SKEW PRODUCTS AND THE SIMILARITY OF VIRTUAL SUBGROUPS

BY

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Introduction

Consider a homomorphism $F: X \times H \to A$, where H and A are locally compact second countable groups and $X \times H$ is the virtual subgroup of H defined by applying Mackey's construction to a measure class preserving ergodic action of H on a finite analytic measure space X, m. We obtain (under some additional restrictions) an exact cohomology sequence (analogous to the first few terms of the Lyndon sequence for group extensions) involving $X \times H$, $W \times A =$ the range closure of F, and the measure groupoid $(X \times A) \times H$ defined by the skew product action (with respect to F) of H on $X \times A$, which plays the role of the kernel of F.

Then we are able to represent similar (in the sense of Ramsay) virtual subgroups of H and A, respectively, as contractions of a virtual subgroup of $H \times A$, obtained by the action of $H \times A$ on $X \times A$ that arises in Mackey's range closure construction. This representation leads easily to our main result—the similarity class of the Radon-Nikodym homomorphism for a virtual subgroup is a similarity invariant.

In Section 1 we introduce a method for dealing with inessential contractions (i.c.'s). Theorem 1.9 provides a method for dealing with homomorphisms as defined by Ramsay in [10], where the composition of homomorphisms is not necessarily defined, even though the composition of the similarity classes of the homomorphisms is defined. Also Theorem 1.9 leads easily to the first few terms of a Lyndon sequence for virtual groups in (2.4) in the special case where H and A are countable and the coefficient group B is an abelian analytic group. The part of the Lyndon sequence that is needed in greater generality is established in Theorems 2.5 and 2.6.

In Sections 3 and 4 we are concerned with similar virtual subgroups of H and A, respectively. 3.1 and 3.2 establish that consideration of a general half of a similarity $\beta: X \times H \to V \times A$ can always be reduced to consideration of the range closure homomorphism $\alpha: X \times H \to W \times A$ defined by a homomorphism $F: X \times H \to A$. Then the results of the previous sections on such an F are applicable. Theorem 3.3 and in more detail Lemma 3.4 provide the means for establishing the applications in Section 4.

The similarity invariance of the similarity class of the Radon-Nikodym homomorphism $[\rho]$ obtained in Theorem 4.4 allows us to classify (proper)

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virtual subgroups as type III (where $[\rho] \neq [1]$) or as type II (where $[\rho] = [1]$), so that similar virtual subgroups are of the same type. Thus we have generalized Proposition 6.1(b) of [4]. In the case where H is unimodular and $X \times H$ is type II we can further classify $X \times H$ as type II_1 (where there is a finite invariant measure in the measure class of m) or type II_∞ (where there is an infinite invariant measure in the measure class of m), but this further classification is not preserved under similarity.

The final application in Section 4 (4.5–4.8) relates the relations of similarity and isomorphism for virtual subgroups of countable groups, through the concept of G-equivalence of Borel sets.

Section 1

In dealing with virtual groups, it is frequently necessary to clip away sets of measure zero from a measure space. This can cause technical difficulties when we are considering a group action on the measure space. To handle such technicalities in a systematic manner we introduce the concept of a partial action of a group.

1.0 DEFINITION. A *partial action* of H on an analytic Borel space X_0 with a finite measure, m_0, consists of a Borel subset $E_0 = (X_0 \times H)_0$ of $X_0 \times H$ and a Borel map $a: E_0 \to X_0$ (to simplify the notation we will write $h(x)$ or hx for $a(x, h^{-1})$) such that:

1.1. For every $x \in X_0$ the x-slice of E_0 ($= E_0|_x$) is conull in H (Haar measure.

1.2(a) If (x, h^{-1}) and $(hx, j^{-1}) \in E_0$ then $(x, (jh)^{-1}) \in E_0$ and $j(hx) = (jh)(x)$.

(b) If $(x, h^{-1}) \in E_0$ then $(hx, h) \in E_0$ and $h^{-1}(hx) = x$.

(c) $(x, 1) \in E_0$ and $1(x) = x$ for all $x \in X_0$.

Conditions 1.2(a), (b), and (c) give $E_0 = (X_0 \times H)_0$ a groupoid structure—we define the composition by $(x, k) \circ (y, j) = (x, kj)$ iff $y = k^{-1}x$ and the inverse by $(x, k)^{-1} = (k^{-1}x, k^{-1})$. The map sending $x \in X_0$ to $(x, 1)$ identifies X_0 with the set of units of $(X_0 \times H)_0$.

We will use standard results on Borel spaces and measures as in Chapter I of [2].

From here on we will assume that the partial action of H on X_0 preserves the measure class of m_0, i.e., for every $h \in H$ and Borel set $B \subseteq X_0$ we have $m_0(B) = 0$ iff $m_0(hB) = 0$, where

$$hB = \{x: (x, h) \in (X_0 \times H)_0 \text{ and } h^{-1}x \in B\}.$$

1.3 *Remarks.* (a) If $B = \{y\}$ then $hB = \{hy\}$ if there exists $(x, h) \in (X_0 \times H)_0$ such that $h^{-1}x = y$; otherwise $hB = \emptyset$.

(b) If we extend the map a to a Borel map $a: X_0 \times H \to X_0$ (arbitrary extension) and X_0 is standard then a satisfies the conditions of Lemma 3.1 in

[10, p. 267] and consequently the partial action of H on X_0 defines a Boolean action of H on $M(X_0, m_0)$, the measure algebra of Borel subsets of X_0 mod null sets.

(c) If $E_0 = X_0 \times H$ then condition 1.1 is satisfied and the conditions 1.2(a), (b), (c) on a establish that the map sending (x, h) into $a(x, h^{-1})$ defines a Borel action of H on X_0.

