INCOMPRESSIBILITY AND FIBRATIONS

BY

JAY E. GOLDFEATHER

0. Introduction

Let $f\colon X \to Y$ and let $A \subseteq Y$ with $i\colon A \to Y$ the inclusion. The map f can be *compressed into* A if there is a map $\bar{f}\colon X \to A$ such that $i \circ \bar{f} \simeq f$. If Y is a CW complex then a map $f\colon X \to Y$ is *incompressible* if it does not compress into any smaller skeleton. In particular, if Y is infinite dimensional, f is incompressible if it does not compress into any finite skeleton.

If the induced homomorphism

$$f_j\colon H_j(X; G) \to H_j(Y; G) \quad (\text{or } f^j\colon H^j(Y; G) \to H^j(X; G))$$

is nonzero for some $j > N$, then f cannot compress into Y^N. However, f_j identically zero for all $j > N$ is not a sufficient condition for f compressing into Y^N.

For example, Weingram [5] has shown that every nontrivial map

$$f\colon \Omega S^{2n+1} \to K(Z_{p^r}, 2n)$$

is incompressible, yet f_j and f^j (any coefficients) are identically zero for all $j > 2npr$.

Suppose nm is even and let $\bar{m}$ be m if n is even and $m/2$ if n is odd. Let $f\colon \Omega S^{n+1} \to K(Z_{p^r}, nm)$ be a nontrivial map so that it represents a cohomology class $x \in H^{nm}(\Omega S^{n+1}; Z_{p^r})$ with $p^{r-j}x = 0$ for some j, $0 \le j < r$. Let $N_k\ (m, s, p)$ be the number of factors of p in $p^{sk}(m!)^k/(km)!$ and let

$$M = \{(m, s, p) \mid \limsup_k N_k(m, s, p) = +\infty\}.$$

The following theorem is proved.

THEOREM 2.2. *If* $(\bar{m}, r - j, p) \in M$, *then* $f\colon \Omega S^{n+1} \to K(Z_{p^r}, nm)$ *is incompressible.*

For example, nontrivial maps

$$f_k\colon \Omega S^{2n+1} \to K(Z_{p^r}, 2np^k) \quad \text{and} \quad g_k\colon \Omega S^{2n+2} \to K(Z_{p^r}, (4n + 2)p^k)$$

are incompressible for all $k = 0, 1, 2, \ldots$. Except for f_0, the incompressibility of these maps are not derivable by the methods of [5].

Sections 3 and 4 deal with applications of Theorem 2.2. In particular, the following are proved:

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COROLLARY 3.7. *Every nontrivial map*

$$f\colon \Omega SM(Z_{p^\infty}, 2n - 1) \to K(Z_{p^r}, 2n)$$

is incompressible.

COROLLARY 3.8. *Let G be finitely generated and abelian. Every nontrivial map $\Omega SM(Q/Z, 2n - 1) \to K(G, 2n)$ is incompressible.*

Let $(S^n)_m$ denote the mth reduced product of S^n and let $Im_k(Y, X; G) =$ and $\{x \in H_k(X; G) \mid x = f_*(y)$ for some $y \in H_k(Y; G)$ and for some $f\colon Y \to X\}$.

THEOREM 4.1. *Let X be a finite H-space and let*

$$y \in Im_{nm}((S^n)_m, X; Z)$$

where nm is even.

(a) *y cannot be of infinite order.*
(b) *Suppose $p_1^{\alpha_1} \cdots p_k^{\alpha_k}(y) = 0$ where p_i is prime. Then for each $i = 1, \ldots, k$ we have $\alpha_i < \bar{m}$ where $\bar{m}$ is m if n is even and $m/2$ if n is odd.*

THEOREM 4.3. *Let X be a $(2n - 2)$-connected finite dimensional H-space (not of finite type). Then $\Pi_{2n-1}(X)$ cannot contain Z_{p^∞} as a summand for any prime p.*

All spaces will be assumed to be homotopic to simply connected CW complexes and $H^*(X)$, $H_*(X)$ will be understood to have coefficient group the integers.

This paper contains results from part of the author's doctoral dissertation. I would like to thank Stephen Weingram for his guidance as my thesis advisor.

1. Fibrations and incompressibility

When studying the problem of whether a map $f\colon X \to B$ compresses into $A \subseteq B$, it suffices to assume that f is a fiber map. If not, there exists a fibration

$$F \longrightarrow X' \xrightarrow{f'} B$$

and a homotopy equivalence $v\colon X' \to X$ such that $f' = f \circ v$ and it follows that f' compresses into A if and only if f does. Let X_A denote $f^{-1}(A)$ and f_A denote f restricted to X_A.

DEFINITION 1.1. Let $A \subseteq X$ with $i\colon A \to X$ the inclusion. A map $r\colon X \to A$ is a coretraction if $i \circ r \simeq \mathrm{id}_X$.

PROPOSITION 1.2. *Let*

$$F \xrightarrow{i} X \xrightarrow{f} B$$

be a fibration and let A be a subspace of B. Let $j_A\colon X_A \to X$ be induced by the inclusion $j\colon A \to B$. Then f can be compressed into A if and only if j_A admits a coretraction $r_A\colon X \to X_A$.

Proof. Suppose $r_A\colon X \to X_A$ exists such that $j_A r_A \simeq \mathrm{id}_X$. Then

$$j \circ (f_A \circ r_A) \simeq f \circ j_A \circ r_A \simeq f$$

so that f compresses into A.

