CLASS NUMBER FORMULAE FOR BICUBIC FIELDS

BY

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1. Introduction

While the study of biquadratic fields dates back to Dirichlet [4], except for the paper of Cassels and Guy [3], the bicubic fields seem to have been totally ignored. By a bicubic field we shall mean an extension of the rational numbers of degree 9 which is obtained by adjoining the cube roots of two rational integers. We shall also think of the normal closure of such a field as a bicubic field.

The bicyclic biquadratic fields have been studied extensively by Kubota [7], [8], Kuroda [9], Wada [14], and others. In particular it is relatively easy to determine the units and class numbers for such fields when one knows the units and class numbers of its quadratic subfields. In this article we achieve similar results for bicubic fields.

As an application of our results we shall explicitly determine when a bicubic field (and its normal closure) has class number divisible by 3.

2. Notation and terminology

The following notation will be used throughout this article.

$\zeta = e^{2\pi i/3}$.

m, n: cube free positive integers ($\neq 1$) which generate distinct pure cubic fields.

m_i ($i = 1, 2, 3, 4$): $m_1 = m$, $m_2 = n$, m_3 (respectively m_4) is the cube free kernel of mn (respectively m^2n).

$K = Q(\sqrt[3]{m}, \sqrt[3]{n})$: real bicubic field.

$L = Q(\zeta, \sqrt[3]{m}, \sqrt[3]{n})$: normal closure of K.

$k_i = Q(\sqrt[3]{m_i})$ for $i = 1, 2, 3, 4$: pure cubic subfield of K.

$K_i = Q(\zeta, \sqrt[3]{m_i})$: normal closure of k_i for $i = 1, 2, 3, 4$.

$k = Q(\zeta)$.

H, h: class number of L, K respectively.

H_i, h_i: class number of K_i, k_i respectively.

$G = G(L/Q)$: Galois group of L/Q.

σ_i ($i = 1, 2, 3, 4$): nontrivial elements of G chosen so that σ_i fixes K_i and $\sigma_3 = \sigma_1^2\sigma_2$, $\sigma_4 = \sigma_1\sigma_2$.

τ: nontrivial element of G which fixes K.

$\hat{E}$: group of units of L.

$\hat{\varepsilon}$: subgroup of $\hat{E}$ generated by the units of K_1, K_2, K_3, K_4.

$\hat{\varepsilon}_0$: subgroup of $\hat{\varepsilon}$ generated by the units of k_1, k_2, k_3, k_4 and their conjugates.

Received September 29, 1975.

$\hat{\varepsilon}_i$ ($i = 1, 2, 3, 4$): subgroup of $\hat{E}$ generated by the units of K, K^{σ_i}, K_i.
$\hat{e}$: group of units of K.
$\hat{e}_0 = \hat{e} \cap \hat{\varepsilon}_0$: subgroup of $\hat{e}$ generated by the units of k_1, k_2, k_3, k_4.
$\hat{U}_i$ ($i = 1, 2, 3, 4$): unit group of K_i.
$\hat{u}_i$ ($i = 1, 2, 3, 4$): subgroup of $\hat{U}_i$ generated by the units of k_i and its conjugates.
R_-: regulator of the field $-$.
R_i ($i = 1, 2, 3, 4$): regulator of the field K_i.
χ_-: character of G induced by the principal character of $G(L/-)$ where $-$ is a subfield of L. (See Lang [11, p. 236] for definitions.)
$\chi_i = \chi_{K_i}$ for $i = 1, 2, 3, 4$.

3. Class number relations

The main goal of this section is to prove the following results.

THEOREM I. *The following class numbers relations hold:*

(1) $3^5 H = (\hat{E}: \hat{\varepsilon}) H_1 H_2 H_3 H_4$,
(2) $3^2 t_i H h_i^2 = (\hat{E}: \hat{\varepsilon}_i) h^2 H_i$,
(3) $3H_i = (\hat{U}_i: \hat{u}_i) h_i^2$,
(4) $3^3 t_i H = (\hat{E}: \hat{\varepsilon}_i)(\hat{U}_i: \hat{u}_i) h^2$ *where* $t_i = 1$ *or* 3.

Proof. Equation (3) is proved in Honda [6] and Barrucand and Cohn [2]. Equation (4) is obtained by substituting (3) in (2). We now proceed to prove equations (1), (2), and give an alternate proof of (4). The following relations on the induced characters are readily verified:

(5) $\chi_L - \chi_1 - \chi_2 - \chi_3 - \chi_4 + 3\chi_k = 0$,
(6) $\chi_L - 2\chi_K - \chi_1 + 2\chi_{k_1} = 0$,
(7) $\chi_L - 2\chi_K - \chi_k + 2\chi_Q = 0$.

It is an immediate consequence of equation (12) of Kuroda [10] that the following equations hold:

(8) $HR_L = H_1 H_2 H_3 H_4 R_1 R_2 R_3 R_4$,
(9) $H h_i^2 R_L R_{k_i}^2 = h^2 H_i R_K^2 R_i$,
(10) $HR_L = h^2 R_K^2$.

The theorem now follows from the following results on the regulators.

THEOREM II. *The following are valid:*

(11) $(\hat{U}_i: \hat{u}_i) R_i = 3R_{k_i}^2$,
(12) $(\hat{E}: \hat{\varepsilon}) R_L = 3^5 R_1 R_2 R_3 R_4$,
(13) $(\hat{E}: \hat{\varepsilon}_i) R_L R_{k_i}^2 = 3^2 t_i R_K^2 R_i$ *where* $t_i = 3$ *or* 1 *according as the fundamental unit of* k_i *is the cube of a unit of* K *or not.*

Combining equations (11), (12), and (13) with equations (8), (9), and (10) completes the proof of Theorem I. It only remains to prove Theorem II. Equation (11) is proved in Honda [6] and the proof will not be reproduced here.

Before proving equations (12) and (13) we need to obtain some results on the units of L.

THEOREM III. *If $i = 1, 2, 3$, or 4 and $x \in L$ then $x^3 = x_1x_2x_3$ with $x_1 \in K$, $x_2 \in K^{\sigma_i}$, $x_3 \in K_i$. Also $x^3 = y_1y_2y_3y_4$ with $y_j \in K_j$. Moreover each x_i and y_j is a unit whenever x is a unit.*

Proof.

$$x^3 = \frac{(xx^{\tau})(xx^{\sigma_1\tau})(xx^{\sigma_1^2\tau})}{(xx^{\sigma_1}x^{\sigma_1^2})^{\tau}}$$

where $y = xx^{\sigma_1^2\tau} \in K^{\sigma_1^2}$. But

$$y = \frac{y^{1+\sigma_1+\sigma_1^2}}{y^{\sigma_1}y^{\sigma_1^2}}$$

with $y^{1+\sigma_1+\sigma_1^2} \in K_1$ and $y^{\sigma_1} \in K$, $y^{\sigma_1^2} \in K^{\sigma_1}$. This proves the first result for $i = 1$ and replacing σ_1 with σ_i proves the result for $i = 2, 3$, and 4. The second result is clear because

$$x^3 = \frac{x^{1+\sigma_1+\sigma_1^2}x^{1+\sigma_2+\sigma_2^2}x^{1+\sigma_1\sigma_2+\sigma_1^2\sigma_2^2}}{(x^{\sigma_1}x^{\sigma_1\sigma_2})^{1+\sigma_1^2\sigma_2+\sigma_1\sigma_2^2}}.$$

The assertion about units is immediate.

