1 motivates our next section.

2. A decomposition theorem for continuous linear functionals

We shall formulate now, in a general setting, the decomposition problem from above. Let X be a normed linear space and let $X_1, \ldots, X_k$ be closed linear subspaces of X. The problem is to find a necessary and sufficient condition in order to have the equality $X' = X_1^\perp + \cdots + X_k^\perp$. Such a condition is given in the following theorem.

THEOREM 1. *In order to have the equality* $X' = X_1^\perp + \cdots + X_k^\perp$ *it is necessary and sufficient to have an inequality of the form*

$$\|x\| \leq M[d(x, X_1) + \cdots + d(x, X_k)] \tag{1}$$

where M is a positive constant and $d(x, X_j)$ is the distance from x to the set X_j, $1 \leq j \leq k$.

Proof. Let us suppose first that $X' = \sum_{j=1}^k X_j^\perp$ and let us find a constant $M > 0$ such that inequality (1) is satisfied. It is well known that the space $X_j^\perp$ is isometrically isomorphic to the dual $(X/X_j)'$ of the quotient space X/X_j. Under this isomorphism, to an element $u_j \in X_j^\perp$ corresponds the element $\tilde{u}_j \in (X/X_j)'$ defined by $\tilde{u}_j(\xi) = u_j(x)$, $\xi \in X/X_j$, $x \in \xi$. On the other hand the equality $X' = \sum_{j=1}^k X_j^\perp$ implies the surjectivity of the (continuous linear) application $\oplus_{j=1}^k X_j^\perp \to X'$ defined by $\oplus_{j=1}^k u_j \to \sum_{j=1}^k u_j$. Therefore, by the open mapping theorem, there exists a constant $M > 0$ such that for any $u \in X'$ we can find a representation $u = \sum_{j=1}^k u_j$, where $\sum_{j=1}^k \|u_j\| \leq M\|u\|$. By using such a representation for any $u \in X'$ we shall obtain successively

$$\begin{aligned}
|u(x)| &\leq \sum_{j=1}^k |u_j(x)| \\
&= \sum_{j=1}^k |\tilde{u}_j(x + X_j)| \\
&\leq \sum_{j=1}^k \|u_j\| \, \|x + X_j\|_{X/X_j} \\
&= \sum_{j=1}^k \|u_j\| \, d(x, X_j) \\
&\leq \left(\sum_{j=1}^k \|u_j\|\right)\left(\sum_{j=1}^k d(x, X_j)\right) \\
&\leq M\|u\| \sum_{j=1}^k d(x, X_j).
\end{aligned}$$

Since $\|x\| = \sup_{\|u\| \le 1} |u(x)|$, we have

$$\begin{aligned} \|x\| &= \sup_{\|u\| \le 1} |u(x)| \\ &\le \sup_{\|u\| \le 1} \left(M\|u\| \sum_{j=1}^{k} d(x, X_j) \right) \\ &= M \sum_{j=1}^{k} d(x, X_j) \end{aligned}$$

and inequality (1) is satisfied. Conversely, let us prove that $X' = \sum_{j=1}^{k} X_j^{\perp}$ by assuming that inequality (1) is satisfied. It is easy to see that inequality (1) may be written in the form $\|x\| \le M \sum_{j=1}^{k} \|x + X_j\|_{X/X_j}$. Thus for any $u \in X'$, we have

$$\begin{aligned} |u(x)| &\le \|u\| \, \|x\| \\ &\le M\|u\| \sum_{j=1}^{k} \|x + X_j\|_{X/X_j} \\ &= M\|u\| \left\| \bigoplus_{j=1}^{k} (x + X_j) \right\| . \end{aligned}$$