Suppose f can be compressed into A. Then there exists $\bar{f}\colon X \to A$ and a homotopy $h_t\colon X \to B$ such that $h_0 = f$ and $h_1 = i \circ \bar{f}$. By the covering homotopy property, there is a homotopy $\bar{h}_t\colon X \to X$ such that $\bar{h}_0 = \mathrm{id}_X$ and $f \circ h_i = i \circ \bar{f}$. Since Image $(i \circ \bar{f}) \subseteq A$ it follows that Image $(\bar{h}_1) \subseteq X_A$. Hence $j_A \circ \bar{h}_1 \simeq \bar{h}_0 = \mathrm{id}_X$. Let $r_A = \bar{h}_1$. ■

DEFINITION 1.3. Let G be an abelian group and p a prime. An element $g \in G$ has *p-depth* $\geq k$ if for some nonnegative integer s,

$$g = p^k g_1 + v \quad \text{where } p^s v = 0 \text{ and } p^{s+k} g_1 \neq 0.$$

If g has p-depth $\geq k$ but not p-depth $\geq k + 1$, then g has p-depth k and this will be denoted by $p[g] = k$.

LEMMA 1.4. *Let $h\colon G_1 \to G_2$ be a homomorphism of abelian groups and suppose $h(g) \in G_2$ has infinite order for some $g \in G_1$. Then $p[g] = k$ implies $p[h(g)] \geq k$.*

Proof. Let $g = p^k g_1 + v$ where $p^s v = 0$ and $p^{s+k} g_1 \neq 0$ for some integer s. Then $h(g) = p^k h(g_1) + h(v)$ so that $p^s h(v) = 0$ and since $h(g)$ has infinite order, $p^{s+k} h(g_1) \neq 0$. Hence $p[h(g)]$ is at least k. ■

THEOREM 1.5. *Let*

$$F \xrightarrow{\ i\ } X \xrightarrow{\ f\ } B$$

be a fibration and let $j_A\colon X_A \to X$ be the inclusion induced by the inclusion $j\colon A \to B$. Suppose there exists $x \in H^*(X)$ such that x has infinite order, $p[x] = k$ but $p[j_A^*(x)] > k$. Then f cannot be compressed into A.

Proof. In the light of Proposition 1.2, it suffices to show j_A does not admit a coretraction r_A.

Suppose such a coretraction existed. Then $j_A \circ r_A \simeq \mathrm{id}_X$ so that $\mathrm{id}_X^* = r_A^* j_A^*$. Then $x = \mathrm{id}_X^*(x) = r_A^*(j_A^*(x))$. By Lemma 1.4, since x has infinite order,

$$k = p[x] = p[r_A^* j_A^*(x)] \geq p[j_A^*(x)].$$

$p[j_A^*(x)] > k$ contradicts this, so no coretraction exists. ■

In order to make use of Theorem 1.5, it is necessary to know something about $H^*(X_A)$. Under certain conditions, this information can be obtained by examining $i^*\colon H^*(X) \to H^*(F)$, so we proceed in this direction.

DEFINITION 1.6. Let $h: G_1 \to G_2$ be a homomorphism of abelian groups and let $\{x_k\}$ be a sequence of distinct elements of G_1. h is said to p-twist $\{x_k\}$ if the following hold.

(a) For all k, $p[x_k] \leq M$ for some nonnegative integer M.

(b) $p[h(x_k)] = \sigma_k$ where $\lim_k \sup \sigma_k = +\infty$.

If x_k is of infinite order for all k, then h *freely p-twists* $\{x_k\}$ if in addition $h(x_k)$ is of infinite order for all k.

Before stating the main theorem of this section, note that if

$$F \xrightarrow{i} X \xrightarrow{f} B$$

is a fibration and $X[N] = f^{-1}(B^N)$ then

$$\begin{tikzcd} F \arrow[r, "\approx"] \arrow[d, "i_N"] & F \arrow[d, "i"] \\ X[N] \arrow[r, "j_N"] \arrow[d, "f_N"] & X \arrow[d, "f"] \\ B^N \arrow[r] & B \end{tikzcd}$$

is a map of fibrations.

THEOREM 1.7. *Let*

$$F \xrightarrow{i} X \xrightarrow{f} B$$

be a fibration with the following properties:

(a) *$H_*(B)$ is a finite p-group in each degree greater than* 0.

(b) *There exists a sequence $\{x_k\}$ such that $x_k \in H^{t_k}(X)$ and each x_k is of infinite order.*

(c) *There are an infinite number of integers N such that*

$$\text{Ker}\,(H^{t_k}(X[N]) \to H^{t_k}(F))$$

is a finite group for all k.

(d) *i^* freely p-twists $\{x_k\}$.*

Then f is not compressible into any finite skeleton.

Before proving this theorem, it is necessary to recall some facts about the Serre cohomology spectral sequence of a fibration $F \to E \to B$ [4].

(1) $H^n(E)$ is filtered by

$$H^n(E) = D^{0,n} \supseteq D^{1,n-1} \supseteq \cdots \supseteq D^{n,0} \supseteq D^{n+1,-1} = 0$$

where $D^{j,n-j} = \text{Ker}\,(H^n(E) \to H^n(E[j-1]))$.

(2) $E_2^{j,n-j} = H^j(B; H^{n-j}(F))$.
(3) $E_\infty^{j,n-j} = D^{j,n-j}/D^{j+1,n-j+1}$.
(4) $0 \to D^{1,n-1} \to H^n(X) \to E_\infty^{0,n} \to 0$ is exact.
(5) $E_\infty^{0,n} = \operatorname{Im}(H^n(E) \to H^n(F))$.

Let $w_k = j_N^*(x_k) \in H^{t_k}(X[N])$ so that $i_N^*(w_k) = p^{\sigma_k}y_k + v_k$ where y_k is of infinite order and for each k, there exists s_k such that $p^{s_k}v_k = 0$. (This just specifies $i^* = i_N^* j_N^*$ as a free p-twisting.)

LEMMA 1.8. *Under the conditions of Theorem* 1.7, *there exists an integer* $v(N)$ *and for each* k, *an element* $z_k \in H^{t_k}(X[N])$ *such that* $i_N^*(z_k) = p^{v(N)}y_k$ *for all* k.