Suppose that X_i is a conull Borel subset of X_0. We can form the contraction of the groupoid

$(X_0 \times H)_0$ to $X_i = \{(x, h) \in (X_0 \times H)_0 \colon x \text{ and } h^{-1}x \in X_i\} = (X_i \times H)_i = E_i$

with the inherited groupoid structure. Then E_i with the restriction of a to $a\colon (X_i \times H)_i \to X_i$ defines a partial action of H on X_i, except possibly for the condition 1.1.

1.4 Lemma. *If X_1 is a conull Borel subset of X_0 then there is a conull Borel subset X_2 of X_1 such that E_2 with the restriction of a to $a\colon (X_2 \times H)_2 \to X_2$ defines a partial action of H on X_2. X_2 can be chosen to be a standard Borel space.*

Proof. We construct a sequence $\{U_i\}$ of Borel subsets of X_0 by choosing U_1 to be a conull Borel subset of X_1 which is standard as a Borel space (see [2, Chapter I, Section 2]) and

$$U_{i+1} = \{x \in U_i \colon \text{the } x\text{-slice } (U_i \times H)_i|_x \text{ is conull in } H\}.$$

Assuming U_i to be a conull Borel subset of X_0, we find $(U_i \times H)_i$ to be a conull Borel subset of $X_0 \times H$ since, for every $h \in H$, the h-slice $(U_i \times H)_i|^h = U_i \cap hU_i$ is conull in X_0. Then U_{i+1} is conull in X_0 by the Fubini Theorem. By induction on i all the U_i's are conull Borel subsets of X_0. Then $X_2 = \bigcap_{i=1}^{\infty} U_i$ satisfies the required conditions (cf. top of page 275, [10]).

We can relate partial actions of H to the universal H-space introduced by Mackey in [7]. Let $X = L^2_{\text{loc}}(H)$ be the set of locally square integrable functions on H mod equality a.e. The seminorms $\| \enspace \|_K$, $\|f\|_K^2 = \int_K |f|^2$, K a compact subset of H, define a polonaise topology on X and we define the Borel action of H on X by $h(f)(k) = f(h^{-1}k)$.

1.5 Theorem. *There is a one to one Borel map $\psi\colon X_0 \to X$ such that if $(x, h^{-1}) \in (X_0 \times H)_0$ then $h\psi(x) = \psi(hx)$.*

Proof. The proof is similar to that of Lemma 2 of [7] and Lemma 3.2 in [10]. Assume $X_0 \subseteq [0, 1]$, and for $x \in X_0$ we define $\psi(x) = \tilde{x} \in X$ by $\tilde{x}(h) = h(x)$ if $(x, h^{-1}) \in (X_0 \times H)_0$ and 0 otherwise. The proof that ψ is Borel is as described in [10]. Suppose $\psi(x) = \psi(y)$. Then $\tilde{x}(h) = \tilde{y}(h)$ on a conull Borel subset H_0 of H. Then we have (x, h^{-1}) and $(y, h^{-1}) \in (X_0 \times H)_0$ and $h(x) = h(y)$ for all $h \in (E_0|_x)^{-1} \cap (E_0|_y)^{-1} \cap H_0$, which is conull in H by Condition 1.1. Hence $x = y$ by Condition 1.2.

The Borel action of H on X preserves the measure class of $m = \psi_*(m_0)$. We can identify X_0 with the conull analytic subset $\psi(X_0)$ of X (via ψ). With this identification, we obtain the following:

1.6 THEOREM. *$(x, h) \in (X_0 \times H)_0$ iff x and $h^{-1}x \in X_0$.*

Proof. The "only if" part is obvious. The proof of the "if" part is related to that of Lemma 5.2 in [10]. We assume x and $h^{-1}x \in X_0$. Then by Condition 1.1 the sets

$$\{k \in H: (x, hk) \in (X_0 \times H)_0\} \quad \text{and} \quad \{k \in H: (h^{-1}x, k) \in (X_0 \times H)_0\}$$

are conull, so there exists an element k in their intersection. Then the composition $(x, hk) \circ (h^{-1}x, k)^{-1}$ is defined and equals $(x, h) \in (X_0 \times H)_0$.

From here on we will assume that the action of H on X is ergodic, i.e., the measure algebra $M(X, m)^H$ of H invariant elements in $M(X, m)$ is trivial. We form the virtual group (or ergodic groupoid) $X \times H$, by Mackey's construction, as in [8] or [10] (see these references for the standard terminology regarding virtual groups). Then the inessential contraction (i.c.) of $X \times H$ to a conull Borel subset X_i of X_0 is the same (algebraically) as $(X_i \times H)_i$. Accordingly, we will regard $(X_i \times H)_i$ as a virtual group, with the Borel and measure class structure it inherits as a subset of $X \times H$, $m \times$ Haar measure on H. We use the definitions of strict homomorphism, homomorphism, strict similarity, and similarity as given in [10, 6.1–6.14].

Consider a homomorphism $F: X \times H \to A$. Then there is a conull Borel subset X_0 of X such that the restriction $F: (X_0 \times H)_0 \to A$ is a strict homomorphism, i.e., F is a Borel map and $F(x, h)F(h^{-1}x, j) = F(x, hj)$ for x, $h^{-1}x$, and $(hj)^{-1}x \in X_0$, h and $j \in H$. By Lemma 1.4 we can choose X_0 so that $(X_0 \times H)_0$ also satisfies Condition 1.1. Then F defines a partial action of $G = H \times A$ on $Y_0 = X_0 \times A$ ($m \times m'$, m' is a finite measure equivalent to Haar measure on A) as follows:

$$\text{(1.7)} \qquad ((X_0 \times A) \times G)_0 = \{((x, a), (h, b)): h^{-1}x \text{ and } x \in X_0\},$$
$$(h, b)(x, a) = (hx, baF(x, h^{-1})). \qquad \text{(cf. [8])}$$

By Theorem 1.5 we can regard the partial action of G on Y_0 as the restriction to Y_0 of a Borel action of G on an analytic measure space Y, which preserves the measure class of a finite measure m'' on Y. The restriction of this action to H $(= H \times \{1\})$ is called the skew product action of H, since it generalizes the skew product transformation of Anzai in [1], as discussed in [8].