COROLLARY I. *$\hat{E}^3 \subset \hat{\varepsilon}$ and $\hat{E}^3 \subset \hat{\varepsilon}_i$ for $i = 1, 2, 3, 4$.*

Proof. Immediate by applying Theorem III to the elements of $\hat{E}$.

COROLLARY II. *There exist fundamental units e and e^{σ_j} of the fields K and K^{σ_j} $(j = 1, 2, 3, 4)$ such that e/e^{σ_j} is a root of unity.*

Proof. Let e_1, e_2, e_3, e_4 be a system of fundamental units of K and ε_i be a fundamental unit of k_i $(i = 1, 2, 3, 4)$. It follows from Theorem III that

$$e_i^3 = \varepsilon_1^{a_{i1}}\varepsilon_2^{a_{i2}}\varepsilon_3^{a_{i3}}\varepsilon_4^{a_{i4}}.$$

Now we may triangulate the matrix $A = (a_{ij})$ to obtain a new set of fundamental units for K which we shall continue to denote by e_1, e_2, e_3, e_4; i.e., we may assume from the beginning that A is a triangular matrix. Thus $e_1^3 = \varepsilon_1^{a_{11}}$ and we may adjust e_1 so that $a_{11} = 1$ or 3. Setting $e = e_1$ we have $(e^{\sigma_1})^3 = \varepsilon_1^{a_{11}}$. Thus e/e^{σ_1} is a root of unity. The same proof holds for $j = 2, 3, 4$.

We now return to Theorem II.

Proof of Theorem II. Let $E_1, \ldots, E_8$ be a system of fundamental units of L and $\varepsilon_{2i-1}, \varepsilon_{2i}$ be fundamental units for K_i ($i = 1, 2, 3, 4$). Let the automorphisms

$$1, \sigma_1, \sigma_1^2, \sigma_2, \sigma_2^2, \sigma_1\sigma_2, \sigma_1^2\sigma_2^2, \sigma_1^2\sigma_2$$

be denoted by $\alpha_1, \ldots, \alpha_8$ respectively. Set $E_{ij} = |E_i^{\alpha_j}|$ for $1 \leq i, j \leq 8$, and similarly define ε_{ij}. Now $E_i^3 = \varepsilon_1^{a_{i1}} \cdots \varepsilon_8^{a_{i8}}$ and $|(a_{ij})| = (\hat{\varepsilon}: \hat{E}^3)$ by Lemma VIII of [12]. Thus

$$\begin{aligned} 3^8 R_L &= |(2 \log E_{ij}^3)_{8\times 8}| = |(a_{ij})|\,|(2 \log \varepsilon_{ij})| \\ &= (\hat{\varepsilon}: \hat{E}^3)|(2 \log \varepsilon_{ij})| \\ &= (\hat{\varepsilon}: \hat{E}^3)3^5 R_1 R_2 R_3 R_4 \quad \text{(computation omitted).} \end{aligned}$$

Thus

$$R_L = (\hat{\varepsilon}: \hat{E}^3)3^{-3} R_1 R_2 R_3 R_4 = (\hat{E}: \hat{\varepsilon})^{-1} 3^5 R_1 R_2 R_3 R_4$$

or $(\hat{E}: \hat{\varepsilon})R_L = 3^5 R_1 R_2 R_3 R_4$.

To prove (13) (when $i = 1$) take $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ as a basis for the units of K where $\varepsilon_1 = e$ in Corollary II and $\varepsilon_5 = \varepsilon_2^{\sigma_1}$, $\varepsilon_6 = \varepsilon_3^{\sigma_1}$, $\varepsilon_7 = \varepsilon_4^{\sigma_1}$, and ε_1^t, ε_8 form a system of fundamental units for K_1, where $t = 1$ or 3 according as ε_1 is in k_1 or not. It follows from results of Barrucand and Cohn [2] that such an ε_8 exists. Thus $\varepsilon_1, \ldots, \varepsilon_8$ form a basis for the unit group $\hat{\varepsilon}_1$. With notation as above,

$$\begin{aligned} 3^8 R_L &= (\hat{\varepsilon}_1: \hat{E}^3)|(2 \log \varepsilon_{ij})_{8\times 8}| \\ &= (\hat{\varepsilon}_1: \hat{E}^3)18 R_K^2 (\log \varepsilon_8 + 2 \log \varepsilon_8^{\sigma_2})/\log \varepsilon_1. \end{aligned}$$

Thus if $\varepsilon_0 = \varepsilon_1^t$ then

$$(\hat{E}: \hat{\varepsilon}_1)R_L \log \varepsilon_1 \log \varepsilon_0 = 18 R_K^2 \log \varepsilon_0 (\log \varepsilon_8 + 2 \log \varepsilon_8^{\sigma_2}) = 9 R_K^2 R_1$$

so $1/t(\hat{E}: \hat{\varepsilon}_1)R_L R_{k_1}^2 = 9 R_K^2 R_1$. Hence

$$(\hat{E}: \hat{\varepsilon}_i)R_L R_{k_i}^2 = 3^2 t_i R_K^2 R_i \quad \text{for } i = 1, 2, 3, 4.$$

We conclude this section by giving necessary and sufficient conditions for the class numbers H of L and h of K to be divisible by 3.

THEOREM IV. *$3 \mid H$ if and only if $3 \mid h$ if and only if $3 \mid h_i$ for some $i =$ 1, 2, 3, or 4. Moreover H and h are relatively prime to 3 precisely when $m = p \equiv 2, 5 \pmod 9$ is prime and $n = 3$.*

Proof. It is proved in Honda [6] that $3 \mid H_i$ if and only if $3 \mid h_i$. Thus if $3 \mid H_1$ then $3 \mid h_1$ and so there exists an abelian unramified extension M of k_1 of degree 3. Now K is a nonnormal extension of k_1 of degree 3 so $M \cap K = k_1$. Thus MK is an abelian unramified extension of K of degree 3 and since $(L: K) = 2$ it follows $MKL = ML$ is an abelian unramified extension of degree 3 over L. Thus 3 divides both h and H. Moreover the above argument shows that $3 \mid h$ implies $3 \mid H$.