Therefore, by applying the Hahn-Banach theorem, we deduce that, for any $u \in X'$, there exists a continuous linear functional U on $\oplus_{j=1}^{k} (X/X_j)$ such that $U(\oplus_{j=1}^{k} (x + X_j)) = u(x)$, $x \in X$. For such a functional U we can find $U_j \in (X/X_j)'$ such that

$$U\left(\bigoplus_{j=1}^{k} (x + X_j) \right) = \sum_{j=1}^{k} U_j(x + X_j).$$

Taking into account the isomorphism $X_j^{\perp} \cong (X/X_j)'$, we deduce that there exist $u_j \in X_j^{\perp}$ such that $U_j = \tilde{u}_j$, $1 \le j \le k$. Consequently we have

$$\begin{aligned} u(x) &= U\left(\bigoplus_{j=1}^{k} (x + X_j) \right) \\ &= \sum_{j=1}^{k} U_j(x + X_j) \\ &= \sum_{j=1}^{k} \tilde{u}_j(x + X_j) \\ &= \sum_{j=1}^{k} u_j(x) \end{aligned}$$

for $x \in X$; that is, $u = \sum_{j=1}^{k} u_j$, $u_j \in X_j^{\perp}$, $1 \le j \le k$ and the proof is finished.

If $n = 2$ and X is a Banach space then inequality (1) has a simple "geometric" interpretation.

PROPOSITION 2. *If X is a Banach space and X_1, X_2 are closed linear subspaces of X, then an inequality of the form*

$$\|x\| \leq M[d(x, X_1) + d(x, X_2)]$$

is satisfied if and only if $X_1 + X_2$ is closed and $X_1 \cap X_2 = \{0\}$.

Proof. If the inequality $\|x\| \leq M[d(x, X_1) + d(x, X_2)]$ is satisfied, then for any $x_1 \in X_1$, $x_2 \in X_2$ we have

$$\|x_1\| \leq M[d(x_1, X_1) + d(x_1, X_2)] = M\, d(x_1, X_2) \leq M\|x_1 - x_2\|$$

and analogously $\|x_2\| \leq M\|x_1 - x_2\|$. Therefore, for any $x_1 \in X_1$, $x_2 \in X_2$ we have

$$\|x_1\| + \|x_2\| \leq 2M\|x_1 - x_2\|$$

and this inequality easily implies that $X_1 + X_2$ is closed and $X_1 \cap X_2 = \{0\}$. Conversely, let us suppose that $X_1 + X_2$ is closed and $X_1 \cap X_2 = \{0\}$. By the open mapping theorem there exists a constant $M > 0$ such that

$$\|x_1\| + \|x_2\| \leq M\|x_1 + x_2\| \quad \text{for any } x_1 \in X_1,\ x_2 \in X_2.$$

Consider now an arbitrary element $x \in X$. Since

$$d(x, X_j) = \inf\{\|x - x_j\|,\ x_j \in X_j\},$$

for any $\varepsilon > 0$ we can find elements $x_{j,\varepsilon} \in X_j$ such that

$$\|x - x_{j,\varepsilon}\| \leq d(x, X_j) + \varepsilon \quad (j = 1, 2).$$

Then we obtain

$$\begin{aligned}
\|x\| &\leq \|x - x_{1,\varepsilon}\| + \|x_{1,\varepsilon}\| \\
&\leq d(x, X_1) + \varepsilon + M\|x_{1,\varepsilon} - x_{2,\varepsilon}\| \\
&\leq d(x, X_1) + \varepsilon + M[\|x_{1,\varepsilon} - x\| + \|x - x_{2,\varepsilon}\|] \\
&\leq d(x, X_1) + \varepsilon + M[d(x, X_1) + d(x, X_2) + 2\varepsilon] \\
&\leq (1 + M)[d(x, X_1) + d(x, X_2)] + \varepsilon + 2M\varepsilon.
\end{aligned}$$

Thus for any x and any $\varepsilon > 0$ we have

$$\|x\| \leq (1 + M)[d(x, X_1) + d(x, X_2)] + \varepsilon + 2M\varepsilon$$

whence, taking the limit when $\varepsilon \to 0$, we deduce

$$\|x\| \leq (1 + M)[d(x, X_1) + d(x, X_2)]$$

and the proof is complete.

3. The main result

Let T be a 2-decomposable operator on the complex Banach space X. We shall prove the dual T' of T is a decomposable operator.