1.8 DEFINITIONS. The ergodic action of A on the analytic finite measure space W, u induced by the Boolean action of A on $M(Y)^H$ (using Mackey's point realization technique, cf. [10]) is called the *range closure action* for F, and the resulting virtual subgroup $W \times A$ of A is called the *range closure* of F, as in [8]. The homomorphism $\alpha: X \times H \to W \times A$ given below is called the *range closure homomorphism* for F.

The inclusion $M(Y)^H \to M(Y)$ defines (by Theorem 3.6 in [10]) an A equivariant, H invariant Borel map $p\colon Y' \to W$, where Y' is a G invariant conull analytic subset of Y. We select an analytic conull subset X_1 of X_0 so that $Y_1 = X_1 \times A \subseteq Y'$ and X_1 has the same properties as required of X_0. Then

$$\alpha\colon (X_1 \times H)_1 \to W \times A; \qquad \alpha(x, h) = (p(x, 1), F(x, h))$$

is a strict homomorphism. Note that

$$F(x, h)p(h^{-1}x, 1) = p(h^{-1}x, F(x, h)) = p(h^{-1}(x, 1)) = p(x, 1).$$

1.9 THEOREM. *Given the homomorphism $F\colon X \times H \to A$, there are homomorphisms ϕ, ψ, and γ such that:*

(a) *ϕ and ψ establish a similarity of $Y \times G$ and $X \times H$.*
(b) *The diagram*

$$\begin{tikzcd} Y \times G \arrow[d, shift right, "\phi"'] \arrow[dr, "\gamma"] & \\ X \times H \arrow[u, shift right, "\psi"'] \arrow[r, "\alpha"] & W \times A \end{tikzcd}$$

commutes mod similarity, where α is the range closure homomorphism for F.
(c) *$(\phi^0)_* m'' \equiv m$ (where $\phi^0\colon Y \to X$ is the restriction of ϕ to units).*
(d) *$(\gamma^0)_* m'' \equiv u$.*

Proof. (a) The similarity homomorphisms are given by

$$\phi\colon (Y_0 \times G)_0 \to (X_0 \times H)_0; \qquad \phi((x, a), (h, b)) = (x, h)$$

and

$$\psi\colon (X_0 \times H)_0 \to (Y_0 \times G)_0; \qquad \psi(x, h) = ((x, 1), (h, F(x, h)).$$

It is easy to verify that ϕ is a strict homomorphism. ψ is a Borel map and direct calculations show that it is algebraically a homomorphism. In general, ψ is not a homomorphism in the sense introduced by Mackey in [8], since the restriction to units $\psi^0\colon X_0 \to Y_0$; $\psi^0(x) = (x, 1)$ has too "thin" an image. However, ψ^0 does satisfy Ramsay's condition in [10, Definition 6.1], since a $(Y_0 \times G)_0$ saturated subset B of Y_0 satisfies the equation $B = B' \times A$ where $B' = (\psi^0)^{-1}(B)$ is an $(X_0 \times H)_0$ saturated subset of X_0. Hence B' is null if B is null, so ψ is a strict homomorphism. $\phi \circ \psi$ is the identity map and $\psi \circ \phi$ is strictly similar to the identity map via the Borel map $\theta\colon Y_0 \to (Y_0 \times G)_0$; $\theta(x, a) = ((x, a), (1, a))$, i.e.,

$$\theta(x, a) \circ (\psi \circ \phi)((x, a), (h, b)) = ((x, a), (h, b)) \circ \theta(h^{-1}x, b^{-1}aF(x, h)).$$

(b) The map $\gamma\colon (Y_1 \times G)_1 \to W \times A$; $\gamma(y, (h, b)) = (p(y), b)$ is a strict homomorphism (note $\gamma^0 = p$ on Y_1), and

$$\gamma \circ \psi(x, h) = (p(x, 1), F(x, h)) = \alpha(x, h) \quad \text{for all } (x, h) \in (X_1 \times H)_1.$$

(c) and (d) are easily established for the given ϕ and γ.

1.10 *Remarks.* (a) The similarity of $Y \times G$ and $X \times H$ is also evident from [10, Theorem 6.17], since the contraction of $(Y_0 \times G)_0$ to $X_0 \times \{1\}$ (with the appropriate measure class) is isomorphic as an ergodic groupoid to $(X_0 \times H)_0$.

(b) Replacing $X \times H$, α by $Y \times G$, γ by use of 1.9(a) has two advantages. First, use of the homomorphism γ avoids the technical problems that arise when a homomorphism has too "thin" an image (cf. 1.9(d)). Second, there is an algebraic simplification, so that roughly speaking the formation of the skew product action of H is similar to the restriction from $Y \times G$ to $Y \times H$, i.e., the diagram

$$\begin{tikzcd} Y \times H \arrow[r, "I"] \arrow[d, equal] & Y \times G \arrow[d, shift right, "\phi"'] \\ Y \times H \arrow[r, "p_F"] & X \times H \arrow[u, shift right, "\psi"'] \end{tikzcd}$$

commutes mod similarity, where $I(y, h) = (y, (h, 1))$ and $p_F((x, a), h) = (x, h)$.