Conversely if $3 \mid H$ and $3 \nmid H_i$ ($i = 1, 2, 3, 4$) then it follows from Honda [6, p. 8] that $m = p \equiv 2, 5 \pmod 9$, $n = 3$, $mn = 3p$, and $mn^2 = 9p$. Under these conditions, the number of ambiguous classes, $\mathscr{A}_{L/K_1}$, of L over K_1 is given by

$$\mathscr{A}_{L/K_1} = 3^{q^*-3}H_1.$$

where $q^* = 1, 2$, or 3 (see Hasse [5, p. 98]). Since this number must be an integer we must have $q^* = 3$ and $\mathscr{A}_{L/K_1} = H_1$ is relatively prime to 3. By decomposing the ideal class group of L into orbits under σ_1 it is easy to see this implies $3 \nmid H$. This contradicts our hypothesis so $3 \mid H_i$ for at least one $i = 1, 2, 3$, or 4.

Finally we note that the explicit conditions for H and h to be relatively prime to 3 are now immediate from the result of Honda mentioned above.

The following corollary is now easy to obtain.

COROLLARY I. *If* $H \not\equiv 0 \pmod 3$ *then* $(\hat{E}: \hat{\varepsilon}) = 3^5$ *and* $(\hat{E}: \hat{\varepsilon}_i) = 3^2 t_i$.

Proof. Immediate from equations (1) and (2) since all class numbers are relatively prime to 3.

4. Units

In this section we obtain considerable information about the unit groups $\hat{E}$ of L and $\hat{e}$ of K.

THEOREM V. $(\hat{E}: \hat{\varepsilon}) = 3^a$ *and* $(\hat{E}: \hat{\varepsilon}_i) = 3^b$ *with* $a \leq 6$ *and* $b \leq 4$.

Proof. As was noted in Corollary I to Theorem III, $\hat{E}^3 \subset \hat{\varepsilon}$ and $\hat{E}^3 \subset \hat{\varepsilon}_i$. Thus we need only prove the inequalities on a and b. If necessary we may change notation so that some prime of K_1 which does not divide (3) will ramify in L. If the equation $E^3 = \varepsilon$ has a solution with $E \notin K_1$ and $\varepsilon \in K_1$ then $L = K_1(E)$ and so no prime of K_1 not dividing (3) can ramify in L. Thus no such E can exist. Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_8$ be a basis for $\hat{\varepsilon}$ where $\varepsilon_1, \varepsilon_2 \in K_1; \ldots; \varepsilon_7, \varepsilon_8 \in K_4$ and $E_1, \ldots, E_8$ be a basis for $\hat{E}$. (Here we are interested in only nontorsion units so we ignore roots of unity as much as possible). Since $\hat{E}^3 \subset \hat{\varepsilon}$ we have

$$(\varepsilon_1, \ldots, \varepsilon_8) = (E_1, \ldots, E_8)A$$

where A is an 8×8 integral matrix. Now elementary row operations on A correspond to a change of basis for $\hat{E}$, thus we may assume A is in upper triangular form with positive diagonal entries which must be either 1 or 3. As already observed no noncube unit of K_1 is in $\hat{E}^3$ so that $a_{11} = a_{22} = 1$. Thus $(\hat{E}: \hat{\varepsilon}) = \det(A) = 3^a$ with $a \leq 6$.

To prove the second inequality we first show that the equation

$$E^3 = \omega\varepsilon \tag{14}$$

has no solution with $E \in \hat{E}$, $\varepsilon \in \hat{\varepsilon}_i$, $\varepsilon \notin \hat{\varepsilon}_i^3$ and $\omega \in L$ a root of unity. If (14) has a solution, then multiply the equation by its complex conjugate to obtain

$$(E\bar{E})^3 = \varepsilon^2 \tag{15}$$

which contradicts the assumption $\varepsilon \notin \hat{\varepsilon}_i^3$. As in the proof of Theorem II we may choose a basis $\varepsilon_1, \ldots, \varepsilon_8$ for the group $\hat{\varepsilon}_1$ with $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ as a fundamental set of units of K, $\varepsilon_5 = \varepsilon_2^{\sigma_1}$, $\varepsilon_6 = \varepsilon_3^{\sigma_1}$, $\varepsilon_7 = \varepsilon_4^{\sigma_1}$, and ε_1, ε_8 a set of fundamental units for K_1. Using a matrix argument as above we see that $a_{11} = a_{22} = a_{33} = a_{44} = 1$. Thus $(\hat{E}: \hat{\varepsilon}_i) = \det(A) = 3^b$ with $b \leq 4$.

THEOREM VI. *A basis for $\hat{e}$ can be chosen in one of four possible ways. If e_1, e_2, e_3, e_4 is a basis for $\hat{e}$ and $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ is a basis for $\hat{e}_0$ with $\varepsilon_i \in k_i$ then the four possibilities are characterized as follows:*

(1) $(\hat{e}: \hat{e}_0) = 27$ *and* $e_1 = \varepsilon_1$, $e_2^3 = \varepsilon_1^{a_1}\varepsilon_2$, $e_3^3 = \varepsilon_1^{b_1}\varepsilon_3$, $e_4^3 = \varepsilon_1^{c_1}\varepsilon_4$.
(2) $(\hat{e}: \hat{e}_0) = 9$ *and* $e_1 = \varepsilon_1$, $e_2 = \varepsilon_2$, $e_3^3 = \varepsilon_1^{a_1}\varepsilon_2^{a_2}\varepsilon_3$, $e_4^3 = \varepsilon_1^{b_1}\varepsilon_2^{b_2}\varepsilon_4$.
(3) $(\hat{e}: \hat{e}_0) = 3$ *and* $e_1 = \varepsilon_1$, $e_2 = \varepsilon_2$, $e_3 = \varepsilon_3$, $e_4 = \varepsilon_1^{a_1}\varepsilon_2^{a_2}\varepsilon_3^{a_3}\varepsilon_4$.
(4) $(\hat{e}: \hat{e}_0) = 1$ *and* $e_1 = \varepsilon_1$, $e_2 = \varepsilon_2$, $e_3 = \varepsilon_3$, $e_4 = \varepsilon_4$.
Here a_i, b_i, c_i are nonnegative integers less than 3 for each i.

Finally we shall give examples to show that the first three "kinds" of unit structure actually exist in nature. We expect the fourth kind exists also, but no such example exists with all of m_1, m_2, m_3, m_4 less than 100.

Proof. It follows from Theorem III that $\hat{e}^3 \subset \hat{e}_0$ so $(\hat{e}: \hat{e}_0) = 3^a$ with $a \leq 4$. Now we have $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = (e_1, e_2, e_3, e_4)A$ where A is an integral 4×4 matrix. As in the previous theorem we may assume that A is an upper triangular matrix with 1's and 3's on the diagonal. If $(\hat{e}: \hat{e}_0) = 3^4$ then $a_{11} = 3$ and $\varepsilon_1 = e_1^3$. Thus $K = k_1(e_1)$ and so only prime divisors of (3) in k_1 can ramify in K. But we may renumber the fields k_1, k_2, k_3, k_4 so this is not true. Thus $a \leq 3$.