Proof. Let $\{E_r, d_r\}$ be the spectral sequence of $F \to X \to B$ and $\{'E_r, 'd_r\}$ the spectral sequence of $F \to X[N] \to B^N$. Then the map $j\colon B^N \to B$ induces a map $E_r \to {}'E_r$ which commutes with the differential.

Since $H^*(B)$ is a finite p group in each degree, for each k there exists an integer $\lambda(k)$ such that $p^{\lambda(k)}E_2^{k,0} = 0$. Then $p^{\lambda(k)}E_j^{k,t} = 0$ for all $j \geq 2$ and $t \geq 0$ where

$$\bar{\lambda}(k) = \max(\lambda(k), \lambda(k+1)).$$

(This is just a consequence of the Universal Coefficient Theorem.) Consider the differential $'d_2\colon {}'E_2^{0,t_k} \to {}'E_2^{2,t_k-1}$. By naturality, $p^{\bar{\lambda}(2)}\,'d_2(y_k) = 0$ for all k. Hence $p^{\lambda(2)}y_k$ is a 2-cycle for all k. Similarly $p^{\bar{\lambda}(2)+\bar{\lambda}(3)}y_k$ is a 3-cycle under $'d_3$ for all k and in general if $v(N) = \bar{\lambda}(2) + \cdots + \bar{\lambda}(N)$, $p^{v(N)}y_k$ is a permanent cycle in $\{'E_r, 'd_r\}$ ($'dj = 0$ for $j > N$). Since $p^{v(N)}y_k \in E_\infty^{0,t_k}$ for all k and

$$E_\infty^{0,t_k} = \operatorname{Im}(i_N^*\colon H^{t_k}(X[N]) \to H^{t_k}(F))$$

there exists $z_k \in H^{t_k}(X[N])$ such that $i_N^*(z_k) = p^{v(N)}y_k$. ■

Proof of Theorem 1.7. It suffices to show that f does not compress into B^N for any N satisfying (c).

Let M be such that $p[x_k] \leq M$ for all k. By Theorem 1.5, it suffices to show that for every N satisfying (c), there exists k such that $p[j_N^*(x_k)] \geq M + 1$.

Let $i^*(x_k) = p^{\sigma_k}y_k + v_k$ and $w_k = j_N^*(x_k)$ as in Lemma 1.8. Since $i_Z^*(z_k) = p^{v(N)}y_k$,

$$i_N^*(w_k - p^{\sigma_k - v(N)}z_k) = v_k.$$

But

$$0 \longrightarrow D^{1,t_k-1} \xrightarrow{\alpha} H^{t_k}(X[N]) \xrightarrow{i_N^*} E_\infty^{0,t_k} \longrightarrow 0$$

is exact, so that

$$p^{s_k}(w_k - p^{\sigma_k - v(N)}z_k) = \alpha(\bar{v}_k) \quad \text{for some } \bar{v}_k \in D^{1,t_k-1}.$$

Condition (c) says that D^{1,t_k-1} is a finite p group for all k so that $p^{j_k}\bar{v}_k = 0$ for some $j_k \geq 0$. Hence $p^{j_k+s_k}(w_k - p^{\sigma_k - v(N)}z_k) = 0$ which implies that

$$w_k = p^{\sigma_k - v(N)}z_k + u_k = 0 \quad \text{where } p^{j_k+s_k}u_k = 0.$$

Since $\limsup \sigma_k = +\infty$ and $v(N)$ is fixed, we may choose k so that $\sigma_k - v(N) \geq M + 1$. Then $p[j_N^*(x_k)] = p[w_k] \geq M + 1$. ∎

2. Incompressibility conditions for maps $\Omega S^{n+1} \to K(Z_{p^r}, nm)$

In [5], Weingram proved that any nontrivial map

$$f\colon \Omega S^{2n+1} \to K(Z_{p^r}, 2n)$$

is incompressible. The proof utilizes the fact that any such map is homotopic to an H map and so induces a homomorphism of rings in homology. In this section a more general theorem is proved using cohomological techniques so that incompressibility conditions can be established for maps

$$f\colon \Omega S^{n+1} \to K(Z_{p^r}, nm)$$

which are not in general homotopic to H maps. In view of the following proposition, attention will be focused on the situation when nm is even.

PROPOSITION 2.1. *If m is odd and p is an odd prime, any map*

$$f\colon \Omega S^{n+1} \to K(Z_{p^r}, nm)$$

compresses into the $(nm + 1)$*-skeleton.*

Proof. If n is even, all maps $\Omega S^{n+1} \to K(Z_{p^r}, nm)$ are trivial, so assume n is odd. For odd primes, $S^{nm}_{(p)}$ (S^{nm} localized at p) is an H-space of dimension $nm + 1$. The natural map $Z \to Z_{(p)}$ induces an epimorphism

$$\operatorname{Hom}(Z_{(p)}, Z_{p^r}) \twoheadrightarrow \operatorname{Hom}(Z, Z_{p^r})$$

and hence an epimorphism $H^{nm}(S^{nm}_{(p)}; Z_{p^r}) \to H^{nm}(S^{nm}; Z_{p^r})$. Let $(S^n)_m$ denote the James mth reduced product space of S^n and let $g\colon (S^n)_m \to S^{nm}$ be the map pinching the $(nm - 1)$-skeleton to a point. Then g induces an epimorphism

$$H^{nm}(S^{nm}; Z_{p^r}) \to H^{nm}((S^n)_m; Z_{p^r}).$$

Since $S^{nm}_{(p)}$ is an H-space, and the attaching maps for constructing ΩS^{n+1} from $(S^n)_m$ are higher order Whitehead products, any map $h\colon (S^n)_m \to S^{nm}_{(p)}$ extends to

$$\bar{h}\colon \Omega S^{n+1} \to S^{nm}_{(p)}.$$

Hence $\bar{h}^*\colon H^{nm}(S^{nm}_{(p)}; Z_{p^r}) \twoheadrightarrow H^{nm}(\Omega S^{n+1}; Z_{p^r})$ is an epimorphism so that any map $f\colon \Omega S^{n+1} \to K(Z_{p^r}, nm)$ factors through $S^{nm}_{(p)}$ which is $(nm + 1)$-dimensional. By the cellular approximation theorem, f compresses into the $(nm + 1)$-skeleton. ∎

Assuming, then, that nm is even, the procedure will be to show when the conditions of Theorem 1.7 are satisfied for $X = \Omega S^{n+1}$ and $B = K(Z_{p^r}, nm)$. We begin by recalling some facts about $H^*(\Omega S^{n+1})$ and $H_*(K(Z_{p^r}, nm))$.