Section 2

Suppose that B is a Borel group and $\mathscr{F}$ is a measure groupoid, i.e., $\mathscr{F}$ satisfies all the conditions for an ergodic groupoid except for the ergodic condition (cf. [11]). Then cohomology groups $H^n(\mathscr{F}; B)$, $n \geq 0$, can be defined as in [12] if B is abelian, and we define $H^1(\mathscr{F}; B) = \{\text{homomorphisms } f\colon \mathscr{F} \to B\}$ mod similarity, if B is nonabelian. For convenience, we will write $H^n(\mathscr{F})$ instead of $H^n(\mathscr{F}; B)$. The definitions and results of [12], before 3.51, apply to the above setting. In particular, if $\mathscr{S}$ is an inessential contraction of $\mathscr{F}$ then the inclusion map $\mathscr{S} \to \mathscr{F}$ induces an isomorphism of the cohomology groups if B is abelian (see [12, Theorem 3.5]), and induces a bijective map in the case where B is nonabelian. Accordingly, we will identify $H^n(\mathscr{F})$ and $H^n(\mathscr{S})$ in the following discussion.

To obtain Theorem 3.51 of [12] (that $H^0(\mathscr{F}; B) \cong B$) it is necessary that $\mathscr{F}$ be ergodic and that the Borel structure of B be countably generated, so that we can regard B as a subset of $[0, 1]$ (Definition 2.0 of [12] should be corrected to include this requirement on the coefficient group). From here on we will assume that B is analytic.

Consider the homomorphism $F\colon X \times H \to A$, as in Section 1. The homomorphism $\phi\colon (Y_0 \times G)_0 \to (X_0 \times H)_0$, from 1.9(a), satisfies the conditions for a strict homomorphism in [12, Definition 3.3], by 1.9(c). Hence ϕ induces group homomorphisms

$$\phi^{n*}\colon H^n(X \times H) \to H^n(Y \times G)$$

if B is abelian, and induces a map

$$\phi^*\colon H^1(X \times H) \to H^1(Y \times G)$$

if B is nonabelian. Unfortunately, ψ (from 1.9(a)) does not in general satisfy the conditions for Definition 3.3 in [12]. However, using some ideas from

[10], in particular the proof of Lemma 6.6 in [10], we obtain the following result.

2.1 THEOREM. *ϕ^{n*}; $H^n(X \times H) \to H^n(Y \times G)$ is an isomorphism for $n \geq 0$ in the case where B is abelian and ϕ^*: $H^1(X \times H) \to H^1(Y \times G)$ is bijective for the case where B is nonabelian.*

Proof. Omitted—for the case $n = 1$, see [10, 6.12–6.13].

Let $\mathscr{M} = \mathscr{M}(Y; B)$ be the set of all Borel maps f: $Y \to B$ mod equality a.e, with the group structure defined by pointwise composition. G acts on $\mathscr{M}$ (as a group of automorphisms) by $gf(y) = f(g^{-1}y)$. We observe that $p^*: \mathscr{M}(W; B) \to \mathscr{M}^H$, the H invariant elements in $\mathscr{M}$, where p: $Y' \to W$ is defined in 1.8, is an A equivariant isomorphism.

We will assume that H and A are countable groups and that B is abelian in obtaining the exact sequences 2.2, 2.3, and 2.4. Then $\mathscr{M}$ is a G module. The group cohomology for a G module is as defined in [9]. We regard H as a normal subgroup of $G = H \times A$ and identify A with the quotient group $(H \times A)/H$. Then the first five terms of the Lyndon spectral sequence (see [5], or [9, p. 354]), appear in our setting as the exact sequence

$$\begin{aligned} 0 \longrightarrow H^1(A; \mathscr{M}^H) &\xrightarrow{p_2^*} H^1(G; \mathscr{M}) \xrightarrow{i^*} H^1(H; \mathscr{M})^A \\ &\longrightarrow H^2(A; \mathscr{M}^H) \xrightarrow{p_2^*} H^2(G; \mathscr{M}) \end{aligned} \tag{2.2}$$

where $p_2(h, b) = b$ and $i(h) = (h, 1)$.

For H and A countable, we can transform the above sequence to a cohomology sequence for virtual groups, using the law of exponents as in [13], to obtain the following exact sequence:

$$\begin{aligned} 0 \longrightarrow H^1(W \times A) &\xrightarrow{\gamma^*} H^1(Y \times G) \xrightarrow{I^*} H^1(Y \times H)^A \\ &\longrightarrow H^2(W \times A) \xrightarrow{\gamma^*} H^2(Y \times G) \end{aligned} \tag{2.3}$$

where γ is as in 1.9(b), and $I(y, h) = (y, (h, 1))$.

Now we replace $H^n(Y \times G)$ by $H^n(X \times H)$ via the isomorphisms ϕ^{n*} of Theorem 2.1 to obtain a "skew product exact sequence"

$$\begin{aligned} 0 \longrightarrow H^1(W \times A) &\xrightarrow{\alpha^*} H^1(X \times H) \xrightarrow{p_F^*} H^1(Y \times H)^A \\ &\longrightarrow H^2(W \times A) \xrightarrow{\alpha^*} H^2(X \times H). \end{aligned} \tag{2.4}$$

relating the cohomology for $X \times H$, $Y \times H$ (the kernel of F), and $W \times A$ (the range closure of F). The result looks like the first few terms of a "Lyndon sequence for virtual groups."

Now we return to the general setting—H and A are not necessarily discrete and B is not necessarily abelian. To interpret the first two terms of the sequence (2.4) in our general setting we will prove the following theorem.