We may also assume the entrees of A satisfy $-a_{ii} < a_{ji} \leq 0$ for $j \neq i$. If $(\hat{e}: \hat{e}_0) = 27$ then $e_2^{a_{23}}$ is in $\hat{e}_0$ which implies $a_{23} \equiv 0 \pmod 3$, so $a_{23} = 0$. Similarly $a_{24} = a_{34} = 0$. Thus Kind 1 conditions are satisfied. If $(\hat{e}: \hat{e}_0) = 9$ then we may assume $a_{11} = a_{22} = 1$. As above $a_{34} = 0$, so Kind 2 conditions are fulfilled. If $(\hat{e}: \hat{e}_0) = 3$ then we may take $a_{11} = a_{22} = a_{33} = 1$ and Kind 3 obviously holds. Finally if $(\hat{e}: \hat{e}_0) = 1$ then Kind 4 is clearly satisfied.

We shall later give examples to show that the first three kinds of fields actually exist.

In [2] Barrucand and Cohn give a classification of pure cubic fields k_1 and K_1. They classify these fields into four "types" in Theorem 15.6 and conjecture that Type II fields do not exist. We shall first give a simple proof of this conjecture and then proceed to relate the "kind" of K (see Theorem VI above) to the type of the subfields k_1, k_2, k_3, k_4. We establish the following notation for the remainder of this section.

ε_i: fundamental unit of k_i ($i = 1, 2, 3, 4$).

B_i: unique primitive integer of K_i such that $\varepsilon_i = B_i/B_i^\sigma$ where $\sigma = \sigma_1$ for $i \neq 1$ and $\sigma = \sigma_2$ for $i = 1$.

$N(B_i) = N_{K_i/k}(B_i)$.

A_i: unique primitive integer of K_i such that $B_i = A_i/A_i^\sigma$ (only defined when $N(B_i) = 1$).

For the moment we shall only be concerned with the fields k_1 and K_1 so we shall write ε, B, A instead of ε_1, B_1, A_1. Now k_1 is of Type II if and only if $N(B) = 1$ and $A\bar{A} = e\beta^3(\sqrt[3]{m})^t$ where e is a unit of K_1, $\beta \in K_1$, and $t = 0, 1,$ or 2 (see [2, p. 235]).

THEOREM VII. *Type II fields do not exist.*

Proof. Suppose K_1 is a Type II field; then

$$\varepsilon = B/B^\sigma = \frac{AA^{\sigma^2}}{(A^\sigma)^2} = \frac{N(A)}{(A^\sigma)^3}.$$

Taking complex conjugates and multiplying the two equations together we obtain

$$\varepsilon^2 = \varepsilon\bar{\varepsilon} = \frac{N(A\bar{A})}{(A^\sigma\bar{A}^\sigma)^3} = \frac{N(A\bar{A})}{\gamma^3}$$

where $\gamma \in k_1$. Thus

$$\varepsilon = \frac{(\varepsilon\gamma)^3}{N(A\bar{A})} = \frac{\alpha^3}{N(A\bar{A})}$$

with $\alpha = \varepsilon\gamma \in k_1$. Now by Type II hypothesis $A\bar{A} = e\beta^3(\sqrt[3]{m})^t$ so

$$\varepsilon = \frac{\alpha^3}{N(e\beta^3\sqrt[3]{m^t})} = \frac{\alpha^3}{N(\beta^3\sqrt[3]{m^t})} = \left(\frac{\alpha}{N(\beta)\sqrt[3]{m^t}}\right)^3$$

since $N(e) = 1$ by Type II hypothesis. Thus $\sqrt[3]{\varepsilon} \in K_1$ which is impossible since ε is the fundamental unit of k_1 and $(K_1 : k_1) = 2$.

In view of this result the classification given in [2] can be considerably simplified. We restate their Theorem 15.6 as:

THEOREM VIII. *Cubic fields k_1 and K_1 can be classified into three types depending only on B.*

Type I. $N(B) = 1$.

Type III. $N(B)$ *is not a unit.*

Type IV. $N(B) = \zeta^a$ *with* $a = 1$ *or* 2.

We now proceed to investigate the relationship between the "kind" of K and the "type" of the subfields k_1, k_2, k_3, k_4. The main reason for carrying out this investigation is to obtain more explicit information concerning the structure of the unit group $\hat{e}$ of K. We shall continue to use the notation established earlier in this section.

THEOREM IX. *The field k_1 is of Type I if and only if $\varepsilon = \alpha^3/r$ with $\alpha \in k_1$ and $r \in Z$. Here r must divide $9m^2$.*

Proof. If k_1 is of Type I then as in the proof of Theorem VII, $\varepsilon = \alpha^3/N(A\bar{A})$ with $\alpha \in k_1$. Now $r = N(A\bar{A}) = N_{K/Q}(A) \in Q$ so $\varepsilon = \alpha^3/r$ with $\alpha \in k_1$, $r \in Q$.
Conversely if $\varepsilon = \alpha^3/r = B/B^\sigma$ then

$$B^\sigma = \frac{r}{\alpha^3} B \quad \text{and} \quad B^{\sigma^2} = \frac{r^2 B}{(\alpha^{1+\sigma})^3} B$$

so that

$$N(B) = B^{1+\sigma+\sigma^2} = \frac{r^3}{(\alpha^{2+\sigma})^3} B^3. \tag{16}$$

Thus $B^3/N(B) = \varepsilon/\varepsilon''$ (where $\varepsilon'' = \varepsilon^{\sigma^2}$) is a cube in K_1. Hence k_1 is not of Type III. Now equation (16) shows $N(B)$ is a cube in K_1 so it can not be a cube root of unity different from 1, i.e., k_1 is not of Type IV. Thus k_1 is of Type I.
Finally note that $K_1(\sqrt[3]{\varepsilon}) = K_1(\sqrt[3]{r})$ so only prime divisors of three in K_1 can ramify in the extension $K_1(\sqrt[3]{r})/K_1$. Thus r divides $9m^2$.

COROLLARY. *The fundamental unit ε_1 of k_1 can be a cube of a unit of K only if k_1 is of Type I.*

Proof. If $e^3 = \varepsilon_1$ with $e \in K$ then $K = k_1(\sqrt[3]{n}) = k_1(e) = k_1(\sqrt[3]{\varepsilon_1})$. Thus $\varepsilon_1 = \alpha^3 n$ with $\alpha \in k_1$ so k_1 is of Type I by Theorem IX.
In the following results we shall let $\varepsilon_i' = \varepsilon_i^\sigma$ and $\varepsilon_i'' = \varepsilon_i^{\sigma^2}$ for $i = 1, 2, 3, 4$ where as usual $\sigma = \sigma_2$ for $i = 1$ and $\sigma = \sigma_1$ for $i \neq 1$.