(1) If n is even, $H^*(\Omega S^{n+1})$ is a divided power ring with generators $x^{(k)}$ in dimension nk satisfying the relation

$$x^{(k)}x^{(s)} = \binom{k}{s} x^{(k+s)}.$$

(2) If n is odd, $H^*(\Omega S^{n+1})$ contains a divided power ring with generators $x^{(k)}$ in dimension $2nk$ satisfying the relation in (1).

(3) In either (1) or (2), $x^{(k)}$ is not divisible by p so that for all k, $p[x^{(k)}] = 0$.

(4) If μ: $H^*(\Omega S^{n+1}) \to H^*(\Omega S^{n+1}; Z_{p^r})$ is the coefficient reduction map, then $H^*(\Omega S^{n+1}; Z_{p^r})$ is (contains) a divided power ring with generators $x_r^{(k)} = \mu(x^{(k)})$ for n even (n odd).

(5) $H_*(K(Z_{p^r}, nm))$ is a finite p group in each degree.

The key to applying Theorem 1.7 is to verify that i^* is a free p-twisting of $\{x_k\}$. This would be straightforward if for example $i^*(x_1) = py_1$ and $x_k = x_1^k$. Then $i^*(x_k) = p^k y_1^k$ so that $\sigma_k \geq k$ and hence is unbounded. The problem arises when $x_1^k = a_k x_k$. Then if $i^*(x_1) = py_1$, $i^*(x_k) = p^{k-p[a_k]}y_k + v_k$ where $a_k v_k = 0$. It is possible that lim sup $(k - p[a_k]) \neq \infty$.

Let $S_p(m)$ denote the number of factors of a prime p in $m!$ Note that $S_p(m) = \sum_{i=1}^{p} [m/p^i]$ where [ ] denotes the greatest integer.

Let $N_k(m, s, p) = sk + kS_p(m) - S_p(km)$ so that $N_k(m, s, p)$ is the number of factors of p in $p^{ks}(m!)^k/(km)!$. Let

$$\mathscr{M} = \{(m, s, p) \mid \limsup_k N_k(m, s, p) = +\infty\}.$$

Weingram [5] showed that $(1, s, p) \in \mathscr{M}$ for all s and primes p. The proof that there are other triples (m, s, p) in $\mathscr{M}$ is number theoretic and is contained in the appendix.

Let mn be even and let f: $\Omega S^{n+1} \to K(Z_{p^r}, mn)$ be a nontrivial map. If ι is the fundamental class in $H^{nm}(K(Z_{p^r}, nm); Z_{p^r})$, and $\bar{m}$ is m if n is even and $m/2$ if n is odd, then $f^*(\iota) = up^j x_r^{(m)}$ where u is a unit in Z_{p^r} and $0 \leq j < r$.

THEOREM 2.2. *If* $(\bar{m}, r - j, p) \in \mathscr{M}$, *then* f: $\Omega S^{n+1} \to K(Z_{p^r}, nm)$ *is incompressible.*

The proof will follow two lemmas. Unless it is necessary to specify more precisely, Ω will denote ΩS^{n+1} and K will denote $K(Z_{p^r}, nm)$.

LEMMA 2.3. *Let G be an extension of Z by* Z_{p^r}. *That is,*

$$0 \longrightarrow Z \xrightarrow{\alpha} G \xrightarrow{\beta} Z_{p^r} \longrightarrow 0$$

is exact. Then:

(a) $G \cong Z \oplus G'$ *where* $p^k G' = 0$ *for some* $k \leq r$.

(b) *If G is a nontrivial extension (i.e.,* $G \not\cong Z \oplus Z_{p^r}$*) then* $\alpha(1) = p^s y + v$ *where* $0 < s \leq r$ *and* $p^{r-s}v = 0$.

Proof. (a) is obvious. Let y be a free generator in G and let $\alpha(1) = my + v$. $\beta(p^r y) = 0$ implies there is a k such that $\alpha(k) = p^r y$. Hence $p^r y = kmy + kv$ so that $km = p^r$. This implies $m = p^s$ and $k = p^{r-s}$ where $0 < s \leq r$. ($s = 0$ implies the sequence splits so that G would be a trivial extension.) Also $p^{r-s}v = 0$ since β is a monomorphism on the torsion subgroup of G. ■

LEMMA 2.4. *A nontrivial map $f\colon \Omega \to K$ satisfies conditions* (a), (b), *and* (c) *of Theorem* 1.7.

Proof. (a) is just statement (5) at the beginning of this section. For the sequence in (b) take the generators $x^{(k)}$.

Note that t_k is even. Condition (c) is equivalent to saying there are infinitely many N such that D^{1,t_k-1} in the filtration of $H^{t_k}(\Omega[N])$ is a finite p group for all k. But D^{1,t_k-1} is a finite p group if E_∞^{j,t_k-j} is a finite p group for $j = 1, \ldots, N$. But E_∞^{j,t_k-1} is a finite p group if E_2^{j,t_k-1} is a finite p group for $j = 1, \ldots, N$. Since $E_2^{j,t_k-j} = H^j(K^N; H^{t_k-j}(F))$ where F is the fiber of the map $f\colon \Omega \to K$, and $H^j(K^N)$ is a finite p group for $j = 1, \ldots, N-1$ by (a) and the Universal Coefficient Theorem, it is only necessary to check if $H^N(K^N, H^{t_k-N}(F))$ is a finite p group. But (a) implies $H^*(F; Q) \cong H^*(\Omega; Q)$ so $H^*(F)$ is a finite p group in all degrees which are not multiples of n. There are infinitely many N such that $t_k - N$ is not a multiple of n for all k and (c) follows. ■

Proof of Theorem 2.2. In view of Lemma 2.4, it suffices to prove that if $i\colon F \to \Omega$ is the inclusion of the fiber then i^* is a free p-twisting of $x^{(k)}$.