2.5 THEOREM. *The map $\alpha^*: H^1(W \times A) \to H^1(X \times H)$ is injective.*

Proof. In view of Theorem 1.9(b) it suffices to prove that

$$\gamma^*: H^1(W \times A) \to H^1(Y \times G)$$

is injective. Recall the notation from 1.8–1.9. Suppose $[S]$ and $[T] \in H^1(W \times A)$ and $\gamma^*([S]) = \gamma^*([T])$. Then $T \circ \gamma$ and $S \circ \gamma$ are defined, by 1.9(d), and are similar homomorphisms. Then there is a conull Borel subset Y_2 of Y_1 and a Borel map $L: Y_2 \to B$ such that

$$T \circ \gamma(y, h, b) = L(y)S \circ \gamma(y, h, b)L((h, b)^{-1}y)^{-1}$$

for all (y, h, b) such that y and $(h, b)^{-1}y \in Y_2$. Setting $b = 1$, we obtain $L(y) = L(h^{-1}y)$ for y and $h^{-1}y \in Y_2$. Then there is a conull Borel subset W_0 of W such that S and T are strict on the i.c. to W_0 and such that there is a Borel map $L': W_0 \to B$ such that $L'(p(y)) = L(y)$. Then

$$T(p(y), b) = L'(p(y))S(p(y), b)L(b^{-1}p(y))^{-1}$$

for all $(p(y), b)$ such that $p(y)$ and $b^{-1}p(y) \in W_0$. Hence S is similar to T.

This result can be applied as described in the introduction to [14] to show that the 1-cohomology group for a certain virtual group (with B the circle group) is very "large," i.e., it contains every compact abelian second countable group.

To interpret the second and third terms of the sequence (2.4) in the general setting, we define the kernel of

$$p_F^*: H^1(X \times H) \to H^1(Y \times H)$$

to be $(p_F^*)^{-1}([1])$ where $[1]$ is the similarity class of $1: Y \times H \to B$; $1(y, h) = 1$. We obtain half of the expected exactness result below.

2.6 THEOREM. *The kernel of p_F^* contains $\alpha^*(H^1(W \times A))$.*

Proof. In view of Theorem 1.9(b) it suffices to prove that the kernel of $I^*: H^1(Y \times G) \to H^1(Y \times H)$ contains $\gamma^*(H^1(W \times A))$. Suppose $M: Y \times G \to B$ is a homomorphism in the similarity class $\gamma^*([T])$ for some homomorphism $T: W \times A \to B$. $T \circ \gamma$ is defined, by 1.9(d), so M is similar to $T \circ \gamma$. Then $M \circ I \in I^*([M])$ and $M \circ I$ is similar to $T \circ \gamma \circ I$, which is $= 1$ on $Y \times H$. Hence $[M]$ is in the kernel of I^*.

We use this result in the next section. Further generalization of (2.4) is possible, but is not needed for this paper.

Section 3

A homomorphism $\beta: X \times H \to V \times A$ is called *half of a similarity* if the similarity class $[\beta]$ is invertible in the category of similarity classes of homomorphisms between virtual subgroups (cf. [10, Definition 6.14]). Given a half of a similarity β, we define $F\colon X \times H \to A$ by $F(x, h)$ = the second component of $\beta(x, h)$. Then F is a homomorphism and we can form the range closure homomorphism $\alpha\colon X \times H \to W \times A$ for F, as in 1.8.

3.1 THEOREM. *The Borel map* $g\colon X_0 \times A \to V$; $g(x, a) = a\beta(x, 1)$ *induces an A equivariant Boolean isomorphism* $g^*\colon M(V) \to M(W)$.

Proof. See the proof of Theorem 7.11 in [10, pp. 299–303].

3.2 *Remark.* By Theorem 3.5 of [10] the map g^* is induced by an A equivariant Borel map $\tilde{g}\colon W' \to V'$, where W' and V' are A equivariant conull analytic subsets of W and V respectively. Then $\tilde{g}$ defines an isomorphism (mod i.c.'s) $g'\colon W \times A \to V \times A$; $g'(w, b) = (g(w), b)$ for w and $b^{-1}w \in W'$, and the diagram

$$\begin{tikzcd} X \times H \arrow[rr, "\beta"] & & V \times A \\ & W \times A \arrow[ul, "\alpha"] \arrow[ur, "g'"'] & \end{tikzcd}$$

commutes mod similarity.

Consequently, the case where $\beta\colon X \times H \to V \times A$ is half of a similarity can be examined in the setting of Sections 1 and 2, replacing β by $\alpha\colon X \times H \to W \times A$, where α is the range closure homomorphism for a homomorphism $F\colon X \times H \to A$.

3.3 THEOREM. *Given a homomorphism* $F\colon X \times H \to A$, *the following conditions on* $\alpha\colon X \times H \to W \times A$, *the range closure homomorphism for F, are equivalent.*

(1) α *is half of a similarity.*

(2) $\alpha^*\colon H^1(W \times A; H) \to H^1(X \times H; H)$ *is surjective.*

(3) *There is a homomorphism* $T\colon W \times A \to H$ *such that the action of* $A \times H = G'$ *on U* (*defined by T as in* 1.7) *is isomorphic* (mod 0) *to the action of G on Y* (*identifying G' and G by interchange of components*) *and the diagram*

$$\begin{tikzcd} Y \times G \arrow[d, "\phi"'] \arrow[r, phantom, "\cong"] & U \times G' \arrow[d, "\phi_0"'] \\ X \times H \arrow[u, bend right, "\psi"'] \arrow[r, "\alpha"'] & W \times A \arrow[u, bend right, "\psi_0"'] \end{tikzcd} \qquad (as\ in\ 1.9(b))$$

commutes mod similarity.