THEOREM X. *Let k_1 be a Type III field. Then:*

(a) $e^3 = \zeta^a\varepsilon_1/\varepsilon_i'$ *has no solution e in L.*
(b) $e^3 = \zeta^a(\varepsilon_1\varepsilon_2/\varepsilon_1'\varepsilon_2'')$ *has a solution e in L if and only if k_2 is a Type III and* $\zeta^a N(B_1) = N(B_2)$.
(c) $e^3 = \zeta^a(\varepsilon_1\varepsilon_2/\varepsilon_1'\varepsilon_2')$ *has a solution e in L if and only if k_2 is of Type III and* $\zeta^a N(B_1) = \overline{N(B_2)}$.
(d) $e^3 = \zeta^a(\varepsilon_1\varepsilon_2\varepsilon_3/\varepsilon_1'\varepsilon_2'\varepsilon_3'')$ *has a solution e in L if and only if*

$$\alpha^3 = \frac{\zeta^a N(B_1)N(B_2)}{N(B_3)}$$

has a solution $\alpha \in k$. Thus either k_2 or k_3 is of Type III.

Proof. (a) If $e^3 = \zeta^a\varepsilon_1/\varepsilon_1' = \zeta^a(N(B_1)/(B_1^\sigma)^3)$ then $L = K_1(e) = K_1(\sqrt[3]{\zeta^a N(B_1)})$ so that $\zeta^a N(B_1) = \alpha^3 n$ or $\alpha^3 n^2$ with $\alpha \in K_1$. There is little difference in the two cases so we shall only consider the former. Now $N_{K_1/Q}(B_1) = N(B_1)\overline{N(B_1)} = (\alpha\bar{\alpha})^3 n^2$. But it follows from Theorem 15.4 of [2] that $N_{K_1/Q}(B_1)$ is the cube of a rational integer. Thus $\sqrt[3]{n}$ is in K_1 contrary to assumption.

(b) If $N(B_1) = \zeta^a N(B_2)$ then

$$\frac{\varepsilon_1\varepsilon_2}{\varepsilon_1'\varepsilon_2''} = \frac{N(B_1)}{(B_1^\sigma)^3}\frac{B_2^3}{N(B_2)} = \zeta^a\left(\frac{B_2}{B_1^\sigma}\right)^3.$$

Conversely if

$$e^3 = \zeta^a\frac{\varepsilon_1\varepsilon_2}{\varepsilon_1'\varepsilon_2''} = \zeta^a\frac{N(B_1)}{N(B_2)}\left(\frac{B_2}{B_1^\sigma}\right)^3$$

then

$$\left(\frac{B_1^\sigma}{B_2}e\right)^3 = \zeta^a\frac{N(B_1)}{N(B_2)} \in k.$$

If $\beta = (B_1^\sigma/B_2)e \notin k$ then $k(\beta) = K_i$ for some $i = 1, 2, 3$, or 4. Thus $\zeta^a(N(B_1)/N(B_2)) = \alpha^3 m_i$ or $\alpha^3 m_i^2$ for some $\alpha \in k$. For convenience assume the former. Taking complex conjugates and multiplying the equations together we obtain

$$(\alpha\bar{\alpha})^3 m_i^2 = \frac{N(B_1)\overline{N(B_1)}}{N(B_2)\overline{N(B_2)}}.$$

But $N(B_1)\overline{N(B_1)}$ and $N(B_2)\overline{N(B_2)}$ are cubes of rational integers so m_i is a cube in Q contrary to assumption. Thus

$$\beta = \frac{B_1^\sigma}{B_2}e \in k \quad\text{and}\quad \zeta^a\frac{N(B_1)}{N(B_2)} = \beta^3.$$

But both $N(B_1)$ and $N(B_2)$ are cube free integers of k so $\beta^3 = 1$ and $\zeta^a N(B_1) = N(B_2)$. Thus k_2 must also be of Type III.

(c) We need only observe that $\overline{N(B_2)} = N(\overline{B_2})$ and

$$\varepsilon_2/\varepsilon_2' = \overline{\left(\frac{\varepsilon_2}{\varepsilon_2''}\right)} = \overline{\left(\frac{B_2^3}{N(B_2)}\right)} = \frac{(\overline{B_2})^3}{N(\overline{B_2})}$$

so that the proof of part (b) applies.

(d) Suppose $\zeta^a(N(B_1)N(B_2)/N(B_3)) = \alpha^3$ for some $\alpha \in k$. Then

$$\zeta^a\frac{\varepsilon_1\varepsilon_2\varepsilon_3}{\varepsilon_1'\varepsilon_2'\varepsilon_3''} = \zeta^a\frac{N(B_1)N(B_2)}{(B_1^\sigma B_2^\sigma)^3}\frac{B_3^3}{N(B_3)} = \left(\frac{\alpha B_3}{B_1^\sigma B_2^\sigma}\right)^3.$$

Conversely if

$$e^3 = \zeta^a\frac{\varepsilon_1\varepsilon_2\varepsilon_3}{\varepsilon_1'\varepsilon_2'\varepsilon_3''} = \zeta^a\frac{N(B_1)N(B_2)}{N(B_3)}\left(\frac{B_3}{B_1^\sigma B_2^\sigma}\right)^3$$

then

$$\left(\frac{B_1^\sigma B_2^\sigma}{B_3}e\right)^3 = \zeta^a\frac{N(B_1)N(B_2)}{N(B_3)} \in k.$$

As in part (b),

$$\alpha = \frac{B_1^\sigma B_2^\sigma}{B_3} e \in k.$$

Since $N(B_1)$ is a cube free integer of k which is not a unit and

$$\alpha^3 = \zeta^a \frac{N(B_1)N(B_2)}{N(B_3)}$$

it is impossible for $N(B_2)$ and $N(B_3)$ to both be units of k. Thus either k_2 or k_3 (possibly both) are of Type III.

COROLLARY I. *Let ε be any unit of K_2. If $e^3 = \zeta^a \varepsilon_1 \varepsilon$ has a solution e in L then k_1 is of Type I.*

Proof. If $e^3 = \zeta^a \varepsilon_1 \varepsilon$ then $(e^{\sigma_2})^3 = \zeta^a \varepsilon_1' \varepsilon$ so that $(e/e^{\sigma_2})^3 = \varepsilon_1/\varepsilon_1'$. Theorem X(a) tells us that k_1 is not of Type III. However

$$(e/e^{\sigma_2})^3 = \varepsilon_1/\varepsilon_1' = \frac{N(B_1)}{(B_1^{\sigma_2})^3}$$

so that $N(B_1)$ is a cube in L and hence k_1 is not of Type IV either.

Before stating more results we need to clean up a few minor details.

Remark A. If $\varepsilon = B/B^\sigma$ then $\varepsilon^2 = \varepsilon\varepsilon'\bar{B}/(\varepsilon\varepsilon'\bar{B})^\sigma$.