The fibration

$$F \xrightarrow{i} \Omega \xrightarrow{f} K$$

induces a fibration

$$\Omega K \xrightarrow{j} F \xrightarrow{i} \Omega.$$

Since Ω is $(n-1)$-connected and ΩK is $(nm-2)$-connected, by Serres exact sequence

$$0 = H^{nm-1}(\Omega; Z_{p^r}) \xrightarrow{i^*} H^{nm-1}(F; Z_{p^r}) \xrightarrow{j^*} H^{nm-1}(\Omega K; Z_{p^r}) \xrightarrow{\tau} H^{nm}(\Omega; Z_{p^r})$$

is exact. But $\tau(\iota) = up^j x_r^{(\bar{m})}$ so that

$$H^{nm-1}(F; Z_{p^r}) = \ker \tau = Z_{p^j}$$

where Z_{p^0} is understood to mean the zero group.

Similarly the sequence

$$0 = H^{nm-1}(\Omega K) \longrightarrow H^{nm}(\Omega) \xrightarrow{i^*} H^{nm}(F) \longrightarrow H^{nm}(\Omega K) \longrightarrow H^{nm+1}(\Omega) = 0$$

is exact, and reduces to $0 \longrightarrow Z \xrightarrow{i^*} H^{nm}(F) \longrightarrow Z_{p^r} \longrightarrow 0$.

By the Universal Coefficient Theorem,

$$Z_{p^j} = H^{nm-1}(F; Z_{p^r}) = H^{nm-1}(F) \otimes Z_{p^r} \oplus \operatorname{Tor}(H^{nm}(F); Z_{p^r})$$

so that the torsion subgroup of $H^{nm}(F)$ has order p^j, $0 \leq j < r$. By Lemma 2.3, $i^*(x^{(\bar{m})}) = p^{r-j}y_{\bar{m}} + v_{\bar{m}}$ where $p^j v_{\bar{m}} = 0$ and $y_{\bar{m}}$ is of infinite order. But

$$(x^{(\bar{m})})^k = \frac{(\bar{m}k)!}{(\bar{m}!)^k} x^{(\bar{m}k)}$$

so that

$$i^*(x^{(\bar{m}k)}) = \frac{p^{k(r-j)}(\bar{m}!)^k}{(\bar{m}k)!} y_{\bar{m}}^k + T_{\bar{m}k} \quad \text{where } p^j(\bar{m}k)!\, T_{\bar{m}k} = 0.$$

Let λ_k be the number of factors of p in

$$\frac{p^{k(r-j)}(\bar{m}!)^k}{(\bar{m}k)!} y_{\bar{m}}^k.$$

Then $\lambda_k \geq N_k(\bar{m}, r - j, p)$ which by hypothesis has lim sup $= \infty$. Hence i^* is a free p-twisting. ∎

3. Moore spaces

Let $M(Z_{p^j}, 2n - 1)$ be a Moore space. In this section it is shown that under certain conditions, given N there is an integer j such that no nontrivial map $\Omega SM(Z_{p^{j+k}}, 2n - 1) \to K(Z_{p^r}, 2n)$ compresses into the N skeleton for all $k \geq 0$. Although it is not proved that such a map cannot compress into a higher dimensional skeleton, this result does imply that every nontrivial map

$$\Omega SM(Z_{p^\infty}, 2n - 1) \to K(Z_{p^r}, 2n)$$

is incompressible. We begin by establishing some conditions to detect whether the composition of incompressible maps is incompressible.

Recall the proof of Theorem 1.7.

(a) $i^*(x_k) = p^{\sigma_k}y_k + v_k$ where $p^{s_k}v_u = 0$ for some integer $s_k \geq 0$.

(b) $j_N^*(x_k) = p^{\sigma_k - v(N)}z_k + u_k$ where $p^{s_k + j_k}u_k = 0$ for some $j_k \leq m_k$ with $p^{m_k}D^{1, t_k - 1} = 0$.

THEOREM 3.1. *Let*

$$F \xrightarrow{i} X \xrightarrow{f} B$$

be a fibration satisfying the conditions of Theorem 1.7 *and let* $r(N)$ *be the smallest integer such that* $p^{r(N)}H^i(B) = 0$ *for* $0 < i \leq N$. *Let* $g\colon Y \to X$. *If for all* k, $p^{s_k + Nr(N)}g^*(x_k) \neq 0$ *and* $p[g^*(x_k)] \leq K$ *for some fixed integer* K, *then* $f \circ g$ *does not compress into* B^N.

Proof. In the filtration of $H^*(X[N])$, $D^{1, t_k - 1}$ is obtained by finding N extensions by groups G_i with $p^{r(N)}G_i = 0$. Hence $p^{Nr(N)}D^{1, t_k - 1} = 0$ so that

$j_k \leq Nr(N)$ for all k. Now

$$\begin{array}{ccc} Y[N] & \xrightarrow{g_N} & X[N] \\ \downarrow{\scriptstyle j_N} & & \downarrow{\scriptstyle j_N} \\ Y & \xrightarrow{g} & X \end{array}$$

is a commutative diagram. Suppose $f \circ g$ compresses into B^N. Then there exists $h: Y \to Y[N]$ such that $j_N h \simeq \mathrm{id}_Y$. Hence

$$g^*(x_k) = h^* j_N^*(g^*(x_k)) = h^* g_N^*(j_N^*(x_k)) = p^{\sigma_k - v(N)} h^* g_N^*(z_k) + h^* g_N^*(u_k).$$

Let $\lambda_k = s_k + Nr(N)$ and $\tau_k = \lambda_k + \sigma_k - v(N)$. Then

$$p^{\lambda_k} h^* g_N^*(u_k) = h^* g_N^*(p^{\lambda_k} u_k) = 0$$

so that $p^{\tau_k} h^* g_N^*(z_k) = p^{\lambda_k} g^*(x_k) \neq 0$. Choose k so large that $\sigma_k - v(N) \geq K + 1$. Then $p[g^*(x_k)] > K$.