Proof. (1) implies that the α^* of (2) is bijective, by [10, 6.12, 6.13]. (3) establishes the chain of similarities $X \times H \sim Y \times G \cong U \times G' \sim W \times A$, and that α is half of a similarity. The difficult part of the proof is to show that (2) implies (3). Assuming α^* is surjective, we see by Theorem 2.6 that p_F^* equals [1] on $H^1(X \times H)$. Since the homomorphism $f\colon Y \times H \to H$; $f(y, h) = h$, satisfies $[f] = p_F^*([f'])$, $f'(x, h) = h$, we have f similar to 1. Hence there is a

Borel subset Y_2 of Y_1 and a Borel map $\theta\colon Y_2 \to H$ such that $h = \theta(y)\theta(h^{-1}y)^{-1}$ for y and $h^{-1}y \in Y_2$. The measure class of $\theta_*(m'')$ on H is H invariant and hence by [6] is equivalent to Haar measure on H. Then $\theta^*\colon M(H) \to M(Y)$; $\theta^*(E) = \theta^{-1}(E)$, is an H equivariant σ-homomorphism. Then by [10, Theorem 3.6], there is an H invariant conull analytic subset Y'' of Y' (see 1.8) and an H equivariant Borel map $\theta\colon Y'' \to H$. We note that $W'' = \theta^{-1}(1)$ is a Borel transversal for the action of H on Y'', and the action of H is strictly free ([11] shows that this is equivalent to (1)). We define $P\colon Y'' \to W''$ by $P(y) = \theta(y)^{-1}y$. P is a Borel map and induces the measure $P_*m'' = u''$ on W''. Then the mapping $I\colon W'' \to W$; $I(w) = p(w)$ defines an A equivariant Boolean isomorphism $I^*\colon M(W) \to M(W'')$. Using Theorem 3.5 of [10] we can find conull subsets W_1 and W_2 of W'' and W respectively, which are standard as Borel spaces, such that $I(W_1) = W_2$ and $I\colon W_1 \to W_2$ is bijective, and so that $(W_2 \times A)_2$ satisfies Condition 1.1. Then $P^{-1}(W_1) = Y_3$ is an H invariant analytic conull subset of Y'' and the diagram

$$\begin{tikzcd} & Y_3 \arrow[dl, "P"'] \arrow[dr, "p"] & \\ W_1 \arrow[rr, "I"'] & & W_2 \end{tikzcd}$$

commutes.

We define the strict homomorphism $T\colon (W_2 \times A)_2 \to H$ by

$$T(p(y), b) = \theta(y)^{-1}\theta(b^{-1}y)$$

(independent of y for fixed $p(y)$). Then T defines a partial action of $A \times H$ on $U_2 = W_2 \times H$ as in Section 1, with the appropriate notation changes:

$$(b, h)(w, k) = (bw, hkT(w, b^{-1})).$$

Then Theorem 1.9 establishes that $(U_2 \times (A \times H))_2$ is similar to $(W_2 \times A)_2$ (via homomorphisms ϕ_0 and ψ_0 corresponding to the ϕ and ψ of Theorem 1.9). We also obtain a partial action of $H \times A$ on U_2, where (h, b) acts as (b, h). Then (3) is obtained as a result of the following lemma.

3.4 Lemma. *The map $p \times \theta\colon Y_3 \to U_2$*

(a) *is bijective,*
(b) *commutes with the partial actions of $H \times A$, and*
(c) *is a Borel and measure class isomorphism.*

Proof. (a) The map sending (w, h) to $hI^{-1}(w)$ is the inverse to $p \times \theta$.
(b) $p \times \theta((h, b)y) = (p(by), \theta(hby))$ and

$$(b, h)(p(y), \theta(y)) = (bp(y), h\theta(y)T(p(y), b^{-1})) = (p(by), \theta(hby))$$

(c) $p \times \theta$ and its inverse are Borel maps. We know that $p_*(m'')$ is in the measure class for W_2, $\theta_*(m'')$ is equivalent to Haar measure on H, and the measure class of $v = (p \times \theta)_*(m'')$ is H invariant. H acts on itself and on the second component of $W_2 \times H$, on the left, and the Borel map $p_2: W_2 \times H \to H$; $p_2(w, h) = h$, is an equivariant fibration in the terminology of [2]. Then, according to Proposition 2.6 of [2, p. 72], v decomposes into a measure $\bar{v} = \theta_*(m'')$ and a collection of measures $\{v_h\}h \in H$, where the v_h's are all equivalent measures on W_2. Then the v_h's are all equivalent to $p_*(m'')$, and hence v is equivalent to $p_*(m'') \times$ Haar measure on H.

Applying Lemma 3.4 we see that the map

$$\Phi: (Y_3 \times (H \times A))_3 \to (U_2 \times (A \times H))_2; \quad \Phi(y, (h, b)) = ((p(y), \theta(y)), (b, h))$$

is an isomorphism of ergodic groupoids.

Section 4

Mackey introduced, as an example in [8], the homomorphism defined by taking Radon-Nikodym derivatives of an ergodic group action, except for some a.e. difficulties. These a.e. problems are now easily settled, with the results of Ramsay on a.e. homomorphisms, in [10].

Consider a measure class preserving ergodic action of H on X, m and a σ-finite (not necessarily finite) measure $\bar{m}$, equivalent to m.

4.0 Theorem. *There is a unique (mod i.c.'s) homomorphism (called the derivative for $X \times H, \bar{m}$) $r: X \times H \to R^+$ such that*

$$\int f(hx)\, d\bar{m}(x) = \int f(x) r(x, h)\, d\bar{m}(x)$$

for every Borel function $f \geq 0$, $h \in H$.

Proof. As in example B, page 198 of [8], there is a Borel function r_0: $X \times H \to R^+$ such that for almost every $h \in H$, $r_0(x, h)$ can be used as the Radon-Nikodym derivative $(dh_*\bar{m}/d\bar{m})(x)$. Then for almost every $(x, h, h') \in X \times H \times H$ we have

$$r_0(x, h) r_0(h^{-1}x, h') = r_0(x, hh'),$$

so r_0 satisfies the conditions for Theorem 5.1 of [10], i.e., r_0 is an almost everywhere homomorphism. Hence we can change r_0 on a null subset of $X \times H$ to obtain r, so that r is a strict homomorphism on some i.c. of $X \times H$. Then

$$\left\{h \in H: \int f(hx)\, d\bar{m}(x) = \int f(x) r(x, h)\, d\bar{m}(x) \text{ for all Borel } f \geq 0\right\}$$

is H invariant (on the left) and conull in H, and hence equals H. If r' is a homomorphism with the same properties as r, then $r = r'$ a.e. on $X \times H$, so $r = r'$ on some i.c. of $X \times H$ by Lemma 5.2 of [10].