Proof. Since $\varepsilon = B/B^\sigma$ we have $\varepsilon = \bar{B}/\overline{B^\sigma} = \bar{B}/(\bar{B})^{\sigma^2}$ and $\varepsilon'' = \varepsilon^{\sigma^2} = (\bar{B})^{\sigma^2}/(\bar{B})^\sigma$. Thus $\varepsilon\varepsilon'' = \bar{B}/(\bar{B})^\sigma$. Therefore

$$\frac{\varepsilon\varepsilon'\bar{B}}{(\varepsilon\varepsilon'\bar{B})^\sigma} = \frac{\varepsilon\varepsilon'}{\varepsilon'\varepsilon''}\varepsilon\varepsilon'' = \varepsilon^2.$$

The significance of this remark is that if we wish to replace ε_i with ε_i^2 in any part of Theorem X then we should replace $N(B_i)$ with $N(\varepsilon_i\varepsilon_i''\bar{B}_i) = \overline{N(B_i)}$.

DEFINITION. If α and β are in k we shall say α and β are *equivalent* and write $\alpha \sim \beta$ if $\alpha = \beta$ or $\alpha = \bar{\beta}$. Moreover we extend this definition multiplicatively.

Remark B. If any ε_i is replaced by ε_i^2 in Theorem X the "only if" statements in parts (b), (c), (d) hold with equality of norms replaced by equivalence.

COROLLARY II. *The equation $e^3 = \varepsilon_1^a \varepsilon_2^b$ with $1 \le a, b \le 2$ can have a solution e in K only when k_1 and k_2 are of Type I.*

Proof. The proof of Corollary I tells us $N(B_1) \sim 1 \sim N(B_2)$. Thus both k_1 and k_2 are of Type I.

COROLLARY III. *If the field k_1 is of Type III then $e^3 = \varepsilon_1^a\varepsilon_2^b\varepsilon_3^c$ with $1 \le a, b, c \le 2$ has no solution e in K unless $N(B_1) \sim N(B_2) \sim N(B_3)$. Thus all of*

k_1, k_2, k_3 are of Type III and m_1, m_2, m_3 must have a common prime divisor $p \equiv 1 \pmod 3$.

Proof. From the above remarks it is clear that we may take $a = b = c = 1$. If $e^3 = \varepsilon_1\varepsilon_2\varepsilon_3$ then $(e^{\sigma_3})^3 = \varepsilon_1'\varepsilon_2''\varepsilon_3$ and $(e^{\sigma_2})^3 = \varepsilon_1'\varepsilon_2\varepsilon_3'$ so that both

$$\frac{\varepsilon_1\varepsilon_2}{\varepsilon_1'\varepsilon_2''} \quad \text{and} \quad \frac{\varepsilon_1\varepsilon_3}{\varepsilon_1'\varepsilon_3'}$$

are cubes of units on L. By Theorem X(b), $N(B_1) \sim N(B_2) \sim N(B_3)$ so k_2 and k_3 are also of Type III. Moreover Theorem 15.4 of [2] shows m_1, m_2, m_3 have a common prime divisor $p \equiv 1 \pmod 3$.

COROLLARY IV. *If k_1 is Type III and $e^3 = \varepsilon_1^a\varepsilon_2^b\varepsilon_3^c\varepsilon_4^d$ has a solution e in K where $1 \leq a, b, c, d \leq 2$ then at least three of the fields k_1, k_2, k_3, k_4 are of Type III. If k_4 is not of Type III then $N(B_4) \sim \zeta^t$ with $t = 0$ or 1 and $N(B_1) \sim \zeta^t N(B_2) \sim \zeta^t N(B_3)$. In the latter case m_1, m_2, m_3 must have a common prime divisor $p \equiv 1 \pmod 3$.*

Proof. As usual we may assume $e^3 = \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4$ so

$$(e^{\sigma_2})^3 = \varepsilon_1'\varepsilon_2\varepsilon_3'\varepsilon_4'', \quad (e^{\sigma_3})^3 = \varepsilon_1'\varepsilon_2''\varepsilon_3\varepsilon_4' \quad \text{and} \quad (e^{\sigma_4})^3 = e_1'\varepsilon_2'\varepsilon_3''\varepsilon_4.$$

Thus all of

$$\frac{\varepsilon_1\varepsilon_3\varepsilon_4}{\varepsilon_1'\varepsilon_3'\varepsilon_4''}, \quad \frac{\varepsilon_1\varepsilon_2\varepsilon_4}{\varepsilon_1'\varepsilon_2''\varepsilon_4'}, \quad \text{and} \quad \frac{\varepsilon_1\varepsilon_2\varepsilon_3}{\varepsilon_1'\varepsilon_2'\varepsilon_3''}$$

are cubes in L. Theorem X tells us that

$$\frac{N(B_1)N(B_3)}{N(B_4)}, \quad \frac{N(B_1)N(B_4)}{N(B_2)}, \quad \text{and} \quad \frac{N(B_1)N(B_2)}{N(B_3)}$$

are equal to cubes in k and that at least two of the fields k_2, k_3, k_4 are of Type III. If k_4 is not of Type III then $N(B_4) = \zeta^t$ with $0 \leq t \leq 2$. Thus $\zeta^t N(B_1) = \alpha_1^3 \overline{N(B_2)}$ and $N(B_1)N(B_3) = \zeta^t\alpha_2^3$ with $\alpha_1, \alpha_2 \in k$. Since $N(B_1)$ and $\overline{N(B_2)}$ are cube free we have $\alpha_1^3 = 1$ and $\zeta^t N(B_1) = \overline{N(B_2)}$. Also $N(B_1)N(B_3)\overline{N(B_3)} = \zeta^t\alpha_2^3\overline{N(B_3)}$. But $N(B_3)\overline{N(B_3)}$ is the cube of a rational integer and $N(B_1)$, $\overline{N(B_3)}$ are cube free integers of k so that $\alpha_2^3 = N(B_3)\overline{N(B_3)}$ and $N(B_1) = \zeta^t\overline{N(B_3)}$. As usual we apply Theorem 15.4 of [2] to prove the last statement.

THEOREM XI. *Assume all fields k_1, k_2, k_3, and k_4 are of Type I or IV and that k_1 is of Type IV. Then:*

(a) *If $e^3 = \varepsilon_1^a\varepsilon_2^b\varepsilon_3^c$ with $1 \leq a, b, c \leq 2$ has a solution e in K then all of k_1, k_2, k_3 are Type IV.*

(b) *If* $e^3 = \varepsilon_1^a\varepsilon_2^b\varepsilon_3^c\varepsilon_4^d$ *with* $1 \leq a, b, c, d \leq 2$ *then exactly three of* k_1, k_2, k_3, k_4 *are of Type IV.*

Proof. (a) As before it suffices to take $a = b = c = 1$. As in the proof of Theorem X and its corollaries, $e^3 = \varepsilon_1\varepsilon_2\varepsilon_3$ implies that $N(B_1)/N(B_2)$ and $N(B_1)N(B_3)$ are cubes in L. But these are cube roots of unity and hence are 1. Since $N(B_1)$ is not 1 neither are $N(B_2)$ nor $N(B_3)$.