This contradicts Theorem 1.5 so that $f \circ g$ cannot compress into B^N. ■

Let

$$g_j: \Omega SM(Z_{p^j}, 2n - 1) \to \Omega S^{2n+1}$$

be induced by $\bar{g}_j: SM(Z_{p^j}, 2n - 1) \to S^{2n+1}$ representing the generator in $\Pi_{2n}(Z_{p^j}; S^{2n+1}) = Z_{p^j}$. Let $w_{k,j} = g_j^*(x^{(k)})$. By comparing the spectral sequences of the path fibrations over S^{2n+1} and $SM(Z_{p^j}, 2n - 1)$ it is easily seen that $w_{k,j}$ generates a copy of Z_{p^j} in $H^{2nk}(\Omega SM(Z_{p^j}, 2n - 1); Z)$.

LEMMA 3.2. *For every map $\bar{f}_j: \Omega SM(Z_{p^j}, 2n - 1) \to K(Z_{p^r}, 2n)$ there exists a map $f: \Omega S^{2n+1} \to K(Z_{p^r}, 2n)$ such that $f \circ g_j \simeq \bar{f}_j$.*

Proof. $\bar{f}^*(\iota) = up^s w_{1,j}$ where u is a unit in Z_{p^j} and $0 \leq s \leq j$. Let $f: \Omega S^{2n+1} \to K(Z_{p^r}, 2n)$ represent

$$up^s x_r^{(1)} \in H^{2n}(\Omega S^{2n+1}; Z_{p^r}). \quad ■$$

Since the map f in Lemma 3.2 satisfies the conditions of Theorem 1.7, we would like to apply Theorem 3.1 to the map g_j. However, without further information on s_k, we cannot determine if $p^{s_k + Nr(N)} w_{k,j} \neq 0$ for any j. The following lemmas show that there is a subsequence $\{k_s\}$ such that $s_{k_s} \leq j$ for all k_s.

LEMMA 3.3. *Let p be prime.*

(a) $$\binom{kp^s}{p^s} \not\equiv 0 \bmod p \text{ for } 0 < k \leq p - 1.$$

(b) $$\binom{p^{s+1}}{p^s} = pu \quad \text{where } (u, p) = 1.$$

(c) $$\binom{t_k}{t_{k-1}} \not\equiv 0 \bmod p \quad \text{if } t_k = p^k + \cdots + p + 1.$$

Proof. An easy exercise in binomial coefficients. ∎

LEMMA 3.4. *Let $h\colon R_1 \to R_2$ be a ring homomorphism. Suppose the torsion parts of R_1 and R_2 considered as groups under $+$ are p groups and*

$$h(x_1) = p^{\alpha_1}y_1 + u_1, \qquad h(x_2) = p^{\alpha_2}y_2 + u_2$$

where for some integers s_1 and s_2, $p^{s_1}u_1 = 0$, $p^{s_2}u_2 = 0$, $(s_2 + \alpha_1) > s_1$, $(s_1 + \alpha_2) > s_2$.

(a) *If $x_1x_2 = mx_3$ where $(m, p) = 1$, then*
$h(x_3) = p^{\alpha_1+\alpha_2}y_3 + u_3$ *where* $p^{s_3}u_3 = 0$, $s_3 = \min(s_1, s_2)$.

(b) *If $x_1x_2 = pmx_3$ where $(m, p) = 1$, then*
$h(x_3) = p^{\alpha_1+\alpha_2-1}y_3 + u_3$ *where* $p^{s_3}u_3 = 0$, $s_3 = 1 + \min(s_1, s_2)$.

Proof. Only (a) is proved as the proof of (b) is similar.

$$\begin{aligned} mh(x_3) &= h(x_1)h(x_2) \\ &= (p^{\alpha_1}y_1 + u_1)(p^{\alpha_2}y_2 + u_2) \\ &= p^{\alpha_1+\alpha_2}y_1y_2 + u_2p^{\alpha_1}y_1 + u_1p^{\alpha_2}y_2 + u_1u_2. \end{aligned}$$

Let $my_3 = y_1y_2$ and $mu_3 = u_2p^{\alpha_1}y_1 + u_1p^{\alpha_2}y_2 + u_1u_2$. If $s_3 = \min(s_1, s_2)$, then $p^{s_3}mu_3 = 0$. Since $(m, p) = 1$, it follows that $p^{s_3}u_3 = 0$. ∎

LEMMA 3.5. *Let*

$$F \xrightarrow{i} \Omega S^{2n+1} \xrightarrow{f} K(Z_{p^r}, 2n)$$

be a fibration where $f^*(\imath) = up^sx_r^{(1)}$ with u a unit in Z_{p^r} and $0 \leq s < r$.

(a) $i^*(x^{(p^t)}) = p^{\alpha_t}y_t + v_t$ *with* $p^{s+t}v_t = 0$.

(b) $i^*(x^{(k_t)}) = p^{\tau_t}\bar{y}_t + u_t$ *where* $k_t = p^t + \cdots + p + 1$, $p^su_t = 0$ *and* $\lim_t \tau_t = +\infty$.