4.1 Theorem. *A homomorphism $\tilde{r}: X \times H \to R^+$ is similar to the derivative r for $X \times H, \bar{m}$ iff there is a (σ-finite) measure $\tilde{m}$ equivalent to $\bar{m}$ such that $\tilde{r}$ is the derivative for $X \times H, \tilde{m}$.*

Proof. ($\Rightarrow$) There is a conull Borel subset X_1 of X and a Borel map $t: X_1 \to R^+$ such that

$$r(x, h) = t(x)r(x, h)t(h^{-1}x)^{-1}.$$

Then let $\tilde{m}(E) = \int_E t(x)\, d\overline{m}(x)$ for a Borel set $E \subseteq X$ ($t = 0$ off X_1) and obtain $\tilde{r}$ as the derivative for $\tilde{m}$.

($\Leftarrow$) Let $t = d\tilde{m}/d\overline{m}$. Then the homomorphism r',

$$r'(x, h) = t(x)\tilde{r}(x, h)t(h^{-1}x)^{-1},$$

equals r a.e. on $X \times H$, so $r' = r$ on an i.c. of $X \times H$. Hence r is similar to $\tilde{r}$.

4.2 *Remarks.* (a) The similarity class of the derivative r for $X \times H$, $\overline{m}$, is uniquely determined by the measure class of m, $C[m]$.

(b) $[r] = [1]$ iff the action of H on X preserves a measure in $C[m]$.

Let $\Delta_H: H \to R^+$ be the modular function for H, so that

$$\int f(kh)\, dh = \int f(h)\Delta_H(k)\, dh \quad \text{and} \quad \int f(h^{-1})\, dh = \int f(h)\Delta_H(h)\, dh,$$

where dh represents right Haar measure on H.

4.3 DEFINITION. The *Radon-Nikodym* (R-N) *homomorphism* for $X \times H$, $\overline{m}$ is the homomorphism $\rho: X \times H \to R^+$; $\rho(x, h) = r(x, h)/\Delta_H(h)$, where r is the derivative for $X \times H$, $\overline{m}$.

To motivate the choice of ρ, consider the unimodular group $H \times R$, R the set of real numbers, $(h, s) \circ (k, t) = (hk, s + \Delta_H(h)t)$ with the action of $H \times R$ on $X \times R$ given by $(h, s)(x, t) = (hx, s + \Delta_H(h)t)$. Then the contraction of $(X \times R) \times (H \times R)$ to $X \times \{0\}$ is naturally isomorphic to $X \times H$, so $(X \times R) \times (H \times R)$ is similar to $X \times H$ by Theorem 6.17 of [10]. This similarity can be established by the homomorphisms $\phi_1(x, t, h, s) = (x, h)$ and $\psi_1(x, h) = (x, 0, h, 0)$. We calculate the derivative $r': (X \times R) \times (H \times R) \to R^+$ for $\overline{m} \times$ Lebesgue measure by

$$\begin{aligned}\int f((h, s)(x, t))\, d\overline{m}(x)\, dt &= \int f(hx, s + \Delta_H(h)t)\, d\overline{m}(x)\, dt \\ &= \int f(x, t)r(x, h)\Delta_H(h)^{-1}\, d\overline{m}(x)\, dt,\end{aligned}$$

so $r'(x, t, h, s) = \rho(x, h)$. Hence the diagram

$$\begin{tikzcd} (X \times R) \times (H \times R) \arrow[r, "r'"] \arrow[d, shift right, "\phi_1"'] & R^+ \arrow[d, equal] \\ X \times H \arrow[u, shift right, "\psi_1"'] \arrow[r, "\rho"] & R^+ \end{tikzcd}$$

commutes mod similarity.

Our main result is that the similarity class of ρ is a similarity invariant.

4.4 THEOREM. *If* $\alpha: X \times H \to W \times A$ *is half of a similarity and* ρ *(resp.* ρ_3*) is the R-N homomorphism for* $X \times H, m$ *(resp.* $W \times A, u$*) then* $\alpha^*[\rho_3] = [\rho]$.

Proof. We construct the action of $G = H \times A$ on Y as in Section 1, ($F(x, h)$ is the 2nd component of $\alpha(x, h)$), and compute the R-N homomorphism ρ_1 for $Y \times G, m \times m_A$ (m_A is right Haar measure on A), which is equivalent to the measure $m \times m'$ used in Section 1. Then

$$\begin{aligned}\int f(x, a) r_1(x, a, h, b)\, dm(x)\, dm_A(a) &= \int f((h, b)(x, a))\, dm(x)\, dm_A(a)\\ &= \int f(hx, baF(x, h^{-1}))\, dm(x)\, dm_A(a)\\ &= \int f(x, a) r(x, h) \Delta_A(b)\, dm(x)\, dm_A(a),\end{aligned}$$

so $\rho_1(x, a, h, b) = r(x, h)/\Delta_H(h) = \phi^* \circ \rho(x, a, h, b)$ on some i.c. of $Y \times G$. A similar argument for $W \times A$ yields $\rho_2 = \phi_0^* \circ \rho_3$ on some i.c. of $U \times (A \times H)$. Using Theorem 3.3 and Lemma 3.4 we can regard U_2 as a conull Borel subset of Y, via $(p \times \theta)^{-1}$, and the partial actions of $H \times A$ on Y_3 and U_2 as restrictions from the action of $H \times A$ on Y. Then the calculation of ρ_1 and ρ_2 is just the calculation of the R-N homomorphisms for two equivalent measures on Y, which yields similar homomorphisms. Then

$$\Phi^*[\rho_2] = [\rho_1] \quad \text{and} \quad \alpha^*[\rho_3] = \psi^* \circ \Phi^* \circ \phi_0^*[\rho_3] = \psi^*[\rho_1] = [\rho].$$

From here on we will assume that H and A are countable groups. Then any partial action of H, A or $G = H \times A$ is an action in the usual sense. Accordingly, we will ignore the complications due to taking inessential contractions and will regard sets as equal if they differ by a null set, functions as equal if they are equal a.e., and an ergodic groupoid "isomorphism" means "isomorphism (mod i.c.'s)."