THEOREM 2. *If T is 2-decomposable, then T' is decomposable.*

Proof. By using Proposition 1 and Theorem 1 it will be sufficient to prove the following statement: for any finite family $\{F_1, \ldots, F_k\}$ of closed sets with void intersection there exists a constant $M > 0$ such that

$$\|x\| \leq M[d(x, X_T(F_1)) + \cdots + d(x, X_T(F_k))], \quad x \in X.$$

We shall proceed by a *reductio ad absurdum.* Let us suppose this statement is not true. Then there exists a sequence $(x_n^0) \subset X$ satisfying the following conditions: $\|x_n^0\| = 1$, $n \in N$, $d(x_n^0, X_T(F_j)) \to 0$ when $n \to \infty$, $1 \leq j \leq k$. Consequently for every j, $1 \leq j \leq k$, there exists a sequence $(x_{j,n}^0) \subset X_T(F_j)$ such that for $n \to \infty$, $\|x_n^0 - x_{j,n}^0\| \to 0$; furthermore these sequences are bounded because $\|x_n^0\| = 1$, $n \in N$. Taking into account that $x_{j,n}^0 \in X_T(F_j)$ and letting

$$G_j = C \backslash F_j \quad \text{and} \quad f_{j,n}(z) = R(z; T \mid X_T(F_j))x_{j,n}^0 \quad \text{for } z \in G_j,$$

we obtain $x_{j,n}^0 = (z - T)f_{j,n}(z)$, $z \in G_j$; moreover the sequences of analytic functions $(f_{j,n})$ are uniformly bounded on compact sets. We can put now this situation in a more adequate framework. Let us consider the space $l_\infty(X)$ of all X-valued bounded sequences, and its quotient space $l_\infty(X)/c_0(X)$ by the subspace $c_0(X)$ of all sequences convergent to 0. Therefore an element of $l_\infty(X)/c_0(X)$ is a class, modulo sequences convergent to zero, of X-valued bounded sequences. Denote by $\tilde{x}^0$ the class defined by the sequence (x_n^0) and by $\tilde{f}_j$ the function defined on G_j to $l_\infty(X)/c_0(X)$ by

$$\tilde{f}_j(z) = (f_{j,n}(z)) + c_0(X), \quad z \in G_j.$$

Since, by definition, $f_{j,n}(z) = R(z; T \mid X_T(F_j))x_{j,n}^0$, it is easy to see that the function $z \to (f_{j,n}(z))$ is an analytic function on G_j to $l_\infty(X)$ and therefore $\tilde{f}_j$ is an analytic function on G_j. Moreover, let us remark that for $z \in G_j \cap G_l$, we have $\tilde{f}_j(z) = \tilde{f}_l(z)$. Indeed we know that as $n \to \infty$, then $\|x_n^0 - x_{j,n}^0\| \to 0$ and $\|x_n^0 - x_{l,n}^0\| \to 0$, hence $\|x_{j,n}^0 - x_{l,n}^0\| \to 0$. On the other hand, for $z \in G_j \cap G_l$ we have

$$\begin{aligned} f_{j,n}(z) - f_{l,n}(z) &= R(z; T \mid X_T(F_j))x_{j,n}^0 - R(z; T \mid X_T(F_l))x_{l,n}^0 \\ &= R(z; T \mid X_T(F_j \cup F_l))(x_{j,n} - x_{l,n}). \end{aligned}$$

Thus as $n \to \infty$, then $f_{j,n}(z) - f_{l,n}(z) \to 0$ uniformly on compact sets and consequently $\tilde{f}_j(z) = \tilde{f}_l(z)$, $z \in G_j \cap G_l$, as desired. Let $\tilde{T}$ be the (continuous linear) operator defined by T on $l_\infty(X)/c_0(X)$ by