(b) If $e^3 = \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4$ then as in the proof of Corollary IV to Theorem X,

$$\frac{N(B_1)N(B_3)}{N(B_4)} \quad \text{and} \quad \frac{N(B_1)N(B_2)}{N(B_3)}$$

are cubes in L and hence are 1. If $N(B_i) = \zeta^{a_i}$ for $i = 1, 2, 3, 4$ then

$$a_1 + a_3 + 2a_4 \equiv 0 \pmod 3 \quad \text{and} \quad a_1 + a_2 + 2a_3 \equiv 0 \pmod 3).$$

Now $a_1 \not\equiv 0 \pmod 3$. First suppose $a_2 \not\equiv 0 \pmod 3$. If $a_1 \equiv a_2 \pmod 3$ then $a_3 \equiv 2a_1$ and $a_4 \equiv 0 \pmod 3$. If $a_1 \equiv 2a_2$ then $a_3 \equiv 0$ and $a_4 \equiv a_1 \pmod 3$. If $a_2 \equiv 0 \pmod 3$ then $a_3 \equiv a_1$ and $a_4 \equiv 2a_1 \pmod 3$. Thus in any case exactly one of the fields k_1, k_2, k_3, k_4 is of Type I and the remaining three are of Type IV.

We are now in a position to state the relationship between the "kind" of K and the "type" of its subfields k_1, k_2, k_3, k_4.

THEOREM XII. (a) *If K is Kind* 1 *then either all the fields* k_1, k_2, k_3 *and* k_4 *are Type I or exactly three, say* k_2, k_3, k_4, *are of Type I and* $k_1 = Q(\sqrt[3]{3})$.

(b) *If K is Kind* 2 *then the fields* k_1, k_2, k_3, k_4 *satisfy one of the following conditions:*

(i) *at least two are Type I;*
(ii) *at least three are Type IV and the remaining (if any) is Type I;*
(iii) *exactly three are Type III and one is Type I.*

(c) *If K is Kind* 3 *then* k_1, k_2, k_3, k_4 *must satisfy at least one of the following conditions:*

(i) *at least one is Type I;*
(ii) *at least three are of the same type.*

(d) *If K is Kind* 4 *then there is no apparent restriction on the type of* k_1, k_2, k_3, k_4.

Proof. (a) It is immediate from Theorem VI and Corollaries I and II to Theorem X that at least three of the fields k_1, k_2, k_3, k_4 are of Type I and if k_1 is not of Type I then the equations $e_i^3 = \varepsilon_i$ $(i = 2, 3, 4)$ all have solutions e_i in K. Thus $K = k_i(e_i)$ for $i = 2, 3, 4$ and so only prime divisors of three in k_i

can ramify in the extension K/k_i ($i = 2, 3, 4$). If $p \neq 3$ is a prime divisor of m_1 then p does not ramify in one of the field k_j with $j = 2, 3$, or 4. Thus the prime divisors of p in k_j must ramify in K contrary to the observation above. Thus m_1 has no prime divisors $p \neq 3$ and so $k_1 = Q(\sqrt[3]{3})$.

(b), (c), and (d). Follow from analyzing the various possible cases of Theorem VI using the corollaries to Theorem X and Theorem XI.

Finally we conclude this section by considering some examples.

THEOREM XIII. *Let* $p, q \equiv 2$ *or* $5 \pmod 9$ *be primes.*

(a) *If* $m_1 = 3$, $m_2 = p$, $m_3 = 3p$, $m_4 = 9p$ *then* K *is of Kind* 1 *and* ε_1, $\sqrt[3]{\varepsilon_2}$, $\sqrt[3]{\varepsilon_3}$, $\sqrt[3]{\varepsilon_4}$ *are a system of fundamental units for* K.

(b) *Let* $m_1 = p$, $m_2 = q$, $m_3 = pq$, $m_4 = p^2q$ *and if necessary interchange* m_3 *and* m_4 *so that* $m_3 \equiv \pm 1 \pmod 9$. *Then* K *is Kind* 1 *and we may choose a fundamental system of the form* ε_1, $\sqrt[3]{\varepsilon_1^a\varepsilon_2}$, $\sqrt[3]{\varepsilon_3}$, *and* $\sqrt[3]{\varepsilon_1^b\varepsilon_4}$ *where* $a = 1$ *or* 2 *and* $b = 0, 1,$ *or* 2.

Proof. (a) Here k_1 is of Type IV and k_2, k_3, k_4 are of Type I (see [2, p. 236]). By Theorem IX, $\varepsilon_i = \alpha_i^3/r_i$ for $i = 2, 3, 4$ where $\alpha_i \in k$ and $r_i \in Z$ divides $9p^2$. Thus $\sqrt[3]{r_i} \in K$ and hence $\sqrt[3]{\varepsilon_i} \in K$ for $i = 2, 3, 4$. It is clear from Theorem VI that ε_1, $\sqrt[3]{\varepsilon_2}$, $\sqrt[3]{\varepsilon_3}$, and $\sqrt[3]{\varepsilon_4}$ form a fundamental system of units for K.

(b) Here all four cubic subfields are of Type I so Theorem IX tells us $\varepsilon_i = \alpha_i^3/r_i$ for $i = 1, 2, 3, 4$. Moreover $r_i \neq m_i, m_i^2$, and r_i is a "principal divisor" of the discriminant of k_i for each $i = 1, 2, 3, 4$. It follows that 3 is a principal divisor of k_1 and k_2, that p and q are principal divisors in k_3 and that either $3, p, 3p$, or $3q$ is a principal divisor in k_4. Thus $\sqrt[3]{\varepsilon_1^a\varepsilon_2}$, $\sqrt[3]{\varepsilon_3}$, $\sqrt[3]{\varepsilon_1^b\varepsilon_4}$ are in K where $a = 1$ or 2 and $b = 0, 1$, or 2. Theorem VI again applies to complete the proof.

COROLLARY. *In part* (a) *above*, $(\hat{E}: \hat{\varepsilon}) = 3^5$ *and* $(\hat{E}: \hat{\varepsilon}_1) = 3^2$.

Proof. Immediate from Theorem I since all class numbers all relatively prime to 3.