Given the half of a similarity $\alpha: X \times H \to W \times A$, and notation as in 3.3 and 3.4, consider the (nonnull) subsets

$$X^1 = X \times \{1\} \quad \text{and} \quad W^1 = (p \times \theta)^{-1}(W \times \{1\})$$

of $Y = X \times A$. The map

$$X \to X^1; \quad x \to (x, 1)$$

identifies X with X^1 and the map

$$W^1 \to W; \quad (x, a) \to p(x, a)$$

identifies W^1 with W.

We use the definition of G-equivalence as in [3].

4.5 DEFINITION. The nonnull Borel subsets E and F of Y are G-equivalent iff there exist sequences of sets $\{E_i\}$ and $\{F_i\}$, and a sequence $\{g_i\}$ in G such that $E_i \cap E_j = F_i \cap F_j = \emptyset$ whenever $i \neq j$, $\bigcup E_i = E$, $\bigcup F_i = F$, and $g_i(E_i) = F_i$.

4.6 *Remark.* If E is G-equivalent to F then $\tilde{g}: E \to F$; $\tilde{g}(x) = g_i(x)$ iff $x \in E_i$, is a Borel and measure class isomorphism (mod 0).

4.7 THEOREM. *The half of a similarity $\alpha: X \times H \to W \times A$ is similar to an isomorphism iff X^1 and W^1 are G-equivalent.*

Proof. ($\Leftarrow$) From the definition of G-equivalence we have $X^1 = \bigcup X_i$, $W^1 = \bigcup W_i$, $g_i = (k_i, b_i)$, where $g_i(X_i) = W_i$. We define $\theta_1: X \to W \times A$ by $\theta_1(x) = (p(x, b_i), b_i)$ iff $(x, 1) \in X_i$. Then

$$\beta(x, h) = \theta_1(x)\alpha(x, h)\theta_1(h^{-1}x)^{-1}$$

defines an isomorphism $\beta: X \times H \to W \times A$. In fact, the diagram

$$\begin{array}{ccc} X^1 & \xrightarrow{\tilde{g}} & W^1 \\ \uparrow & & \downarrow \\ W & \xrightarrow[\beta^0]{} & W \end{array}$$

commutes, where β^0 is the restriction of β to units.

($\Rightarrow$) Suppose $\beta(x, h) = \theta_1(x)\alpha(x, h)\theta_1(h^{-1}x)^{-1}$ is an isomorphism, for some $\theta_1: X \to W \times A$. Since G is countable, we write $G = \{g_i\}$, $g_i = (k_i, b_i)$. Let

$$X_i = \{(x, 1): (k_i, b_i)(x, 1) \in W^1 \text{ and } p_2 \circ \theta_1(x) = b_i\}$$

(where $p_2(w, a) = a$) and $W_i = (k_i, b_i)X_i$. Then X^1 and W^1 are G-equivalent.

4.8 COROLLARY. *If neither of the actions of H on X and of A on W preserve a finite measure and $\alpha: X \times H \to W \times A$ is half of a similarity then α is similar to an isomorphism.*

Proof. Since the actions are of infinite type the corresponding sets X^1 and W^1 are G-equivalent by [3, Corollary 3, p. 416], and the result follows by Theorem 4.7.

REFERENCES

1. H. ANZAI, *Ergodic skew product transformations on the torus*, Osaka Math. J., vol. 3 (1951), pp. 83–89.
2. L. AUSLANDER AND C. C. MOORE, *Unitary representations of solvable Lie groups*, Mem. Amer. Math. Soc., No. 62, 1966.
3. N. DANG-NGOC, *On the classification of dynamical systems*, Ann. Inst. Henri Poincaré, vol. IX (1973), pp. 397–425.
4. P. FORREST, *On the virtual groups defined by ergodic actions of R^n and Z^n*, Advances in Math., vol. 14 (1974), pp. 271–308.

5. R. C. Lyndon, *The cohomology theory of group extensions*, Duke Math. J., vol. 15 (1948), pp. 271–292.
6. G. W. Mackey, *Induced representations of locally compact groups I*, Ann. of Math., vol. 55 (1952), pp. 101–139.
7. ———, *Point realizations of transformation groups*, Illinois J. Math., vol. 6 (1962), pp. 327–335.
8. ———, *Ergodic theory and virtual groups*, Math. Ann., vol. 166 (1966), pp. 187–207.
9. S. MacLane, *Homology*, Die Grundlehren der math. Wissenschaften, Band 114, Academic Press, New York; Springer-Verlag, Berlin, 1963.
10. A. Ramsay, *Virtual groups and group actions*, Advances in Math., vol. 6 (1971), pp. 253–322.
11. ———, *Subobjects of virtual groups*, preprint.
12. J. J. Westman, *Cohomology for ergodic groupoids*, Trans. Amer. Math. Soc., vol. 146 (1969), pp. 465–471.
13. ———, *Cohomology for the ergodic actions of countable groups*, Proc. Amer. Math. Soc., vol. 30 (1971), pp. 318–320.
14. ———, *Virtual group homomorphisms with dense range*, Illinois J. Math., vol. 20 (1976), pp. 41–47.

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