$$\tilde{T}[(x_n) + c_0(X)] = (Tx_n) + c_0(X).$$

Then we have $\tilde{x}^0 = (z - \tilde{T})\tilde{f}_j(z)$, $z \in G_j$ and $\tilde{f}_j(z) = \tilde{f}_l(z)$, $z \in G_j \cap G_l$. Since $\bigcup_{j=1}^k G_j = C$, we can define an analytic function $\tilde{f}$ on C to $l_\infty(X)/c_0(X)$ by $\tilde{f}(z) = \tilde{f}_j(z)$ if $z \in G_j$ and we obtain $\tilde{x}^0 = (z - \tilde{T})\tilde{f}(z)$, $z \in C$. By taking a circumference Γ contained in the resolvent set of $\tilde{T}$ and surrounding the spectrum of $\tilde{T}$, we deduce

$$\tilde{x}^0 = \frac{1}{2\pi i}\int_\Gamma R(z; \tilde{T})\tilde{x}^0\, dz = \frac{1}{2\pi i}\int_\Gamma \tilde{f}(z)\, dz = 0.$$

We have obtained a contradiction because on the one hand $\tilde{x}^0$ is defined as $(x_n^0) + c_0(X)$ where $\|x_n^0\| = 1$, $n \in N$, and on the other hand $\tilde{x}^0 = 0$. The proof is concluded.

4. Applications

Let us first give some simple corollaries of our main theorem.

COROLLARY 1. *If T is decomposable, then T' is also decomposable.*

COROLLARY 2. *On a reflexive Banach space, any 2-decomposable operator is decomposable.*

Proof. If T is a 2-decomposable operator on a reflexive Banach space X, then by Theorem 2, T' is decomposable and by Corollary 1, T'' is also decomposable. Since $T = T''$, the proof is finished.

The last consequence is a characterization for the quasinilpotent equivalence of 2-decomposable operators, similar to that recalled in Section 1 for decomposable operators.

PROPOSITION 3. *If T, $S \in L(X)$ are 2-decomposable operators, then T is quasinilpotent equivalent to S if and only if for any closed set $F \subset C$, the corresponding spectral spaces are equal, that is, $X_T(F) = X_S(F)$.*

Proof. Let us note first that T is quasinilpotent equivalent to S if and only if T' is quasinilpotent equivalent to S'. This statement is a consequence of the following equalities:

$$\left\| \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} T'^k S'^{n-k} \right\| = \left\| \sum_{p=0}^{n} \binom{n}{p} (-1)^{n-p} S^p T^{n-p} \right\|$$

and

$$\left\| \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} S'^k T'^{n-k} \right\| = \left\| \sum_{p=0}^{n} \binom{n}{p} (-1)^{n-p} T^p S^{n-p} \right\|.$$

If T and S are 2-decomposable and $X_T(F) = X_S(F)$ for any closed set $F \subset C$, then we have $X_T(\mathscr{C}F) = X_S(\mathscr{C}F)$ and therefore $X_T(\mathscr{C}F)^\perp = X_S(\mathscr{C}F)^\perp$. By applying the duality of spectral spaces we obtain $X'_{T'}(F) = X'_{S'}(F)$ for any closed set $F \subset C$. Now, by Theorem 2, T' and S' are decomposable and consequently T' is quasinilpotent equivalent to S' whence T is quasinilpotent to S. Conversely, if T, S are 2-decomposable and T is quasinilpotent equivalent to S, then $\sigma_T(x) = \sigma_S(x)$ for any $x \in X$ (see Section 1 and [2, Chapter 1, Theorem 2.4]) and thus $X_T(F) = X_S(F)$ for any closed set $F \subset C$. This finishes the proof.

REFERENCES

1. E. J. ALBRECHT AND F. -H. VASILESCU, *On spectral capacities*, Rev. Roum. Math. Pures Appl., vol. 19 (1974), pp. 701–705.

2. I. Colojoară and C. Foiaş, *Theory of generalized spectral operators*, Gordon and Breach, New York, 1968.
3. Ş. Frunză, *A duality theorem for decomposable operators*, Rev. Roum. Math. Pures Appl., vol. 16 (1971), pp. 1055–1058.
4. S. Plafker, *On decomposable operators*, Proc. Amer. Math., Soc., vol. 24 (1970), pp. 215–216.
5. F. -H. Vasilescu, *On the residual decomposability in dual spaces*, Rev. Roum. Math. Pures Appl., vol. 16 (1971), pp. 1573–1587.

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