5. A class number formula for K

THEOREM XIV. *The class number* h *of* K *satisfies the relation*

$$3^3h = (\hat{e}: \hat{e}_0)h_1h_2h_3h_4.$$

Proof. We may always number our fields k_i ($i = 1, 2, 3, 4$) so that $t_1 = 1$ in Theorem I. In fact if one of the fields k_i is of Type III then we may take this to be k_1. Let $s = 0, 1, 2, 3$, or 4 be the number of fields k_i of Type III. Letting $t = 0$ or 1 according as k_1 is of Type III or not, we have $(\hat{\varepsilon}_1: \hat{\varepsilon}_0) = 3^t(\hat{e}: \hat{e}_0)^2$ and $(\hat{\varepsilon}: \hat{\varepsilon}_0) = 3^{4-s}$.

Thus

$$(\hat{E}:\hat{\varepsilon}) = 3^{t+s-4}(\hat{e}:\hat{e}_0)^2(\hat{E}:\hat{\varepsilon}_1).$$

Theorem I tell us that $3^3H = 3^t(\hat{E}:\hat{\varepsilon}_1)h^2$ and

$$\begin{aligned} 3^5H &= (\hat{E}:\hat{\varepsilon})H_1H_2H_3H_4 \\ &= 3^{t+s-4}(\hat{e}:\hat{e}_0)^2(\hat{E}:\hat{\varepsilon}_1)3^{-s}h_1^2h_2^2h_3^2h_4^2 \\ &= 3^{t-4}(\hat{e}:\hat{e}_0)^2(\hat{E}:\hat{\varepsilon}_1)h_1^2h_2^2h_3^2h_4^2. \end{aligned}$$

Thus $3^6h^2 = (\hat{e}:\hat{e}_0)^2h_1^2h_2^2h_3^2h_4^2$ yielding the desired result.

COROLLARY I. *If h is not divisible by three then K is of Kind* 1 *and so* $(\hat{e}:\hat{e}_0) = 27$.

Proof. Theorem IV tells us that all of h_1, h_2, h_3, and h_4 are relatively prime to three. Thus Theorem XIV gives $(\hat{e}:\hat{e}_0) = 27$ and Theorem VI tell us K is Kind 1.

COROLLARY II. *If all but one of the class numbers h_1, h_2, h_3, h_4 are relatively prime to three then K is of Kind* 1.

Proof. Say h_1 is not prime to three. If $3^a \parallel h_1$ then it is clear from Theorem XIV that $3^{a+1} \nmid h$. However it is easy to show by class field theory that $h_1 \mid h$, i.e., simply take the Hilbert class field $\bar{k}_1$ of k_1 and note that $\bar{k}_1K$ is an abelian unramified extension of K of degree h_1. Thus $3^a \parallel h$ and so $(\hat{e}:\hat{e}_0) = 27$ and K is of Kind 1.

Combining our last theorem with Lemma 5 of [3] gives an interesting result. First some additional notation is required.

T: ring of integers of K.

S: smallest subring of K containing $1, \sqrt[3]{m_1}, \sqrt[3]{m_2}, \sqrt[3]{m_3}, \sqrt[3]{m_4}$.

$s_i = 3$ or 1: according as $m_i \equiv \pm 1 \pmod 9$ or not, for $i = 1, 2, 3, 4$.

COROLLARY III. *For any field K, $(T:S) = 3^3s_1s_2s_3s_4$.*

Proof. Lemma 5 of [3] says

$$(T:S)h = (\hat{e}:\hat{e}_0)h_1h_2h_3h_4s_1s_2s_3s_4 = 3^3hs_1s_2s_3s_4$$

by Theorem XIV. This proves the corollary.

We conclude this article with a short table of class numbers for some bicubic fields. In the table the types of the subfields k_1, k_2, k_3, k_4 are listed in consecutive order. The class number and a system of fundamental units is given for each field K. The class numbers of the fields k_1, k_2, k_3, k_4 were obtained from the table of Selmer [13].

CLASS NUMBER AND UNITS

m_1	m_2	m_3	m_4	h_1	h_2	h_3	h_4	Type				h	Units				
2	3	6	12	1	1	1	1	I	IV	I	I	1	$\sqrt[3]{\varepsilon_1}$	ε_2	$\sqrt[3]{\varepsilon_3}$	$\sqrt[3]{\varepsilon_4}$	
2	5	10	20	1	1	1	3	I	I	I	I	3	ε_1	$\sqrt[3]{\varepsilon_1\varepsilon_2}$	$\sqrt[3]{\varepsilon_3}$	$\sqrt[3]{\varepsilon_1\varepsilon_4}$	
2	7	14	28	1	3	3	3	I	III	I	III	3	ε_1	ε_2	$\sqrt[3]{\varepsilon_1\varepsilon_3}$	ε_4	
2	11	22	44	1	2	3	1	I	I	I	I	6	ε_1	$\sqrt[3]{\varepsilon_1\varepsilon_2}$	$\sqrt[3]{\varepsilon_1^2\varepsilon_3}$	$\sqrt[3]{\varepsilon_4}$	
2	13	26	52	1	3	3	3	I	III	III	I	3	ε_1	ε_2	ε_3	$\sqrt[3]{\varepsilon_1\varepsilon_4}$	
2	15	30	60	1	2	3	3	I	I	I	I	18	ε_1	$\sqrt[3]{\varepsilon_1\varepsilon_2^2}$	$\sqrt[3]{\varepsilon_2\varepsilon_3}$	$\sqrt[3]{\varepsilon_4}$	
2	17	34	68	1	1	3	3	I	IV	I	I	3	ε_1	ε_2	$\sqrt[3]{\varepsilon_3}$	$\sqrt[3]{\varepsilon_4}$	
2	19	38	76	1	3	3	6	I	III	I	I	18	ε_1	ε_2	$\sqrt[3]{\varepsilon_1\varepsilon_3}$	$\sqrt[3]{\varepsilon_1\varepsilon_4}$	
3	5	15	45	1	1	2	1	IV	I	I	I	2	ε_1	$\sqrt[3]{\varepsilon_2}$	$\sqrt[3]{\varepsilon_3}$	$\sqrt[3]{\varepsilon_4}$	
3	7	21	63	1	3	3	6	IV	III	III	III	6	ε_1	ε_2	ε_3	$\sqrt[3]{\varepsilon_1^a\varepsilon_2^b\varepsilon_3^c\varepsilon_4^d}$	$0 \le a \le 2$ $1 \le b, c, d \le 2$
3	10	30	90	1	1	3	3	IV	I	I	I	3	ε_1	ε_2	$\sqrt[3]{\varepsilon_3}$	$\sqrt[3]{\varepsilon_4}$	
3	11	33	99	1	2	1	1	IV	I	I	I	2	ε_1	$\sqrt[3]{\varepsilon_2}$	$\sqrt[3]{\varepsilon_3}$	$\sqrt[3]{\varepsilon_4}$	
5	6	30	150	1	1	3	3	I	I	I	I	9	ε_1	$\sqrt[3]{\varepsilon_1\varepsilon_2}$	$\sqrt[3]{\varepsilon_1^2\varepsilon_2}$	$\sqrt[3]{\varepsilon_4}$	

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