Proof. (a) From the proof of Theorem 2.2 we have $i^*(x^{(1)}) = p^{r-s}y_0 + v_0$ where $p^sv_0 = 0$. Hence by repeated applications of 3.3 and 3.4,

$$i^*(x^{(p)}) = p^{p(r-s)-1}y_1 + v_1 \quad \text{where } p^{s+1}v_1 = 0.$$

Suppose inductively that $i^*(x^{(p^{t-1})}) = p^{\alpha_t-1}y_{t-1} + v_{t-1}$ where

$$\alpha_{t-1} = p^{t-1}(r - s) - (p^{t-1} - 1)/p - 1, \quad \text{and} \quad p^{s+t-1}v_{t-1} = 0.$$

By repeated applications of 3.3 and 3.4, $i^*(x^{(p^t)}) = p^{\alpha_t}y_t + v_t$ where

$$\alpha_t = p\alpha_{t-1} - 1 = p^t(r - s) - (p^t - 1)/(p - 1) \quad \text{and} \quad p^{s+t}v_t = 0.$$

(b) $x^{(k_0)} = x^{(1)}$ so that $i^*(x^{(k_0)}) = p^{\alpha_0}y_0 + u_0$ where $p^su_0 = 0$. Suppose inductively that $i^*(x^{(k_{t-1})}) = p^{\tau_{t-1}}\bar{y}_{t-1} + u_{t-1}$ where $\tau_{t-1} = \alpha_{t-1} + \tau_{t-2}$, $\tau_0 = \alpha_0$, and $p^su_{t-1} = 0$. By 3.3, since

$$x^{(k_{t-1})} \cdot x^{(p_t)} = \binom{k_t}{k_{t-1}} x^{(k_t)},$$

we have $i^*(x^{(k_t)}) = p^{\alpha_t + \tau_{t-1}}\bar{y}_t + u_t$ where $p^s u_t = 0$ provided $\tau_{t-1} > t$. But $\tau_t = \sum_{i=0}^{t} \alpha_i$ so this is easily verified as well as the fact that $\lim_t \tau_t = +\infty$. ∎

THEOREM 3.6. *Let $f_j\colon \Omega SM(Z_{p^j}, 2n-1) \to K(Z_{p^r}, 2n)$ be a nontrivial map so that $f_j^*(\iota) = up^s w_{1,j}$ where u is a unit and $s < \min(j, r)$. Let $r(N)$ be the smallest integer such that $p^{r(N)}H^i(K(Z_{p^r}, 2n)) = 0$ for $0 < i \leq N$. Then for all $j \geq Nr(N) + s + 1$, f_j is not compressible into the N skeleton.*

Proof. By Lemma 3.5(b), $s_{k_t} + Nr(N) = s + Nr(N)$. But if $j \geq Nr(N) + s + 1$. $p^j w_{1,j} \neq 0$ and so Theorem 3.1 applies. ∎

COROLLARY 3.7. *Any nontrivial map $f\colon \Omega SM(Z_{p^\infty}, 2n-1) \to K(Z_{p^r}, 2n)$ is incompressible.*

Proof. Such a nontrivial map implies that there exists an integer k such that if $h_j\colon \Omega SM(Z_{p^j}, 2n-1) \to \Omega SM(Z_{p^\infty}, 2n-1)$ is the natural inclusion, then fh_j is nontrivial for all $j \geq k$. f compressible contradicts Theorem 3.6. ∎

COROLLARY 3.8. *Let G be finitely generated and abelian. Every nontrivial map $\Omega SM(Q/Z, 2n-1) \to K(G, 2n)$ is incompressible.*

Proof. Since $Q/Z \cong \oplus_{p \in P} Z_{p^\infty}$, such a nontrivial map implies the existence of a nontrivial map $\Omega SM(Z_{p^\infty}, 2n-1) \to K(Z_{p^r}, 2n)$ for some prime p. ∎

4. Incompressibility and H-spaces

In this section some of the previous results are applied to deduce certain properties of H-spaces. We begin by stating a well-known result of James.

THEOREM (James [2]). *X is a retract of ΩSX if and only if X is an H-space.*

Let $\mathrm{Im}_k(Y; X; G) = \{x \in H_k(X; G) \mid x = f_*(y)$ for some $y \in H_k(Y; G)$ and some $f\colon Y \to X\}$. Note that $\mathrm{Im}_k(S^k, X; Z)$ is the image of the Hurewicz map in dimension k.

THEOREM 4.1. *Let X be a finite H-space and let $y \in \mathrm{Im}_{nm}((S^n)_m, X; Z)$ where nm is even.*

(a) *y cannot be of infinite order.*
(b) *Suppose $p_1^{\alpha_1} \cdots p_k^{\alpha_k} y = 0$ where p_i is prime. Then for each $i = 1, \ldots, k$ we have $\alpha_i < \bar{m}$ where $\bar{m}$ is m if n is even and is $m/2$ if n is odd.*

Proof. (a) Suppose y is of infinite order, where $y = f^*(x)$ for some $f\colon (S^n)_m \to X$. Then, for all primes p, there is an integer j such that the coefficient reduction map $\mu\colon H_{nm}(X; Z) \to H_{nm}(X; Z_{p^j})$ is such that $\mu(y) \neq 0$. Hence

$$f^*\colon H^{nm}(X; Z_{p^j}) \to H^{nm}((S^n)_m; Z_{p^j})$$

is nontrivial. Choose p so that $m \leq p - 1$. Then the composition

$$(S^n)_m \xrightarrow{f} X \longrightarrow K(Z_{p^j}, nm)$$

is nontrivial. Since X is an H-space and the obstruction to extending f to $\bar{f}$: $\Omega S^{n+1} \to X$ are higher order spherical Whitehead products, the composition

$$\Omega S^{n+1} \xrightarrow{\bar{f}} X \longrightarrow K(Z_{p^j}, nm)$$

is nontrivial. By A.3 in the appendix, $(m, 1, p) \in \mathscr{M}$ and hence this composition is incompressible by Theorem 2.2. This contradicts the finite dimensionality of X.

(b) Suppose $p_1^{\alpha_1} \cdots p_k^{\alpha_k} y = 0$ and $\alpha_i \geq \bar{m}$ for some i. Then there is an integer j and a map g: $X \to K(Z_{p_i^j}, nm)$ such that $(g \circ f)^*(\iota) = p^{j-\alpha_i} x_j^{(m)}$. But by A.3, $(m, \alpha_i, p) \in \mathscr{M}$ so that $g \circ f$ is incompressible. Again this contradicts the finite dimensionality of X. ∎

Remark. If $m = 1$, the above theorem says that the image of the Hurewicz map in even dimensions is zero for finite simply connected H-spaces. (See Browder [1] and Weingram [5].)

THEOREM 4.2. *Let X be a $(2n - 2)$-connected finite H-space of dimension N. Let $r(N)$ be the smallest integer such that $p^{r(N)} H^i(K(Z_p, 2n)) = 0$, $0 < i \leq N$. Let $j \geq Nr(N) + 1$. Then $\Pi_{2n-1}(X)$ has no p torsion of order greater than p^{j-1}.*

Proof. Suppose $x \in \Pi_{2n-1}(X)$ is such that $p^m x = 0$, $m \geq j$, but $p^{m-1} x \neq 0$. Let f: $S^{2n-1} \to X$ represent x. Then f lifts to $\bar{f}$: $M(Z_{p^m}, 2n - 1) \to X$ and since $\Pi_{2n-1}(X) \cong H_{2n-1}(X)$ via the Hurewicz map, the induced map

$$f^*: H^{2n}(X; Z_p) \to H^{2n}(M(Z_{p^m}, 2n - 1); Z_p)$$

is nonzero. Then

$$M(Z_{p^m}, 2n - 1) \xrightarrow{\bar{f}} X \longrightarrow K(Z_p, 2n)$$

is nontrivial and since X is an H-space, $\bar{f}$ extends to

$$\Omega SM(Z_{p^m}, 2n - 1) \to X.$$

Hence $\Omega SM(Z_{p^m}, 2n - 1) \to X \to K(Z_p, 2n)$ is nontrivial and by Theorem 3.6 does not compress into the N skeleton. This contradicts that X is N dimensional. ∎

Remark. If it could be proved that $\Omega SM(Z_{p^j}, 2n - 1) \to K(Z_{p^r}, 2n)$ is incompressible for all j, a proof similar to the above would imply that $\Pi_{2n-1}(X)$ has no p torsion if X is a $(2n - 2)$-connected finite H-space. For $n = 2$, and except for elements of order two this has been proved by Lin [3] by entirely different methods.

THEOREM 4.3. *Let X be a $(2n - 2)$-connected finite dimensional H-space (not of finite type). Then $\Pi_{2n-1}(X)$ cannot contain Z_{p^∞} as a summand for any prime p.*

Proof. Suppose $\Pi_{2n-1}(X) = Z_{p^\infty} \oplus G$. Then there is a map

$$f\colon M(Z_{p^\infty}, 2n - 1) \to X$$

such that $f^*\colon H^{2n}(X; Z_p) \to H^{2n}(N(Z_{p^\infty}, 2n - 1); Z_p)$ is nonzero. This implies there is a nontrivial map

$$\Omega SM(Z_{p^\infty}, 2n - 1) \to X \to K(Z_p, 2n).$$

By 3.7 this map is incompressible, contradicting the finite dimensionality of X. ∎

Appendix

Let $N_k(m, s, p) = sk + kSp(m) - S_p(km)$ where p is a prime and $S_p(m)$ is the number of factors of p in $m!$. Let

$$\mathcal{M} = \{m, s, p)) \mid \limsup_k N_k(m, s, p) = +\infty\}.$$

LEMMA A.1. (a) $S_p(p^r) = (p^r - 1)/p - 1$
(b) $S_p(p^r - 1) = (p^r - 1)/(p - 1) - r$
(c) $S_p(p^r m) = m(p^r - 1)/(p - 1) + S_p(m)$
(d) $S_p(m(p^r - 1)) \leq m(p^r - 1)/(p - 1) - r$.

Proof. Only (a) is proved as (b), (c), and (d) follow by similar arguments.

$$S_p(p^r) = \sum_{i=1}^{\infty} [p^r/p^i] = p^{r-1} + \cdots + p + 1 = (p^r - 1)/(p - 1). \qquad ∎$$

PROPOSITION A.2. *If $m/(p - 1) \leq S_p(m) + s$ then $(m, s, p) \in \mathcal{M}$.*

Proof. Let $k_j = p^j - 1$. Then by A.1(d), $S_p(k_j m) \leq m(p^j - 1)/(p - 1) - j$. Hence

$$\begin{aligned} N_{k_j}(m, s, p) &= s(p^j - 1) + (p^j - 1)S_p(m) - S_p(k_j m) \\ &\geq p^j(s + S_p(m) - m/(p - 1)) + j + K \end{aligned}$$

where $K = m/(p - 1) + S_p(m) - s$. Since $m/(p - 1) \leq S_p(m) + s$, $\lim_j N_{k_j} = +\infty$ so $\limsup N_k(m, s, p) = +\infty$. ∎

COROLLARY A.3. (a) *$(p^r, s, p) \in \mathcal{M}$ for all $r \geq 0$, $s \geq 1$ and all primes p.*
(b) *$(m, s, p) \in \mathcal{M}$ if $m \leq s(p - 1)$.*
(c) *For all m and s, there exists a prime q such that $(m, s, p) \in \mathcal{M}$ for all primes $p \geq q$.*

Proof. In each case it is easily verified that $m/(p - 1) \leq S_p(m) + s$ so A.2 applies. ∎

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The University of Wisconsin–Milwaukee
Milwaukee, Wisconsin